Recitation 4.1

Today: Review for Midterm.

Remark about HW2 Q4:

a) $Y \sim N(0,1)$, $x \in \mathbb{R}$. find $\mathbb{E}[(e^{x+Y} - K)^+]$.

Integration $\Rightarrow$ $e^{x+\frac{1}{2}} N(d_1) - K N(d_2)$, $d_1 = 1 + x - \ln K$, $d_2 = d_1 - 1$

b). $Y, X \sim N(0,1)$ independent. $\mathbb{E}[(e^{X+Y} - K)^+ | X](\omega)$

Ind Lemma $\Rightarrow$ $\mathbb{E}[(e^{X+Y} - K)^+ | X](\omega) = g(X(\omega))$

where $g(x) = \mathbb{E}[(e^{x+Y} - K)^+]$.

many people said ; $\mathbb{E}[(e^{X+Y} - K)^+ | X] = e^{x+\frac{1}{2}} N(d_1) - K N(d_2)$.

without saying $X(\omega)$ is a random variable.

Ex 1 a). $f: [0, T] \longrightarrow \mathbb{R}$ deterministic with $\int_0^T f(u)^2 du < \infty$.

Find $\mathbb{E}\left[e^{t\int_0^T f(u)\, dW(u)}\right]$.

what is the distribution of $\int_0^T f(u)\, dW(u)$?

$$\int_0^T f(u)\, dW(u) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\tfrac{Ti}{n}\right)\underbrace{\left(W\left(\tfrac{T(i+1)}{n}\right) - W\left(\tfrac{Ti}{n}\right)\right)}_{N(0, \frac{1}{n})}.$$

$$\overset{d}{=} \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\tfrac{Ti}{n}\right) N\left(0, \tfrac{1}{n}\right) \overset{d}{=} \lim_{n\to\infty} \sum_{i=0}^{n-1} N\left(0, \frac{f\left(\frac{Ti}{n}\right)^2}{n}\right).$$

$$\overset{d}{=} \lim_{n\to\infty} N\left(0, \frac{\sum_{i=1}^{n} f\left(\frac{Ti}{n}\right)^2}{n}\right) \overset{d}{=} N\left(0, \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f\left(\frac{Ti}{n}\right)^2}{n}\right).$$

$$\overset{d}{=} N\left(0, \int_0^T f(u)^2 du\right).$$

We actually proved Ito Isometry for deterministic functions:

<u>Recall</u>: (Ito Isometry). $E\left[\left(\int_0^T X_t\, dW_t\right)^2\right] = E\left[\int_0^T X_t^2\, dt\right]$.

$$M = \int_0^T f(u)\, dW(u).$$

$E[M^2]$. $\quad M \sim N\left(0, \int_0^T f(u)^2\, du\right)$.

so $E[M^2] = \int_0^T f(u)^2\, du$.

$$E\left[e^{t \int_0^T f(u)\, dW(u)}\right] = e^{\frac{t^2}{2}\int_0^T f(u)^2\, du}$$ by formula of a Normal MGF.

b). Is $Z(s) \doteq \exp\Big(\underbrace{t \int_0^s f(u)\, dW(u) - \frac{t^2}{2}\int_0^s f(u)^2\, du}_{X(s)}\Big)$ ⊛ a martingale?

$Z(s) = \exp(X(s))$. $\quad f(s, x) = e^x \qquad f_s = 0.$

$f_x = e^x = f_{xx}$

$dZ_s = e^{X(s)} dX_s + \frac{1}{2} e^{X(s)} d[X, X]_s$

from ⊛ $dX_s = t\, f(s)\, dW(s) - \frac{t^2}{2} f(s)^2\, ds$

$d[X, X]_s = t^2 f(s)^2\, ds$

$$dZ_s = e^{X(s)}\left[t f(s)\, dW(s) - \cancel{\frac{t^2}{2} f(s)^2\, ds}\right]$$

$$+ \cancel{\frac{1}{2} e^{X(s)} t^2 f(s)^2\, ds} = e^{X(s)}\, t f(s)\, dW(s).$$

So Z is a martingale!

<u>Ex2</u>: Find $E\left[\cos\left(e^{W_s^2}\right) \int_0^t \frac{\log(W^2(u)+1)}{W^4(u)+2W^2(u)+1}\, dW(u) \,\middle|\, \mathcal{F}_s\right]$, $s<t$

<u>Answer can be an integral, but should have no expectations.</u>

$$= \cos\left(e^{W_s^2}\right) E\left[\int_0^t \frac{\log(W^2(u)+1)}{W^4(u)+2W^2(u)+1}\, dW(u) \,\middle|\, \mathcal{F}_s\right]$$

$$= \cos\left(e^{W_s^2}\right) \int_0^s \frac{\log(W^2(u)+1)}{W^4(u)+2W^2(u)+1}\, dW(u)$$

Done!

Ex 3. $X(t) = \int_0^t s\, dW_s$, Find $\mathbb{E}[X(t)^n]$

Quick way!

showed $\int_0^t f(s)\, dW_s \sim N(0, \int_0^t f^2(s)\, ds)$

here take $f(s) = s$.

then $\int_0^t s\, dW_s \sim N(0, \frac{t^3}{3})$.

$$\mathbb{E}[X^n] = \begin{cases} 0 & n \text{ is odd.} \\ \sigma^n (n-1)!! & n \text{ is even.} \end{cases}$$ (from Wikipedia)

(Check this!)

Using Ito! $\mathbb{E}[X^n] = \frac{1}{2} n(n-1) \int_0^t s^2\, \mathbb{E}[X^{n-2}]\, ds$.

(good Exercise!)

Ex 4. Show that $Y_t = t^2 W_t^3$, $t \geq 0$ satisfies the SDE:

$$dY_t = \left(2\frac{Y_t}{t} + 3(t^4 Y_t)^{1/3}\right)dt + 3(t Y_t)^{2/3} dW_t$$

with initial condition: $Y(0) = 0$. ✓

define $f(t,x) = t^2 x^3$ $\quad f_t = 2t x^3$, $f_x = 3t^2 x^2$, $f_{xx} = 6t^2 x$.

$$dY_t = 2t W_t^3 dt + 3t^2 W_t^2 dW_t + \frac{1}{2}(6t^2 W_t)dt$$

Aside:

Ito: $Z = f(t, X(t))$ then $dZ = f_t\,dt + f_x\,dX + \frac{1}{2} f_{xx}\,d[X,X]_t$

$Y = g(t, W(t))$ $\quad dW_t = 0\,dt + 1\,dW_t$

$$= \left(2\frac{t^2 W_t^3}{t} + 3t^2 W_t\right)dt + 3t^2 W_t^2 dW_t.$$

$$= \left(2\frac{Y(t)}{t} + 3(t^6 W_t^3)^{1/3}\right)dt + 3(t^3 W_t^3)^{2/3} dW_t.$$

$$= \left(2\frac{Y(t)}{t} + (t^4 Y(t))^{1/3}\right)dt + 3(tY(t))^{2/3} dW_t. \checkmark$$

Ex 5 Is $X_t = e^{W_t + \frac{t}{2}} + e^{W_t - \frac{t}{2}}$ a martingale?

$$W_t \pm \frac{t}{2} \sim N(\pm\tfrac{t}{2}, t)$$

$$E[X(t)] = e^{\frac{t}{2} + \frac{t^2}{2}} + e^{-\frac{t}{2} + \frac{t^2}{2}}$$

$$= e^{\frac{t^2}{2}}\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)$$

Depends on t, not a martingale!

Ex 6. $X(t) = \int_0^t e^{W(u)}\, dW(u)$, $0 < s < t$, Find $E[X_t^2 | \mathcal{F}_s]$.

(Thm $M^2 - [M,M]$ is a martingale if M is a martingale). (Try it!)

Warning! X_t^2 is not a martingale so:

$$E[X_t^2 | \mathcal{F}_s] \neq X_s^2.$$

Instead use Ito. $dX_t = e^{W(t)}\,dW_t$

$$(dX_t^2) = 2X_t\,dX_t + \tfrac{1}{2}(2)e^{2W(t)}\,dt$$

$$\underset{d[X,X](t)}{\parallel}$$

Key step!

$$X_t^2 - X_s^2 = \int_s^t 2X(u)\,e^{W(u)}\,dW(u) + \int_s^t e^{2W(u)}\,du$$

$$\Rightarrow E[X_t^2 \mid \mathcal{F}_s] = X_s^2 + E\left[\int_s^t 2X(u)\,e^{W(u)}\,dW(u) \,\Big|\, \mathcal{F}_s\right]$$

$$+ E\left[\int_s^t e^{2W(u)}\,du \,\Big|\, \mathcal{F}_s\right]$$

$$= X_s^2 + 0 + \int_s^t E[e^{2W(u)} \mid \mathcal{F}_s]\,du \rightarrow$$

We now reduced the problem to finding $E[e^{2W(u)} \mid \mathcal{F}_s]$.

<u>Notice</u>: $E\left[\int_s^t Y_u\,dW_u \,\Big|\, \mathcal{F}_s\right] = E\left[\int_0^t Y(u)\,dW(u) \,\Big|\, \mathcal{F}_s\right] - E\left[\int_0^s Y(u)\,dW(u) \,\Big|\, \mathcal{F}_s\right]$

$$= 0$$

$$E[e^{2W(u)} | \mathcal{F}_s] = E[e^{2(W(u) - W(s)) + 2W(s)} | \mathcal{F}_s]$$

$$= E[e^{2(W(u) - W(s))}] e^{2(W(s))} \qquad 2(W(u) - W(s)) \sim N(0, 4(u-s))$$

then $t=1$.

$$= [e^{\frac{1}{2}(4(u-s))}] e^{2(W(s))} = e^{2(u-s) + 2W(s)}$$

or

$t = 2 \quad N(0, u-s)$.

plugging back in:

$$E[X_t^2 | \mathcal{F}_s] = X_s^2 + \int_s^t e^{2(u-s) + 2W(s)} du.$$

$$= X_s^2 + e^{2W(s)} \cdot \left[\frac{1}{2} e^{2(u-s)} \Big|_{u=s}^{u=t} \right]$$

$$= X_s^2 + \frac{1}{2} e^{2W(s)} \left[e^{2(t-s)} - 1 \right].$$

$W(u) - W(s) \perp \mathcal{F}_s$.

<u>but</u> $W(u)$ may not be.