Review Problems:

① $r \le s \le t$. Compute $E\big(W(r)W(s)W(t)\big)$.

Sol: Trick 1: $E\big(W(r)W(s)W(t)\big) =$

$$E\Big(W(r)\big(W(s)-W(r)+W(r)\big)\big(W(t)-W(s)+W(s)-W(r)+W(r)\big)\Big)$$

... will eventually work. Boring.

Trick 2: $E\big(W(r)W(s)W(t)\big) = E\,E\big(W(r)W(s)W(t) \,\big|\, \mathcal{F}_s\big)$

$$= E\,W(r)W(s)\,E\big(W(t) \,|\, \mathcal{F}_s\big)$$

$$= E\,W(r)W(s)\,W(s)$$

$$= \mathbb{E}\, \mathbb{E}\left( W(r)\, W(s)^2 \,\middle|\, \mathcal{F}_r \right).$$

$$= \mathbb{E}\, W(r)\, \mathbb{E}\left( W(s)^2 \,\middle|\, \mathcal{F}_r \right). \quad \text{---} \quad (*)$$

Need to Compute $\mathbb{E}\left( W(s)^2 \,\middle|\, \mathcal{F}_r \right)$.

Trick 1 : $\mathbb{E}\left( W(s)^2 \,\middle|\, \mathcal{F}_r \right) = \mathbb{E}\left( (W(s) - W(r) + W(r))^2 \,\middle|\, \mathcal{F}_r \right)$

expand & simplify.

(can also use independence lemma).

Trick 2: Itô Formula.

Trick 2: Know $W(s)^2 - s$ is a mg.

$$\Rightarrow E\left( W(s)^2 - s \,\middle|\, \mathcal{F}_r \right) = W(r)^2 - r$$

$$\boxed{\Rightarrow E\left( W(s)^2 \,\middle|\, \mathcal{F}_r \right) = W(r)^2 + (s - r)}$$

Trick 3: Itô: $d\left(W(s)^2\right) = 2\, W(s)\, \cancel{ds}\, dW(s)$

$$+ \frac{1}{2} \cdot 2\, d[W, W](s).$$

(Ito with $f(t, x) = x^2$). $= 2\, W(s)\, dW(s) + ds.$

$$\Rightarrow W(s)^2 - W(0)^2 = 2 \int_0^s W(u)\, dW(u) + \int_0^s 1\, ds.$$

$$\Rightarrow W(s)^2 - W(r)^2 = 2\int_r^s W(u)\,dW(u) + (s-r).$$

$$\Rightarrow W(s)^2 = W(r)^2 + 2\int_r^s W(u)\,dW(u) + (s-r)$$

$$\Rightarrow E\left(W(s)^2 \mid \mathcal{F}_r\right) = E\left(\text{———} \mid \mathcal{F}_r\right)$$

$$= W(r)^2 + \underbrace{2\int_r^r W(u)\,dW(u)}_{0} + s-r$$

$$\boxed{= W(r)^2 + (s-r)}.$$

Substitute in (*):

$$E\big(W(r)W(s)W(t)\big) = E\,W(r)\big(W(r)^2 + (s-r)\big)$$

$$= E\,W(r)^3 + (s-r)\,E\,W(r) = 0$$

Eg 2: $X(t) = \int_0^{W(t)} e^{-s^2}\, ds$.

Goal: find Ito decomposition & compute QV.

$\Big( X = M + B$, $M \to$ cts mg

$B \to$ cts BoV.

(finite 1st var) $\Big)$.

Sol: Let $f(x) = \int_0^x e^{-s^2}\, ds$.

$$\boxed{\begin{aligned} \partial_t f &= 0 \\ \partial_x f &= e^{-x^2} \\ \partial_x^2 f &= -2x\, e^{-x^2} \end{aligned}}$$

$X(t) = f(W(t))$.

$\Rightarrow dX = \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, dt$

$$\Rightarrow X(t) - X(0) = \int_0^t e^{-W(s)^2}\, dW(s) - \int_0^t W(s) e^{-W(s)^2}\, ds.$$

$(X(0) = 0)$ : $\boxed{\cancel{e^{W(t) - \frac{1}{2}t} = M(t)}}$

$$\Rightarrow X(t) = 0 + M(t) + B(t)$$

$$M = \underbrace{\int_0^t e^{-W(s)^2}\, dW(s)}_{Mg} \quad \& \quad B(t) = -\underbrace{\int_0^t W(s) e^{-W(s)^2}\, ds}_{BV}.$$

$$QV : [X, X](t) = \int_0^t \left(e^{-W(s)^2}\right)^2 ds.$$

Recall : $\dfrac{d}{dx} \displaystyle\int_0^x F(y)\,dy = F(x)$.

& $\displaystyle\int_0^x \cancel{d}\, \partial_y F(y)\, dx = F(x) - F(0)$.

---

Eg 2 : $Y(t) = \exp\left( \displaystyle\int_0^t W(s)\, ds \right)$. Find Ito decomp & QV of $Y$.

<u>Claim:</u> $Y$ already has finite first var & QV of $Y = 0$.

Note $\displaystyle\int_0^t W(s)\, ds$ is diff in time & has finite $1^{st}$ var.

$\Rightarrow \exp(\quad)$ & $\Rightarrow QV = 0$.

**Eg 3:** $X(t) = \int_0^t W(s)\, ds$ & $Y(t) = \int_0^t W(s)\, dW(s)$.

Compute $E\big(X(t) \mid \mathcal{F}_s\big)$ & $E\big(Y(t) \mid \mathcal{F}_s\big)$.

Sol: ① $E\big(X(t) \mid \mathcal{F}_s\big) = E\left( \int_0^t W(\tau)\, d\tau \;\middle|\; \mathcal{F}_s \right)$.

$$\overset{(*)}{=} \int_0^t E\big(W(\tau) \mid \mathcal{F}_s\big)\, d\tau.$$

**Reason:** $E\big(Y_1 + Y_2 \mid \mathcal{F}_s\big) = E\big(Y_1 \mid \mathcal{F}_s\big) + E\big(Y_2 \mid \mathcal{F}_s\big)$

$$\int_0^t W(\tau)\,d\tau = \lim_{\|P\|\to 0} \sum \cancel{W(t_i)}\, W(\tau_i)\,(\tau_{i+1}-\tau_i).$$

$$E\left(\int_0^t W(\tau)\,d\tau \,\middle|\, \mathcal{F}_s\right) = E\left(\lim_{\|P\|\to 0} \underline{\hspace{3cm}} \,\middle|\, \mathcal{F}_{\cancel{\tau}s}\right).$$

$$= \lim_{\|P\|\to 0} E\left(\sum W(\tau_i)(\tau_{i+1}-\tau_i) \,\middle|\, \mathcal{F}_{\cancel{\tau}s}\right).$$

$$= \lim_{\|P\|\to 0} \sum E\left(W(\tau_i) \,\middle|\, \mathcal{F}_\tau\right)(\tau_{i+1}-\tau_i)$$

$$= \int E\left(W(\tau) \,\middle|\, \mathcal{F}_s\right) d\tau.$$

$$\Rightarrow E\left(X(t) \mid \mathcal{F}_s\right) = E\left(\int_0^t W(\tau)\, d\tau \;\Big|\; \mathcal{F}_s\right)$$

$$= \int_0^t E\left(W(\tau) \mid \mathcal{F}_s\right) d\tau$$

$$= \int_0^s E\left(W(\tau) \mid \mathcal{F}_s\right) d\tau + \int_s^t E\left(W(\tau) \mid \mathcal{F}_s\right) d\tau$$

$$= \int_0^s W(\tau)\, d\tau + \int_s^t W(s)\, d\tau$$

$$= \int_0^s W(\tau)\, d\tau + W(s)(t-s).$$

Compute $\mathbb{E}\left( \int_0^t W(\tau)\, dW(\tau) \,\middle|\, \mathcal{F}_s \right) =$

$$\cancel{\int_0^t \mathbb{E}\left( W(\tau) \,\middle|\, \mathcal{F}_s \right) dW(\tau)}$$ HOPELESSLY WRONG.

Ito integral wrt $W$ is a mg

$$\Rightarrow \mathbb{E}\left( \int_0^t W(\tau)\, dW(\tau) \,\middle|\, \mathcal{F}_s \right) = \int_0^s W(\tau)\, dW(\tau).$$

Problem: 7.4: Let $M(t) = \int_0^t W(s)\, dW(s)$ (is a mg).

Find a fn $f$ such that the process.

$$\xi(t) = \exp\left(M(t) - \int_0^t f(s, W(s))\, ds\right) \text{ is a mg.}$$

Sol: Let $B(t) = -\int_0^t f(s, W(s))\, ds$ (finite 1st var BV).

Let $\underset{\text{mg}}{X = M} + \underset{\text{BV}}{B}$ (Ito decompostion).

Note $\xi(t) = e^{X(t)}$ & Itô.

$\boxed{g(x) = e^x, \ \partial_x g = e^x \ \& \ \partial_x^2 g = e^x}$

$$d\xi(t) = e^{X(t)} dX(t) + \frac{1}{2} e^{X(t)} d[X,X].$$

$$dX = dM + dB = W(t)\,dW(t) - f(t, W(t))\,dt.$$

$$\Rightarrow d[X,X](t) = W(t)^2 dt$$

$$\Rightarrow d\xi(t) = e^{X(t)} W(t)\,dW(t) - e^{X(t)} f(t, W(t)) \cdot dt$$

$$+ \frac{1}{2} e^{X(t)} W(t)^2 dt$$

Want $\xi$ to be a mg.

Choose $f$ so that the dt terms cancel.

$$\Rightarrow d\xi(t) = e^{X(t)} W(t)\, dW(t)$$

$$+ \left\{ e^{X(t)} \left( \frac{W(t)^2}{2} - f(t, W(t)) \right) dt \right. .$$

Choose $f(t, W(t)) = \dfrac{W(t)^2}{2}$ $\left( \text{or } f(t,x) = \dfrac{x^2}{2} \right)$.

Get $\xi(t) = \exp\left( \int_0^t W(s)\, dW(s) - \frac{1}{2} \int_0^t W(s)^2\, ds \right)$ is a mg.

$\boxed{7.5}$ $\sigma(t) \longrightarrow$ NOT RANDOM. (Eg: $\sigma(t) = 1$).

Define $X(t) = \int_0^t \sigma(u)\, dW(u)$ ($\sigma \to$ not random).

ⓐ Given $\lambda, s, t$, $s \leq t$, compute.

$$E\left[\exp\left(\lambda(X(t) - X(s))\right) \Big| \mathcal{F}_s\right].$$

$Y(t) = f(t, X(t))$

Sol: Let $Y(t) = e^{\lambda X(t)}$, Compute $dY$: $f(t,x) = e^{\lambda x}$

$$dY(t) = \lambda e^{\lambda X(t)} dX(t) + \frac{1}{2}\lambda^2 e^{\lambda X(t)} d[X,X](t).$$

$$\Rightarrow E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = e^{\frac{\lambda^2}{2}\int_s^t \sigma^2(u)\,du}$$

Note $\Rightarrow$ $$E\, e^{\lambda(X(t)-X(s))} = \exp\left(\frac{\lambda^2}{2}\int_s^t \sigma^2(u)\,du\right).$$

$$= \text{MGF of } N\left(0, \int_s^t \sigma^2(u)\,du\right).$$

$$\Rightarrow X(t)-X(s) \sim N\left(0, \int_s^t \sigma^2(u)\,du\right).$$

Let $\varphi(t) = E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$.

Take $E(\quad | \mathcal{F}_s)$ of $(**)$.

$\sigma^2$ not random.

Get $\varphi(t) = 1 + 0 + \frac{\lambda^2}{2}\int_s^t E\left(e^{\lambda(X(u)-X(s))}\sigma^2(u) \,\middle|\, \mathcal{F}_s\right) du$

$$= 1 + \frac{\lambda^2}{2}\int_s^t \sigma^2(u)\varphi(u)\,du$$

$\Rightarrow \partial_t \varphi = \frac{\lambda^2}{2}\sigma(t)^2\varphi(t)$

$$\boxed{\Rightarrow \varphi(t) = 1 \exp\left(\int_s^t \frac{\lambda^2}{2}\sigma(u)^2\,du\right)}$$

Compute: $dX = \sigma \, dW$

$$d[X,X](t) = \sigma(t)^2 \, dt$$

$$\Rightarrow dY(t) = \lambda e^{\lambda X(t)} \sigma(t) \, dW(t) + \frac{\lambda^2}{2} e^{\lambda X(t)} \sigma^2(t) \, dt.$$

$$\Rightarrow Y(t) - Y(s) = \int_s^t \lambda e^{\lambda X(u)} \sigma(u) \, dW(u) + \frac{\lambda^2}{2} \int_s^t e^{\lambda X(u)} \sigma^2(u) \, du$$

$$\Rightarrow e^{\lambda X(t)} - e^{\lambda X(s)} =$$

$$\Rightarrow e^{\lambda(X(t) - X(s))} = 1 + \int_s^t \lambda e^{\lambda(X(u) - X(s))} \sigma(u) \, dW(u)$$

$$+ \frac{\lambda^2}{2} \int_s^t e^{\lambda(X(u) - X(s))} \sigma^2(u) \, du \quad \cdots \ast\ast$$

(b) Compute $E\left(e^{\lambda X(r) + \mu(X(t) - X(s))}\right)$ $\quad r \leq s < t.$

$EE\left( \quad \middle| \mathcal{F}_s \right)$ & use part a.

get $\exp\left(\frac{\lambda^2}{2}\int_0^r \sigma(u)^2\,du\right) \exp\left(\frac{\mu^2}{2} \int_s^t \sigma(u)^2\,du\right).$

$\Longrightarrow$ $X(t) - X(s)$ is ind of $X(r)$.

$\Rightarrow \begin{pmatrix} X(r) \\ X(t)-X(s) \end{pmatrix} \sim N\left(0, \begin{pmatrix} \int_0^r \sigma(u)^2\,du & 0 \\ 0 & \int_s^t \sigma^2(u)\,du \end{pmatrix}\right).$

(d) Say $\sigma^2 = 1$. $\left( \Rightarrow d[X,X] = \sigma^2 dt = dt \Rightarrow [X,X] = t \right)$.

then (c) gives. (1) $X(t) \sim N\left(0, \int_0^t 1\,dt\right)$.

$$= N(0, t).$$

Also $X(t) - X(s) \sim N(0, t-s)$.

& is independent of $X(r)$ for $r \leq s$.

$\Rightarrow X$ is BM!!