$X \longrightarrow$ Itô process.

$$X(t) = \underbrace{X(0)}_{\text{not random.}} + \underbrace{B(t)}_{\substack{\text{bdd variation} \\ \text{(finite } 1^{st} \text{ var)}}} + \underbrace{M(t)}_{\text{Mg}}.$$

$$M(t) = \overbrace{\int_0^t \sigma(s)\, dW(s)}^{\text{Itô integral.}}$$

$$B(t) = \underbrace{\int_0^t b(s)\, ds}_{\text{R-int}}$$

($b \rightarrow$ process (adapted))

$\sigma \rightarrow$ adapted process.

Given a process D, define

$$\int_0^t D(s)\, dX(s) = \underbrace{\int_0^t D(s)\, b(s)\, ds}_{\text{Rieman}} + \underbrace{\int_0^t D(s)\, \sigma(s)\, dW(s)}_{\text{Itô.}}$$

Note: D "nice process" $\longrightarrow$ Know $\int_0^t D(s)\,dW(s)$ is a mg.

But $\int_0^t D(s)\,dX(s)$ need not be a mg.

(is only guaranteed to be a mg if $B \equiv 0$
$(\Leftrightarrow b \equiv 0)$).

Say $f = f(t,x)$ is C^1 ($\partial_t f$ & $\partial_x f$ exist & are cts.

Say $X = X(t)$ is also C^1

$\partial_t f = \dfrac{\partial f}{\partial t} \boxed{= f_t}$

[False for 99.99% of Ito processes)

avoid.

$$Y(t) = f(t, X(t)).$$

$$\partial_t Y = \partial_t f(t, X(t)) + \partial_x f(t, X(t))\, \partial_t X \quad (\text{Chain rule}).$$

$$\Rightarrow Y(t) - Y(0) = \int_0^t \partial_t Y(s)\, ds.$$

$$= \int_0^t \partial_t f(s, X(s))\, ds + \int_0^t \partial_x f(s, X(s))\, \underbrace{\partial_t X(s)\, ds}_{dX}.$$

$$f(t, X(t)) - f(0, X(0)) = \int_0^t \partial_t f(s, X(s))\, ds + \int_0^t \partial_x f(s, X(s))\, dX(s).$$

Itô Formula: Do the same when X is an Itô process.

① f is $= f(t,x) \longrightarrow C^{1,2}$ ($\partial_t f$, $\partial_x f$, $\partial_x^2 f$ exist & are cts).

② $X \longrightarrow$ Itô process.

Then
$$f(t,X(t)) = f(0,X(0)) + \int_0^t \partial_t f(s,X(s))\,ds + \int_0^t \partial_x f(s,X(s))\,dX(s) + \frac{1}{2}\int_0^t \partial_x^2 f(s,X(s))\,d[X,X](s)$$

Notation: "Differential form" of Itô's formula

$$d\left(f(t, X(t))\right) = \partial_t f(t, X(t))\, dt$$
$$+ \partial_x f(t, X(t))\, dX(t) + \frac{1}{2} \partial_x^2 f(t, X(t))\, d[X,X](t).$$

Also: If $dX = b(t)\,dt + \sigma(t)\,dW(t)$

$$\left(\text{i.e. } X(t) - X(0) = \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s)\right).$$

~~Then~~ $\cancel{d\left(f(t, X(t))\right) = \partial_t f}$

Riemann.

then
$$d\big(f(t,X(t))\big) = \partial_t f(t,X(t))\,dt$$

Riemann. Itô

$$+\ \partial_x f(t,X(t))\,b(t)\,dt + \partial_x f(t,X(t))\,\sigma(t)\,dW(t)$$

$$+\ \frac{1}{2}\,\partial_x^2 f(t,X(t))\,\sigma(t)^2\,dt$$

Also called Itô – Doeblin formula.

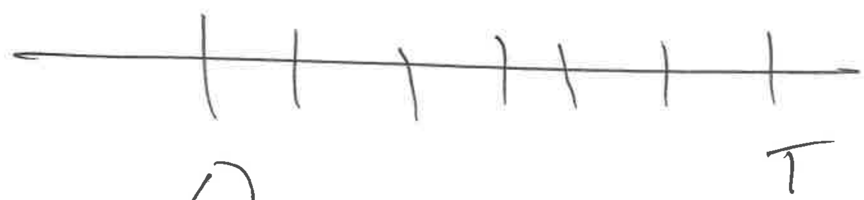
Intuition: Simplest case: $f(t,x) = f(x)$.

& $X = W$

Itô formula: $$d\big(f(W(t))\big) = 0\,dt + \partial_x f(W(t))\,dW + \frac{1}{2}\partial_x^2 f(W(t))\,dt$$

Let $Y(t) = f(W(t))$. Fix T. $P = \{0 = t_0 < t_1 < \cdots t_n = T\}$.

$$Y(T) - Y(0) = \sum Y(t_{i+1}) - Y(t_i)$$
$$= \sum \Delta_i Y$$



$(\Delta_i Y = Y(t_{i+1}) - Y(t_i))$.

$$= \sum f(W(t_{i+1})) - f(W(t_i))$$

$$\overset{\text{(Taylor Exp)}}{=} \sum \left(W(t_{i+1}) - W(t_i)\right) \partial_x f(W(t_i))$$

$$+ \sum \left(W(t_{i+1}) - W(t_i)\right)^2 \frac{\partial_x^2 f(W(t_i))}{2}$$

$$+ \text{Remainder terms. } (\text{small}).$$

① as $\|P\| \to 0$, Remainder terms $\longrightarrow 0$

② $$\lim_{\|P\| \to 0} \sum \left(W(t_{i+1}) - W(t_i)\right) \partial_x f\left(W(t_i)\right) \longrightarrow \int_0^T \partial_x f(W(s))\, dW(s)$$

Ito integral.

(3) $$\sum \frac{\partial_x^2 f(W(t_i))}{2}(\Delta_i W)^2$$

$$= \sum \frac{\partial_x^2 f}{2}\left[(t_{i+1}-t_i) + (\Delta_i W)^2 - (t_{i+1}-t_i)\right].$$

$$= \underbrace{\sum \frac{\partial_x^2 f}{2}(t_{i+1}-t_i)}_{\xrightarrow{\|P\|\to 0}\ \int_0^T \partial_x^2 f(W(s))\,ds} + \underbrace{\sum \frac{\partial_x^2 f}{2}\left((\Delta_i W)^2 - (t_{i+1}-t_i)\right)}_{\to\ 0}.$$

because $\longrightarrow$

Let $\xi_i = (\Delta_i W)^2 - (t_{i+1} - t_i)$

Know $\xi_i \sim N(0, t_{i+1} - t_i)^2 - (t_{i+1} - t_i)$

Check: $E\,\xi_i = 0$ & $\mathrm{Var}(\xi_i) = (t_{i+1} - t_i)^2 \cdot c.$

Note $\sum \mathrm{Var}(\xi_i) = c \sum (t_{i+1} - t_i)^2 \xrightarrow{\|P\| \to 0} 0$

Hence $\sum \frac{\partial_x^2 f}{2} \xi_i \longrightarrow 0$

Var $\to 0$ mean.

QED.

Eg: Compute QV of $W(t)^2$

Guess: $[W^2, W^2](t) = t^2$ (Won't work).

Bad way: $\lim_{\|P\| \to 0} \sum \left(W(t_{i+1})^2 - W(t_i)^2 \right)^2$

Better way: Ito formula: Let $f(t,x) = x^2$.

$\partial_t f = 0$
$\partial_x f = 2x$
$\partial_x^2 f = 2$

Let $Y(t) = W(t)^2, = f(W(t))$

Itô: $dY = \partial_t f(t, W(t))\, dt + \partial_x f(t, W(t))\, dW(t)$
$+ \frac{1}{2} \partial_x^2 f(t, W(t))\, d[W,W](t)$

$$= \partial_t f\, dt + \partial_x f\, dW + \frac{1}{2} \partial_x^2 f\, dt$$

$$= 0\, dt + 2W(t)\, dW(t) + \frac{1}{2} 2\, dt$$

$$\Rightarrow Y(t) - Y(0) = 2\int_0^t W(s)\, dW(s) + t$$

$$\Rightarrow W(t)^2 = 2\int_0^t W(s)\, dW(s) + t$$

$$\boxed{\Rightarrow [W^2, W^2](t) = 4\int_0^t W(s)^2\, ds}$$

[Guess $[W,W] = t^2$ was wrong]

Eg 2: $M(t) = W(t)$

$N(t) = W(t)^2 - t$

Know M, N are mg. Is MN a mg?

Sol: Note $MN = W(t)^3 - t\,W(t)$

↳ apply Ito $\longrightarrow$ $MN = \underbrace{\int (\ )\,dt} + \int (\ )\,dW$

MN will only be a mg if $= 0$

Let $f(t,x) = x^3 - tx$

Then $MN = f(t, W(t))$.

$$\boxed{\begin{aligned} \partial_t f &= -x \\ \partial_x f &= 3x^2 - t \\ \partial_x^2 f &= 6x \end{aligned}}$$

Ito: $d(MN) = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt$

$$= -W(t)\,dt + (3W(t)^2 - t)\,dW + 3W(t)\,dt$$

$$= 2W(t)\,dt + (3W(t)^2 - t)\,dW$$

$$\Rightarrow MN(t) = \underbrace{\int_0^t 2W(s)\,ds}_{\text{Not a mg}} + \underbrace{\int_0^t (3W(s)^2 - \cancel{t}s)\,dW(s)}_{\text{Mg}}$$

$\Rightarrow MN$ is not a mg.

Eg 3: $X(t) = t \sin(W(t))$

Q: Is $X(t)^2 - [X,X](t)$ a mg?

Note: If X is a mg know $X(t)^2 - [X,X](t)$ is mg.

If not $\rightarrow$ dont know!

Sol: Let $f(t,x) = t \sin x$

$$\partial_t f = \sin x , \quad \partial_x f = t \cos x$$

$$\partial_x^2 f = -t \sin x$$

$X(t) = f(t, W(t))$.

$$\Rightarrow dX = \partial_t f \, dt + \partial_x f \, d\overset{W}{\cancel{X}} + \frac{1}{2} \partial_x^2 f \, d[\overset{W}{\cancel{X}}, \overset{W}{\cancel{X}}].$$

$$= \sin W(t) \, dt + t \cos W(t) \, dW \cancel{+} - \frac{1}{2} t \sin(W(t)) \, dt$$

$$= \left(\sin(W(t)) - \frac{t}{2} \sin(W(t)) \right) dt + t \cos W(t) \, dW.$$

Let $g(x) = x^2$. Compute $d(X^2)$.

Ito: $$d(X^2) = 2X \, dX + \frac{1}{2} \cdot 2 \, d[X, X]$$

$$= 2X \Big(\sin(W(t)) \left(1 - \frac{t}{2}\right) dt + 2X \, t \cos W(t) \, dW$$

$$+ t^2 \cos^2 W(t) \, dt$$

Also $d[X,X] = t^2 \cos^2 W(t)\, dt$

not 0

$$\Rightarrow d\left(X^2 - [X,X]\right) = 2X \sin(W(t)) \left(1 - \frac{t}{2}\right) dt$$

$$+ 2Xt \cos(W(t))\, dW$$

$\Rightarrow X^2 - [X,X]$ is not a mg.

~~Eg~~:
$$M(t) = \int_0^t \sigma(s)\, dW(s).$$

Know $[M,M](t) = \int_0^t \sigma(s)^2\, ds.$

(Also $E M^2 = E\left(\int_0^t \sigma(s)\, dW(s)\right)^2 = E\int_0^t \sigma(s)^2\, ds.$).

Let $M(t) = \int_0^t \sigma(s)\, dW(s). \Rightarrow dM = \sigma\, dW$

Check $M^2 - [M,M]$ is a mg.

Pf: Let $f(x) = x^2$, $Y = f(M)$. Compute dY

$$\Rightarrow dY = 2M\,dM + \frac{1}{2}\cdot 2\,d[M,M].$$

$$\Rightarrow d(M^2) = 2M\,dM + d[M,M].$$

$$\Rightarrow d\left(M^2 - [M,M]\right) = 2M\,dM = \underbrace{2M\sigma\,dW}_{Mg}.$$

$$\Rightarrow M^2 - [M,M] \text{ is a } mg.$$