Recitation 3

Today: Ito Calculus

Δ - $\mathcal{F}_t$ adapted process.

Ito Integral: $I(T) = \int_0^T \Delta(t)\, dW_t := \lim_{n\to\infty} \sum_{i=0}^{n-1} \left[\Delta\left(\frac{Ti}{n}\right)\left[W\left(\frac{T(i+1)}{n}\right) - W\left(\frac{Ti}{n}\right)\right]\right]$.

makes sense when $\int_0^T \Delta_t^2\, dt < \infty$. In this case $E\left[\int_0^T \Delta_t^2\, dt\right] = [I, I](T)$.

$$\downarrow$$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1}\left[\Delta\left(\frac{Ti}{n}\right)\left(\frac{T(i+1)}{n} - \frac{Ti}{n}\right)\right]$$

Good Exercise: Compute $\int_0^T W_t\, dW_t$ using the defn above.

~~Hint: $W\left(\frac{Ti}{n}\right) = \frac{1}{2}\left[W\left(\frac{T(i+1)}{n}\right) + W\left(\frac{Ti}{n}\right)\right] - \frac{1}{2}\left[W\left(\frac{T(i+1)}{n}\right) - W\left(\frac{Ti}{n}\right)\right]$.~~

Hint: $W\left(\frac{Ti}{n}\right) = \frac{1}{2}\left[W\left(\frac{T(i+1)}{n}\right) + W\left(\frac{Ti}{n}\right)\right] - \frac{1}{2}\left[W\left(\frac{T(i+1)}{n}\right) - W\left(\frac{Ti}{n}\right)\right]$

Ito Process:

Ito Process: X is an Ito process if we can write.

$$X = X(0) + \int_0^T b(t)\,dt + \int_0^T \sigma(t)\,dW_t$$

1) $X(0) = c \in \mathbb{R}$.
2) $\int_0^T b(t)\,dt < \infty$.
3) $E\left[\int_0^T \sigma(t)\,dW_t\right]^2 < \infty$.

Ito Isometry.
$E\left[\int_0^T \sigma^2(t)\,dt\right]$.
$\longrightarrow [X,X](T) = \int_0^T \sigma^2(t)\,dt$.

Shorthand: $dX_t = \underbrace{b(t)\,dt}_{\text{drift}} + \underbrace{\sigma(t)\,dW_t}_{\text{volatility}}$

$$dX_t \Delta_t = \Delta_t b(t)\,dt + \Delta_t \sigma(t)\,dW_t$$

$$\int_0^T \Delta_t\,dX_t := \int_0^T \Delta_t b(t)\,dt + \int_0^T \Delta_t \sigma(t)\,W_t$$

Ito's lemma: $f(t,x) \in C^{1,2}$ then X_t Ito process.

$\delta f(0, X(0))$.

$$df(t,X(t)) = f_t(t,X(t))\,dt + f_x(t,X(t))\,dX_t + \frac{1}{2} f_{xx}(t,X(t))\,d[X,X]_t.$$

If $dX_t = b(t)\,dt + \sigma(t)\,dW_t$.

$$d[X,X]_t = \sigma^2(t)\,dt \quad = [X,X](T) = \int_0^T \sigma^2(t)\,dt$$

<u>Ex 1</u>: Back to $\int_0^T W_t\,dW_t$. Let's use Ito Lemma to solve this.

guess: $\frac{W_t^2}{2}$. $\quad f(t,x) = \frac{x^2}{2}$, use Ito with $X_t = W_t$

$$f_t = 0, \quad f_x = x, \quad f_{xx} = 1.$$

$$d\left(\frac{W_t^2}{2}\right) = 0\,dt + W_t\,dW_t + \frac{1}{2}\,d[W,W]_t \quad = W_t\,dW_t + \frac{1}{2}\,dt.$$

By Def'n.

$$dW_t = 0\,dt + (1)\,dW_t$$

$$\frac{W_t^2}{2} = \underbrace{f(0,W_0)}_{0} + \int_0^T W_t\,dW_t + \frac{1}{2}\int_0^T dt$$

$$= \int_0^T W_t\,dt + \frac{T}{2} \longrightarrow \int_0^T W_t\,dW_t = \frac{W_t^2}{2} - \frac{T}{2}$$

One benefit: Can evaluate Ito Integrals!

$dS = \mu S dt + \sigma S dW_t$ → Geometric Brownian Motion (GBM).
→ SDE

Remarks: i) We will this to model stock prices.

ii) μ is the "true" return of the stock.

iii) σ^2 is the volatility of the stock

iv) To price options it turns we don't need to actually know the value μ!

Let's solve GBM! if $\sigma = 0$ we have $dS = \mu S dt$.

which is solved by $S = S(0) e^{\mu t}$

A good guess: $S_0 e^{\mu t + \sigma W_t} := f(t, W_t)$ i.e.

$$f(t,x) = S_0 e^{\mu t + \sigma x}.$$

$$f_t = \mu f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f.$$

By Ito: $d(f(t,W_t)) = \mu f dt + \sigma f dW_t + \frac{1}{2}\sigma^2 f dt.$

if $f = S$ $dS = \mu S dt + \sigma S dW_t + \frac{1}{2}\sigma^2 S dt$ Not Quite!

$$S = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$ will work! (Check!).

$$= S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z_1}$$ where $Z \sim N(0,1)$

Ex 2. $dS = \mu S\,dt + \sigma S\,dW$, $f(x) = x^p$. find $d(S^p)$.

$$f_t = 0, \quad f_x = p x^{p-1}, \quad f_{xx} = p(p-1)x^{p-2}$$

$$d(S^p) = df(S) = 0\,dt + p S^{p-1}\,dS + \frac{1}{2}p(p-1)S^{p-2}\underbrace{[\sigma^2 S^2]}_{= [S,S]_t}dt$$

$$= pS^{p-1}(\mu S\,dt + \sigma S\,dW) + \frac{1}{2}p(p-1)S^p\sigma^2\,dt$$

$$= S^p\left[\left[p\mu + \frac{\sigma^2}{2}p(p-1)\right]dt + p\sigma\,dW\right].$$

when is this a martingale? (when is S^p a mart'gale).

$p = 0$, or $p = 1 - \frac{2\mu}{\sigma^2}$ (i.e. when "dt term" is 0).

Ex3 $X(t) = t e^{3W(t)}$. Find b, σ s.t.

$$X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW_s.$$

$$f(t,x) = te^{3x}, \quad f_t = e^{3x}, \quad f_x = 3te^{3x}, \quad f_{xx} = 9te^{3x}$$

$$te^{3W_t} = f(0,0) + \int_0^t e^{3W_s}\,ds + \int_0^t 3s e^{3W_s}\,dW_s$$

$$+ \frac{1}{2}\int_0^t 9s\, e^{3W_s}\,ds \quad (d[W,W]_t = dt)$$

$$= \underbrace{\int_0^t e^{3W_s}\left(1 + \frac{9s}{2}\right)}_{b(s)} ds + \int_0^t \underbrace{3s e^{3W_s}}_{\sigma(s)}\,dW_s.$$

T or F: $Z = X + Y$, Then $[Z,Z] = [X,X] + [Y,Y]$.

$X = Y = W_t$. We know $[W,W](t) = t \Rightarrow [X,X] + [Y,Y] = 2t$

$Z = 2W_t \quad dW_t = 0 + 2\,dW_t \Rightarrow d[Z,Z]_t = 4 \Rightarrow [Z,Z] = 4t$