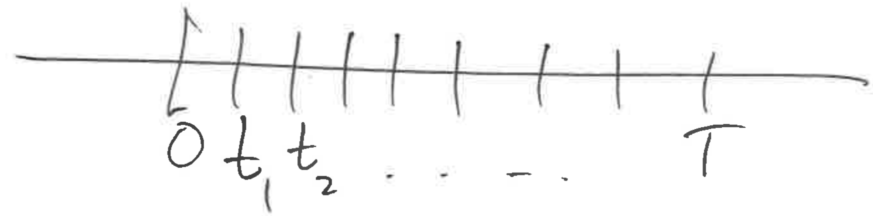
Goal today: Itô integrals / formula

$P = \{0 = t_0 < t_1 \cdots < t_n = T\}$.

<u>Recall:</u> Q.V.



① $W \to$ std B.M.

Compute $\lim\limits_{\|P\| \to 0} \sum |\Delta_i W| = \infty$ (First variation).

$\Delta_i W = W(t_{i+1}) - W(t_i)$,

$\|P\| = \max\limits_i t_{i+1} - t_i$

② Compute $[W, W](T) = \lim\limits_{\|P\| \to 0} \sum\limits_{i=0}^{n-1} |\Delta_i W|^2 = T$

(Quadratic variation).

Itô Integral: $\{\mathcal{F}_t \mid t \geq 0\} \longrightarrow$ Brownian filtration.

$W \longrightarrow$ std BM.

$D \longrightarrow$ some <u>adapted</u> process.

Say $\{0 = t_0 < t_1 < t_2 \cdots\} = P$.

Say we only trade at $t_0, t_1, \ldots$

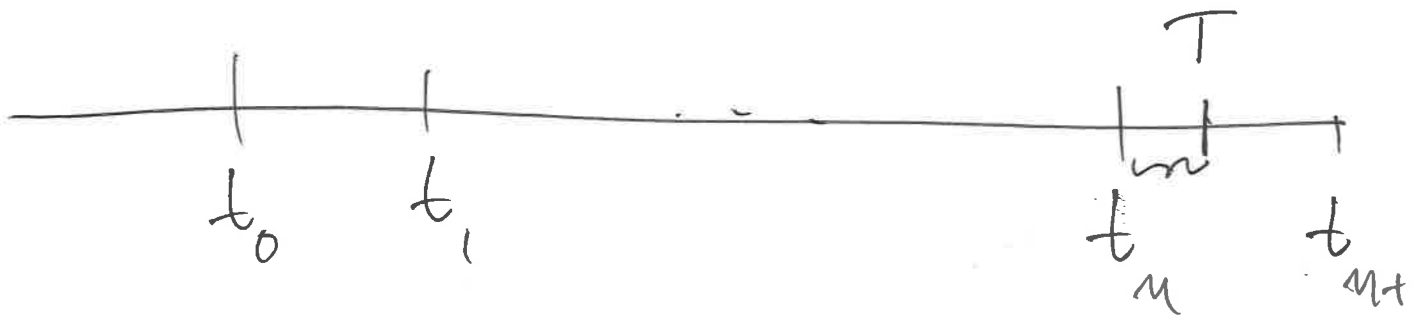
Winnings / Profit up to time $T = t_n$ $= \sum_{i=0}^{n-1} D(t_i)\left(W(t_{i+1}) - W(t_i)\right)$.

$\lim_{\|P\| \to 0}$

need finite 1st var of W. (Dont have this).

Define $$I_P(T) = \sum_{i=0}^{n-1} D(t_i)\,\Delta_i W + D(t_n)\big(W(T) - W(t_n)\big)$$

if $T \in [t_n, t_{n+1})$.



Note I_P is a process.

Claim: (1) $E\, I_p^2(T) = E \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + E\, D(t_n)^2 (T - t_n)$

if $T \in [t_n, t_{n+1})$.

(2) I_p is a cts process.

I_p is a <u>martingale</u>.

(3) $[I_p, I_p](T) = \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + D(t_n)^2 (T - t_n)$

if $T \in [t_n, t_{n+1})$.

Pf: Lets check ① :

Assume $T = t_n$.

$$E\, I_p(T)^2 = E\left(\sum_{0}^{n-1} D(t_i)\, \Delta_i W\right)^2$$

$$= \underbrace{E \sum D(t_i)^2 (\Delta_i W)^2}_{①} + \underbrace{2E \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} D(t_i) D(t_j)\, \Delta_i W\, \Delta_j W}_{②}$$

①: $E \sum D(t_i)^2 (\Delta_i W)^2 = \sum E\, D(t_i)^2 (\Delta_i W)^2$

$$= \sum E\left(E\left(D(t_i)^2 (W(t_{i+1}) - W(t_i))^2 \,\middle|\, \mathcal{F}_{t_i} \right) \right).$$

$$= \sum E\left(D(t_i)^2\, E\left[(W(t_{i+1}) - W(t_i))^2 \,\middle|\, \mathcal{F}_{t_i} \right] \right).$$

$$= \sum E\, D(t_i)^2 (t_{i+1} - t_i),$$

②:

②:

$$E \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} D(t_i) D(t_j) \Delta_i W \Delta_j W$$

$$= \sum_j \sum_{i<j} E\, E\left(D(t_i) D(t_j) \Delta_i W \Delta_j W \,\middle|\, \mathcal{F}_{t_j} \right)$$

$$= \sum_j \sum_{i<j} E\left[D(t_i) D(t_j) \Delta_i W \underbrace{E\left(\Delta_j W \,\middle|\, \mathcal{F}_{t_j} \right)}_{0} \right].$$

$$= 0,$$

QED ①.

(2) $\longrightarrow$ You check (on HW).

(3) $\longrightarrow$ You check:

To check $[I_p, I_p](t) = A(t)$.

Last time: Enough to check $I_p^2 - A$ is a mg.

Note: $(T = t_n)$ $\cancel{I_P(T) =}$

$$[I_P, I_P](T) = \underbrace{\sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i)}_{\downarrow \, \|P\| \to 0 \, \int_0^T D(t)^2\, dt}.$$

Take lim $\|P\| \to 0$.

$$[I_P, I_P](T) \xrightarrow{\|P\| \to 0} \int_0^T D(t)^2\, dt$$

I_P is a mg. "Mg conv" $\Rightarrow$ I_P converges (as martingales) to a process.

the limiting process is called the Itô integral.

Thm: Say D is an adapted process.

① If $\int_0^T D(t)^2\,dt < \infty$ almost surely

Then I_P converges as $\|P\| \to 0$ to a ~~martingale~~ cts process I.

$$I(T) = \lim_{\|P\| \to 0} I_P(T)$$ (Itô integral of D wrt W).

② If $E\int_0^T D(t)^2\,dt < \infty$, then I is a mg

& the QV is $[I, I](T) = \int_0^T D(t)^2\,dt$.

Remk: $\boxed{\text{Crucial } D \text{ is adapted.}}$

Notation: Ito integral of D wrt W

denoted by $\displaystyle\int_0^T D(t)\, dW(t) = \int_0^T D(s)\, dW(s)$.

t: dummy

Remk: If $E\displaystyle\int_0^T D(t)^2 dt < \infty$, then

① $E\left(\displaystyle\int_0^T D(t)\, dW(t)\right) = 0$ (bc $\displaystyle\int_0^T D(t) dW(t)$ is a mg).

② $E\left(\displaystyle\int_0^T D(t)\, dW\right)^2 = E\displaystyle\int_0^T D(t)^2\, dt$ (Itô Isometry).

Proof of (2):

~~Theorem~~ Know QV of $\int_0^T D(t)\,dW(t) = \int_0^T D(t)^2\,dt$

Let $M(T) = \left(\int_0^T D(t)\,dW(t)\right)^2 - \int_0^T D(t)^2\,dt$.

Know M is a mg.

$\Rightarrow E M(T) = E M(0) = 0$

$\Rightarrow E\left[\left(\int_0^T D(t)\,dW(t)\right)^2 - \int_0^T D(t)^2\,dt\right] = 0$

$$\Rightarrow E\left(\int_0^T D(t)\,dW(t)\right)^2 = E\int_0^T D(t)^2\,dt$$

QED.

Properties of Ito Integral:

TRUE ① $\displaystyle\int_0^T (D_1(t) + \alpha D_2(t))\,dW = \int_0^T D_1(t)\,dW + \alpha\int_0^T D_2(t)\,dW$

(α const, D_1, D_2 adapted processes).

FALSE ② If $D_1(t) \le D_2(t)$, Must $\displaystyle\int_0^T D_1(t)\,dW \le \int_0^T D_2(t)\,dW$?

~~False~~. FALSE!

Itô formula:

Say b & σ are two adapted processes.

Define $$X(T) = X(0) + \underbrace{\int_0^T b(t)\,dt}_{\text{Riemann}} + \underbrace{\int_0^T \sigma(t)\,dW(t)}_{\text{Itô}}.$$

Def: X is called an Itô process if

① $X(0)$ is not random.

& ② $E \int_0^T \sigma(t)^2\,dt < \infty$ & $\int_0^T b(t)\,dt < \infty$

<u>Notation:</u> Write "$dX = b(t)\,dt + \sigma(t)\,dW(t)$".

<u>Prop:</u> The QV of X is given by

$$[X, X](T) = \int_0^T \sigma(t)^2\,dt.$$

<u>Proof:</u> Let $B(T) = \int_0^T b(t)\,dt$. (Bounded variation, finite 1st variation).

$M(T) = \int_0^T \sigma(t)\,dW$ (Mg)

$X = X(0) + B + M.$

P a partition . . .

$$\sum (\Delta_i X)^2 = \sum (\Delta_i B)^2 + \sum (\Delta_i M)^2$$

$$+ 2 \sum (\Delta_i M)(\Delta_i B).$$

① $$\lim_{\|P\| \to 0} \sum (\Delta_i M)^2 = [M, M](T)$$

$$= \int_0^T \sigma(t)^2 \, dt.$$

② $\sum (\Delta_i B)^2$:

$$(\Delta_i B)^2 = \left(B(t_{i+1}) - B(t_i)\right)^2 = \left(\int_{t_i}^{t_{i+1}} b(t) \, dt\right)^2.$$

$$\leq \max |b|^2 (t_{i+1} - t_i)^2.$$

$$\Rightarrow \left| \sum (\Delta_i B)^2 \right| \leq \sum \left(\max |b|^2\right) (t_{i+1} - t_i)^2.$$

$$\leq \left(\max |b|^2\right) \max_i (t_{i+1} - t_i) \sum (t_{i+1} - t_i)$$

$$\leq \|P\| \left(\max |b|^2\right) T \xrightarrow[\|P\| \to 0]{} 0$$

<u>You check:</u> $\displaystyle \lim_{\|P\| \to 0} \left| \sum (\Delta_i M)(\Delta_i B) \right| = 0.$

$$\Rightarrow \lim_{\|P\| \to 0} \sum (\Delta_i X)^2 = 0 + \int_0^T \sigma(t)^2 \, dt + 0 .$$

QED.

Note: $X = X(0) + B + M$. $\quad B \to$ Bdd Var. $\quad M \to$ Mg.

then $[X, X] = [M, M]$.

Called the $\boxed{\text{Itô Decomposition of a process.}}$

<u>Def</u>: X is a cts. semi-mg. if

$X = \cancel{X(0)} + B + M$ where

B ~~is~~ has finite 1st variation (& is cts).

M is a cts mg.

<u>Prop</u>: The semi-mg decomposition (Ito-decomposition) is unique.

That is: If ~~if~~ $X = \cancel{X(0)} + B_1 + M_1$

& $X = B_2 + M_2$.

where M_1, M_2 are cts mg & B_1, B_2 are cts finite variation processes.

then $M_1 = M_2$ & $B_1 = B_2$.

Pf: $\cancel{X = M_1 + M_2}$ $X = M_1 + B_1 = M_2 + B_2$.

$$\Rightarrow \underbrace{M_1 - M_2}_{\text{cts mg}} = \underbrace{B_2 - B_1}_{\text{cts, finite 1st var.}}$$

$\Rightarrow QV = 0$.

Let $M = M_1 - M_2 \Rightarrow [M, M] = 0$.

$$E\, M(t)^2 = E\,[M, M](t) = 0 \Rightarrow M = 0$$

$\Rightarrow M_1 = M_2$ & $B_1 = B_2$,

Itô formula: $X \to$ Ito Process.

$$X = X(0) + B + M.$$

Let D some adapted process.

Define $$\int_0^T D(t)\, dX(t) = \underbrace{\int_0^T D(t)\, dB(t)}_{\text{Rieman Integral}} + \underbrace{\int_0^T D(t)\, dM}_{\text{Itô integral}}.$$

If $M(T) = \int_0^T \sigma(t)\, dW(t)$, define $\int_0^T D(t)\, dM = \int_0^T D(t)\, \sigma(t)\, dW$.

Itô Formula: $X = X(0) + B + M = X(0) + \int_0^T b(t)\,dt + \int_0^T \sigma(t)\,dW(t)$.

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^{1,2}$ function.

$f = f(t,x) \longrightarrow$ once diff wrt t, & twice diff wrt x, $\partial_t f, \partial_x f, \partial_x^2 f$ are all cts.

$$f(T, X(T)) = f(0, X(0)) + \int_0^T \partial_t f(t, X(t))\,dt$$

$$+ \int_0^T \partial_x f(t, X(t))\,dX(t) + \frac{1}{2}\int_0^T \partial_x^2 f(t, X(t))\,d[X,X](t)$$