Last time: Conditional Expectation.

$(\Omega, \mathcal{G}, P)$. $\mathcal{F} \subseteq \mathcal{G}$ σ-sub alg.

X is a $\mathcal{G}$-meas RV,

$E(X \mid \mathcal{F}) =$ "best approx of X given $\mathcal{F}$"

R.V. $\mathcal{F}$-meas, & partial av.

~~M~~ $EX \longrightarrow$ number. ($\in \mathbb{R}$, non random).

$E(X|\mathcal{F}) \longrightarrow$ RV.

Claim: $E\big(E(X|\mathcal{F})\big) = EX$

Proof: Partial av, for every $F \in \mathcal{G}$,

$$\int_F E(X|\mathcal{G})\, dP = \int_F X \, dP$$

Choose $F = \Omega$:

$$\underbrace{\int_\Omega E(X|\mathcal{G})\, dP}_{E(E(X|\mathcal{G}))} = \underbrace{\int_\Omega X \, dP}_{EX}$$

QED.

Martingales:

① **Filtration:** We say a family of σ-alg $\{\mathcal{F}_t \mid t \geq 0\}$ is a filtration if whenever $s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

Intuition: $\mathcal{F}_t \longrightarrow$ information up to time t

Eg: If X is a process, the filtration generated by X is defined by

$$\mathcal{F}_t = \sigma\left(\bigcup_{s \leq t} \sigma(X(s))\right).$$

Adapted: ~~Given~~ A process X is ~~said~~ called adapted to the filtration $\{\mathcal{F}_t\}$ if for every t, $X(t)$ is $\mathcal{F}_t$ measurable.

Clearly: If $\{\mathcal{F}_t \mid t \geq 0\}$ = filtration generated by X then X is $\mathcal{F}_t$ meas.

Martingale: We say a process M is a martingale wrt the filtration $\{\mathcal{F}_t \mid t \geq 0\}$. if

① M is adapted (i.e. $M(t)$ is $\mathcal{F}_t$ meas).

& ② Whenever $s \leq t$, $E(M(t) \mid \mathcal{F}_s) = M(s)$.

(Martingale — fair game)

ⓐ Sub martingale : $E(M(t) \mid \mathcal{F}_s) \geq M(s)$

ⓑ Super martigal : $E(M(t) \mid \mathcal{F}_s) \leq M(s)$.

Eg 1: BM is a mg. wrt the filt gen by BM.

Check: Let W be a BM. (std).

$\{\mathcal{F}_t \mid t \geq 0\}$ Browian filtration. $\mathcal{F}_t = \sigma\left(\bigcup_{s \le t} \sigma(W_s)\right)$.

NTS: (1) Adapted (Free).

(2) If $s \le t$, $E\left(W(t) \mid \mathcal{F}_s\right) = W(s)$.

Check: $E\left(W(t) \mid \mathcal{F}_s\right) = E\left(W(t) - W(s) + W(s) \mid \mathcal{F}_s\right)$.

$$= \underbrace{E\left(W(t) - W(s) \mid \mathcal{F}_s\right)}_{0} + \underbrace{E\left(W(s) \mid \mathcal{F}_s\right)}.$$

By independence $W(t)-W(s)$ is ind of $\mathcal{F}_s$

& $W(t)-W(s) \sim N(0, t-s)$.

$$\Rightarrow E\left(W(t)-W(s) \,\middle|\, \mathcal{F}_s\right) = E\left(W(t)-W(s)\right) = 0$$

Also, $E\left(W(s) \,\middle|\, \mathcal{F}_s\right) = W(s)$ ($\because$ $W(s)$ is $\mathcal{F}_s$ meas).

$$\Rightarrow E\left(W(t) \,\middle|\, \mathcal{F}_s\right) = W(s).$$

QED.

Eg 2: Let $\{\mathcal{F}_t \mid t \geq 0\}$ some filt.

Let $\mathcal{F}_\infty = \sigma\left(\bigcup \mathcal{F}_t\right)$. Let Y be any $\mathcal{F}_\infty$ meas RV.

Define $M(t) = E(Y \mid \mathcal{F}_t)$.

Claim M is a mg.

Pf: $E(M(t) \mid \mathcal{F}_s) = E\left(E(Y \mid \mathcal{F}_t) \mid \mathcal{F}_s\right)$

$$\overset{\text{tower}}{=} E(Y \mid \mathcal{F}_s) = M(s).$$

QED.

Stochastic Integration:

$S(t) \longrightarrow$ asset spot price at time t

$\Delta(t) \longrightarrow$ Your position on the asset.

Say you only trade at times $t_0 = 0, t_1, t_2, \dots t_n = T$

Change in wealth / winnings

$$X(T) - X(0) = \sum_{i=0}^{n-1} \Delta(t_i)\left(S(t_{i+1}) - S(t_i)\right).$$

Want to trade ctsly in time.

Guess $X(T) - X(0) = \lim\limits_{\|P\| \to 0} \sum_{i=0}^{n-1} \Delta(t_i)\left(S_{t_{i+1}} - S(t_i)\right)$

$P \longrightarrow$ partition of $[0, T]$. $= \{t_0 = 0 < t_1 < t_2 \cdots < t_n = T\}$

$\|P\| = \max t_{i+1} - t_i$.

$$\lim_{\|P\| \to 0} \sum \Delta(t_i) \left(S(t_{i+1}) - S(t_i) \right). = \text{Riemann integral}$$

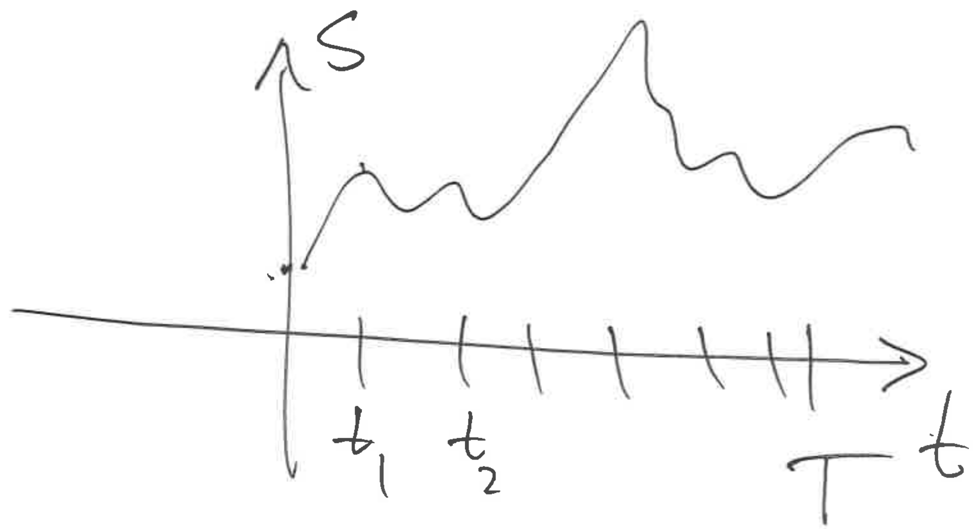
$$\left(\text{Riemann-Stieltjes.} \quad \int_0^T \Delta(t)\, dS(t) \right).$$

DOESNT WORK when $S =$ anything reasonable.

The above will only work if S has $\boxed{\text{finite first variation.}}$

Def: $V_{[0,T]}(S) =$ first variation of S on $[0,T]$.

$$\stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left| S(t_{i+1}) - S(t_i) \right|.$$



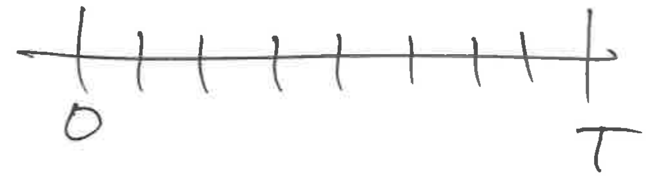
You check: S diff

$\Rightarrow$ S has finite first variation.

(bounded variation).

Claim: B.M does NOT have finite first variation.
(bounded variation).

Pf: $W \to$ std BM. Fix $T > 0$.



Let $N \in \mathbb{N}$. say $t_i = \frac{iT}{N}$ $(i = 0, 1, \dots N)$.

$$\lim_{N \to \infty} E \sum_{i=0}^{N-1} \left| W(t_{i+1}) - W(t_i) \right|.$$

Note $E \sum \left| W(t_{i+1}) - W(t_i) \right| = \sum E \left| W(t_{i+1}) - W(t_i) \right|$.

$$= \sum E \left| W\left(\frac{(i+1)T}{N}\right) - W\left(\frac{iT}{N}\right) \right|.$$

Know $W\left(\frac{(i+1)T}{N}\right) - W\left(\frac{iT}{N}\right) \sim N\left(0, \frac{T}{N}\right)$.

$$\Rightarrow E\left|W\left(\frac{(i+1)T}{N}\right) - W\left(\frac{iT}{N}\right)\right| = E\left|N\left(0, \frac{T}{N}\right)\right|.$$

$$\overset{\text{You check}}{=} C\left(\frac{T}{N}\right)^{1/2}.$$

($C \rightarrow$ some constant).

$$\Rightarrow \sum_{i=0}^{N-1} E\left|W\left(\frac{(i+1)T}{N}\right) - W\left(\frac{iT}{N}\right)\right| = \sum_{i=0}^{N-1} C\left(\frac{T}{N}\right)^{1/2}.$$

$$= \sqrt{NT} \cdot C \xrightarrow{N \to \infty} +\infty.$$

Quadratic Variation:

Def: Let M be any process.

$$[M, M](T) = \text{Quadratic variation of } M \text{ at time } T.$$

$$= \lim_{\|P\| \to 0} \sum \left(M(t_{i+1}) - M(t_i) \right)^2$$

Remark: ① If M has finite first variation, then the Quadratic variation of $M = 0$

② If M has finite Quad variation then the first variation of $M = +\infty$.

<u>Claim:</u> $[W, W](T) = T$ (for a standard BM W).

<u>Check:</u> $t_i = \frac{Ti}{n}$, $n \in \mathbb{N}$.

$$\Delta_i W = W(t_{i+1}) - W(t_i) = W\left(\frac{T(i+1)}{n}\right) - W\left(\frac{Ti}{n}\right).$$

Q-V. $$[W, W](T) = \lim_{n \to \infty} \sum_{i=0}^{n-1} (\Delta_i W)^2$$

Note $$\sum_{i=0}^{n-1} (\Delta_i W)^2 - T = \sum_{i=0}^{n-1} \left((\Delta_i W)^2 - \frac{T}{n} \right).$$

Let $\xi_i = (\Delta_i W)^2 - \frac{T}{n}$

Know ξ_i's are iid, dist $N(0,\frac{T}{n})^2 - \frac{T}{n}$

$$\Rightarrow E\xi_i = E N(0,\tfrac{T}{n})^2 - \frac{T}{n} = 0$$

$$\& \quad E\xi_i^2 = E\left(N(0,\tfrac{T}{n})^2 - \frac{T}{n}\right)^2$$

(You check) $$= \frac{T^2}{n^2}\left(E\left(N(0,1)^4 - 1\right)\right).$$

$$\Rightarrow \sum \xi_i^{\#} = \sum (\Delta_i W)^2 - T$$

Note $$E\left(\sum_{i=0}^{n-1} \xi_i^{\#}\right)^2 = E\sum_{i=0}^{n-1}\xi_i^2 = \sum_{i=0}^{n-1}\frac{T^2}{n^2}\left(EN(0,1)^4 - 1\right),$$

$$= \frac{T^2}{n}\left(EN(0,1)^4 - 1\right) \longrightarrow 0$$

$$\Rightarrow \lim_{n\to\infty} E\left(\sum \xi_i\right)^2 \longrightarrow 0$$

$$\Rightarrow \lim_{n\to\infty} \sum \xi_i \longrightarrow 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\sum (\Delta_i W)^2 - T\right) = 0$$

QED.

<u>Claim:</u> The Process $M(t) = W(t)^2 - [W,W](t)$ is a mg.

Pf: Know $[W,W](t) = t$.

$$\Rightarrow M(t) = W(t)^2 - t.$$

Compute $E(M(t) \mid \mathcal{F}_s)$ (& check $= M(s)$).

$$E(M(t) \mid \mathcal{F}_s) = E(W(t)^2 - t \mid \mathcal{F}_s).$$

$$= E\left((W(t) - W(s) + W(s))^2 \mid \mathcal{F}_s \right) - t$$

$$= E\left((W(t) - W(s))^2 + W(s)^2 + 2W(s)(W(t) - W(s)) \mid \mathcal{F}_s \right) - t$$

$$= t - s + W(s)^2 + 2W(s) E\left(W(t) - W(s) \mid \mathcal{F}_s \right) - t$$

$$= -s + W(s)^2 + 0 = M(s)$$

QED.

Theorem: ① If M is a mg wrt $\{\mathcal{F}_t\}$.

then $E(M(t)^2) < \infty \iff E[M,M](t) < \infty$.

in this case. $M(t)^2 - [M,M](t)$ is also a mg.

(consequently $EM(t)^2 = EM(0)^2 + E[M,M](t)$).

Theorem: ② If $A(t)$ is any cts. increasing adapted process.

such that $A(0) = 0$ & $M(t)^2 - A(t)$ is a mg.

then $A(t) = [M,M](t)$.

Construct the Itô integral.

Main Idea: $\lim\limits_{\|P\|\to 0} \sum_i \Delta(t_i)\left(S(t_{i+1}) - S(t_i)\right)$.

DNC in the usual sense.

However. claim $\sum\limits_{0}^{\infty} \Delta(t_i)\left(S(t_{i+1}\wedge t) - S(t_i \wedge t)\right)$.

become mg.