Rec 2

Today: - Conditional Expectation

- Martingales

Possible

$X^{-1}((-\infty, a)) \in \mathcal{G} \setminus \mathcal{F}$

but not necessarily $\mathcal{F}$-meas.

<u>Recall</u>: $(\Omega, \mathcal{G}, \mathbb{P})$, $\mathcal{F} \subseteq \mathcal{G}$, X r.v. (X $\mathcal{G}$-meas).

Then $E[X|\mathcal{F}]$ is a $\boxed{\text{RANDOM VARIABLE}}$ satisfying

i) $E[X|\mathcal{F}]$ is $\mathcal{F}$-measurable.

ii) (Partial Averaging Condition) $\forall$ $A \in \mathcal{F}$ we have:

$$\int_A E[X|\mathcal{F}]\,dP = \int_A X\,dP.$$

<u>Fact</u>: $E[X|\mathcal{F}]$ is the <u>unique</u> r.v. satisfying (1) and (2)

<u>prop</u>: (1) X $\mathcal{F}$-meas $\Rightarrow$ $E[X|\mathcal{F}] = X$

(2) X indep of $\mathcal{F}$ $\Rightarrow$ $E[X|\mathcal{F}] = E[X]$.

(3) X, Y r.v. then:

(Linearity): $E[X + \alpha Y | \mathcal{F}] = E[X|\mathcal{F}] + \alpha E[Y|\mathcal{F}]$.

(Positivity): $X \leq Y$ then $E[X|\mathcal{F}] \leq E[Y|\mathcal{F}]$

$(\Omega, \mathcal{G}, \mathbb{P})$

(4) X $\mathcal{F}$-meas, Y some r.v. then $E[XY|\mathcal{F}] = X E[Y|\mathcal{F}]$. (TOWK).

(5) (Tower property) $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$:

$$E[X|\mathcal{E}] = E[E[X|\mathcal{F}]|\mathcal{E}]$$

Indep Lemma: X, Y r.v., X is indep of $\mathcal{F}$, Y $\mathcal{F}$-meas, $f: \mathbb{R}^2 \to \mathbb{R}$.

then $E[f(X,Y)|\mathcal{F}] = E[f(x,Y)] = g(Y)$.

suppose X has pdf p_X then:

$$E[f(X,Y)|\mathcal{F}] = \int_{-\infty}^{\infty} f(x,Y)\, p_X(x)\,dx$$

$$\mathbb{1}_B(\omega) = \mathbb{I}_B(\omega) = \begin{cases} 0 & \omega \notin B \\ 1 & \omega \in B \end{cases}$$

EX 1: $(\Omega, \mathcal{G}, P)$, $A, B \in \mathcal{G}$. consider $\sigma(\mathbb{I}_B) = \{\emptyset, B, B^c, \Omega\}$

Then $E[\mathbb{I}_A | \sigma(\mathbb{I}_B)] = P(A|B)\,\mathbb{I}_B + P(A|B^c)\,\mathbb{I}_{B^c}$. Recall: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Notation: X is a r.v. $\sigma(X)$ is the σ-algebra generated by X i.e. smallest σ-algebra s.t. X is still a r.v. under $\sigma(X)$.

We just need to show that $\overbrace{P(A|B)\,\mathbb{I}_B + P(A|B^c)\,\mathbb{I}_{B^c}}^{X}$ satisfies ① and ②.

① Let's show X is $\sigma(\mathbb{I}_B)$ meas.

$$X(\omega) = \begin{cases} P(A|B) & \omega \in B \\ P(A|B^c) & \omega \in B^c \end{cases}$$

$c := \min\{P(A|B), P(A|B^c)\}$

$d = \max\{P(A|B), P(A|B^c)\}$

$\{\omega : X(\omega) < a\} \Rightarrow$

for $a \le c$: $X^{-1}(-\infty, a) = \phi$

$c < a \le d$: $X^{-1}(-\infty, a) = \begin{cases} B & \text{if } c = P(A|B) \\ B^c & \text{if } c = P(A|B^c) \end{cases} \in \sigma(\mathbb{I}_B)$.

$X(\omega) = P(A|B)\ \omega \in B$

$a > d$: $X^{-1}(-\infty, a) = \Omega$.

② Need to show $\int_F X\,dP = \int_F \mathbb{I}_A\,dP \quad \forall\, F \in \sigma(\mathbb{I}_B) = \{\phi, B, B^c, \Omega\}$

ϕ: $\int_\phi X\,dP = 0 = \int_\phi \mathbb{I}_A\,dP$ ✓.

Ω: $\int_\Omega X\,dP = E[X] = P(A|B)P(B) + P(A|B^c)P(B^c) = P(A\cap B) + P(A\cap B^c)$

$= P((A\cap B^c)\cup(A\cap B)) = P(A) = E[\mathbb{I}_A] = \int_\Omega \mathbb{I}_A\,dP$.

B: $\int_B X\,dP = \int_B P(A|B)\,dP = P(A|B)P(B) = P(A\cap B) = \int_B \mathbb{I}_A\,dP$.

why? $\int_B \mathbb{I}_A\,dP = \int_\Omega \mathbb{I}_A \mathbb{I}_B\,dP = E[\mathbb{I}_A\mathbb{I}_B] = P(A\cap B)$

B^c: $\int_{B^c} X\,dP = \int_{B^c} P(A|B^c)\,dP = P(A|B^c)\,P(B^c) = P(A \cap B^c) = \int_{B^c} \mathbb{1}_A \,dP.$

$$\therefore\ E[\mathbb{1}_A | \sigma(\mathbb{1}_B)] = P(A|B)\,\mathbb{1}_B + P(A|B^c)\,\mathbb{1}_{B^c}.$$

Ex2: X, Y independent with $X \sim \exp(\lambda)$, $Y \sim \text{unif}\,[0,\lambda]$. Let $Z = e^{-XY^2}$

Find $E[Z|Y] = E[e^{-XY^2}|Y] := E[e^{-XY^2}|\sigma(Y)]$

Notice we are in the setting of the independence lemma:

By ind. lem: $E[e^{-XY^2}|Y] = E[e^{-xY^2}] = \int_{-\infty}^{\infty} e^{-xY^2} p_X(x)\,dx = \int_0^{\infty} e^{-xY^2} \lambda e^{-\lambda x}\,dx$

$$= \int_0^{\infty} \lambda e^{-x(\lambda+Y^2)}\,dx = \frac{-\lambda}{\lambda+Y^2} e^{-x(\lambda+Y^2)}\Big|_{x=0}^{x=\infty} = \frac{\lambda}{\lambda+Y^2}$$

$$\therefore\ E[e^{-XY^2}|Y] = \frac{\lambda}{\lambda+Y^2}$$

Want: $E[e^{-XY^2}] = E\big[E[e^{-XY^2}|Y]\big] = \int_0^{\lambda} \frac{\lambda}{\lambda+Y^2}\left(\frac{1}{\lambda}\right)dy = \int_0^{\lambda} \frac{1}{\lambda+y^2}\,dy$

$$= \frac{\arctan\left(\frac{y}{\sqrt{\lambda}}\right)}{\sqrt{\lambda}}\Bigg|_0^{\lambda} = \frac{\arctan(\sqrt{\lambda})}{\sqrt{\lambda}}.$$

Can check $E[e^{-XY^2}]$ directly, but it is tedious!

Def $(\Omega, \mathcal{G}, \mathbb{P})$; $\{\mathcal{F}_t\}$ filtration. Then a Stochastic process M is a martingale if

i) M is $\mathcal{F}_t$ -adapted (i.e. $M(t)$ is a r.v. with respect $\mathcal{F}_t$ $\forall t$).

ii) for $t \geq s$ $E[M(t) | \mathcal{F}_s] = M(s)$.

Essential: – Stochastic Integration
– Fundamental Theorem of Asset pricing.

(T)or F : Martingales have constant Expectations: $\forall t \geq s$

$$E[M(t) | \mathcal{F}_s] = M(s) \Rightarrow E[M(t)] = E[E[M(t) | \mathcal{F}_s]] = E[M(s)].$$

T or (F) : Do stochastic processes with constant Expectations have to be Martingales?

Ex: W_t is Brownian Motion ($\{\mathcal{F}_t\}$ is the Filtration generated by BM).

$$X_s = \frac{1+W_s^2}{1+s} \leftarrow X_t = \frac{1+W_t^2}{1+t} \quad \text{then} \quad E[X_t] = \frac{1+E[W_t^2]}{1+t} = \frac{1+t}{1+t} = 1$$

$$E[X_t | \mathcal{F}_s] = E\left[\frac{1+W_t^2}{1+t} \middle| \mathcal{F}_s\right] \overset{\text{Lin}}{=} E\left[\frac{1}{1+t} \middle| \mathcal{F}_s\right] + E\left[\frac{W_t^2}{1+t} \middle| \mathcal{F}_s\right]$$

$= \frac{1}{1+t} + \frac{1}{1+t} E[W_t^2 | \mathcal{F}_s]$.

$E[W_t^2 | \mathcal{F}_s] = E[(W_t - W_s + W_s)^2 | \mathcal{F}_s] = E[(W_t - W_s)^2 | \mathcal{F}_s] + 2E[W_s(W_t - W_s) | \mathcal{F}_s]$
$+ E[W_s^2 | \mathcal{F}_s]$

independent of

__Note__ $W_t - W_s \perp\!\!\!\perp (W_s - W_0)$.

$= E[(W_t - W_s)^2] + 2W_s E[(W_t - W_s)] + W_s^2$

$= t - s + 0 + W_s^2$. // $\frac{(1+W_s^2) + t - s}{1+t}$

$\therefore E[X_t | \mathcal{F}_s] = \frac{1 + t - s + W_s^2}{1+t} = \frac{X_s(1+s) + t - s}{1+t}$. Does this equal X_s?

__Exercise__ (You will see $E[X_t | \mathcal{F}_s] = X_s \iff W_s^2 = s$ which is __not true__).

__Ex__: Show that $X_t = W_t^2 - t$ is a martingale

1) Clearly X_t is $\mathcal{F}_t$ adapted

2) $E[X_t | \mathcal{F}_s] = E[W_t^2 | \mathcal{F}_s] - t \overset{\text{from above}}{=} (t - s + W_s^2) - t = W_s^2 - s = X_s$.

Hence X_t is a martingale.

Check: M, N martingales 1) $M+N$ is a martingale.

2) MN not a martingale

Note/Warning! Martingale and Markov process are not the same.

X stoch process

Markov process: $\forall$ function f: $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = g(X_s)$