Reminder: RECITATION $\longrightarrow$ $\boxed{\text{9:30 Friday}}$

Last Time: Independence.

$X \longrightarrow$ R.V.

Def $\sigma(X) =$ smallest σ-alg containing $\{X \leq \alpha\}$ for all $\alpha \in \mathbb{R}$.

$= \sigma$-alg generated by $\{ \{X \leq \alpha\} \mid \alpha \in \mathbb{R} \}$.

Def: $X_1, X_2, \ldots X_N$ are N random variables.

We say $X_1, \ldots X_N$ are independent if.

for EVERY $A_1 \in \sigma(X_1)$, $A_2 \in \sigma(X_2)$ ... $A_N \in \sigma(X_N)$.

we have $$P(A_1 \cap A_2 \cap \cdots A_N) = P(A_1)\,P(A_2) \cdots P(A_N).$$

<u>You check:</u> $\cancel{\{X_i\}}$ $X_1, X_2, \ldots X_N$ are ind RV's then whenever $A_i \in \sigma(X_i)$, $\{A_i\}$ are ind (as events).

<u>Proposition:</u> ~~X~~ Let $X_1, X_2, \ldots X_N$ be RV's.

The following are equivalent:

① $X_1, X_2, \ldots X_N$ are independent

$\Longleftrightarrow$ (2) for every $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, $\cdots$ $\alpha_N \in \mathbb{R}$,

$$P\left(X_1 \le \alpha_1, X_2 \le \alpha_2, \dots X_N \le \alpha_N\right)$$

$$= P(X_1 \le \alpha_1) P(X_2 \le \alpha_2) \cdots P(X_N \le \alpha_N).$$

$\Longleftrightarrow$ (3) for any bounded cts funtions.

$f_1, f_2, \cdots f_N$, we have

$$E\left(f_1(X_1) f_2(X_2) \cdots f_N(X_N)\right) = E f(X_1) \cdot E f(X_2) \cdot \cdots E f(X_N)$$

$\Longleftrightarrow$ 4^* for every $t_1, t_2, \cdots t_N$ in some small interval around 0, we have

$$E\left(\exp \sum_{j=1}^{N} t_j X_j\right) = \left(E e^{t_1 X_1}\right)\left(E e^{t_2 X_2}\right) \cdots \left(E e^{t_N X_N}\right).$$

Compute covariance of BM.

$W \longrightarrow$ BM. ① W cts process.

② $W(0) = 0$

③ Independent Increments & ④ $W(t) - W(s) \sim N(0, t-s)$.

Compute $E\, W(t)\, W(s)$:

Say $\cancel{s \leq t.}$ $s < t.$

Ind inc $\Rightarrow$ $W(s)$ & $W(t) - W(s)$ are ind.

$$E\big(W(s)\,W(t)\big) = E\, W(s)\big(W(t) - W(s) + W(s)\big).$$

$$= E\, W(s)\big(W(t) - W(s)\big) + E\, W(s)^2.$$

$$= 0 + s$$

$\Big(\because \underbrace{\big(E\,W(s)\big)\, E\big(W(t) - W(s)\big)}_{\text{by ind}}\Big)$ $\Big(\because W(s) \sim N(0, s)\Big).$

$$W(t) - W(s) \sim N(0, t-s).$$

$$W(t-s) \sim N(0, t-s).$$

But $W(t) - W(s)$ need not equal $W(t-s)$.

Martingales: (in continuous time).

↳ Martingale → "Fair Game".

Precise : $E(M_t \mid \mathcal{F}_s) = M_s$.

Conditional Expectation.

Conditional Expectation: $(\Omega, \mathcal{G}, P)$.

$X \longrightarrow$ R.V. (measurable wrt $\mathcal{G}$).

$\mathcal{F} \longrightarrow$ σ-alg & $\mathcal{F} \subset \mathcal{G}$.

$\underset{\substack{\uparrow \\ \text{conditional exp.}}}{E(X \mid \mathcal{F})}$ = "Best approximation of X that only uses information from $\mathcal{F}$"

Eg: 30 cards.

10 Red $\rightarrow$ 5 high, 5 low.

20 Black $\rightarrow$ 4 high, 16 low.

Game: High card $\rightarrow$ +1\$

Low card $\rightarrow$ −1\$.

Dealer only tells you card color.

<u>True Game:</u> $X = \mathbb{1}_H - \mathbb{1}_L$

Approximation based on the info from dealer.

$$Y = 0\,\mathbb{1}_R + \left(\frac{4}{20} - \frac{16}{20}\right)\mathbb{1}_B.$$

$\mathcal{F} = \sigma$-alg from dealer. $= \{\emptyset, R, B, \Omega\}$.

$\mathcal{G} =$ full σ-alg $= \{\emptyset, R, B, RH, RL, \ldots\ldots$

$H, L, \ldots \}$.

Note: (1) Y is a $\mathcal{G}$-measurable RV.

(X is an $\mathcal{F}$-meas RV).

(2) For every $A \in \mathcal{G}$, $\int_A Y \, dP = \int_A X \, dP$.

(Notation: $\int_\Omega X \, dP = EX$;

Define $\int_A X \, dP = E(1_A X)$.

Def (Conditional Expetation).

Ω, $\mathcal{G}$ σ-alg, P prob measure.

X is an $\mathcal{G}$- meas RV.

Suppose $\mathcal{F} \subseteq \mathcal{G}$ is some σ-alg.

Define $E(X|\mathcal{F})$ = cond exp of X given $\mathcal{F}$

to be a $\boxed{\text{RANDOM VARIABLE}}$ such that

(1) $E(X|\mathcal{F})$ is an $\mathcal{F}$- meas R.V.

& (2) for every $A \in \mathcal{F}$ $\int_A E(X|\mathcal{F})\, dP = \int_A X\, dP$.

(partial averaging).

Intuition: $E(X|\mathcal{F})$ = Best approximation of X given only the info $\mathcal{F}$.

Remark: Uniqueness: Any R.V that satisfies must be $E(X|\mathcal{F})$.

Remark 2: In general (2) holds for all $A \in \mathcal{F}$ but need not hold if $A \notin \mathcal{F}$.

Prop: ① If X is $\mathcal{G}$-meas then $E(X|\mathcal{G}) = X$

② If X is independent of $\mathcal{G}$, then $E(X|\mathcal{G}) = EX$.

(Recall X is ind of $\mathcal{G}$ if for any $A \in \sigma(X)$ & $B \in \mathcal{G}$

$$P(A \cap B) = P(A)\,P(B)\ ).$$

You check: ①.

② Say $X = \sum a_i \mathbb{1}_{A_i}$, $A_i \in \mathcal{G}$.

Pick any $F \in \mathcal{G}$

Compute $\displaystyle\int_F X \, dP = \int_F \sum a_i 1_{A_i} \, dP$

$$= \sum a_i P(A_i \cap F) \quad \left(\begin{array}{l} \text{Note } A_i \in \sigma(X) \\ \quad F \in \mathcal{F} \\ \Rightarrow A_i \text{ ind of } F \end{array} \right).$$

~~dx~~

$$= \sum a_i P(A_i) P(F)$$

$$= P(F) \left(\sum a_i P(A_i) \right) = P(F) EX$$

$$= \int_F (EX) \, dP.$$

EX is $\mathcal{F}$ meas $\Rightarrow EX = E(X \mid \mathcal{F})$.

Lemma: (Independence Lemma).

Say X & Y are 2 RV's. ($\mathcal{G}$ - meas).

$\mathcal{F} \subseteq \mathcal{G}$ σ-alg.

Suppose X is ind of $\mathcal{F}$

& Y is meas wrt $\mathcal{F}$.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be any fn.

$$E\left(f(X,Y) \,\middle|\, \mathcal{F}\right) = g(Y) \quad \text{where}$$

$$g(y) = E f(X, y)$$

Remark: If p_X is PDF of X.

then above says.

$$E\left(f(X,Y) \mid \mathcal{F}\right) = \int_{-\infty}^{\infty} f(x,Y)\, p_X(x)\, dx$$

Prop: ① If X, Y RV's, $\alpha \in \mathbb{R}$,

then $E(X+\alpha Y \mid \mathcal{F}) = E(X \mid \mathcal{F}) + \alpha E(Y \mid \mathcal{F})$

② (Positivity) If $X \leq Y \Rightarrow E(X \mid \mathcal{F}) \leq E(Y \mid \mathcal{F})$.

③* If $\underline{X \text{ is } \mathcal{F}\text{ meas}}$, Y anything $\Rightarrow E(XY \mid \mathcal{F}) = X\, E(Y \mid \mathcal{F})$

④ (Tower Property). Say $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$

Then $\underline{E}(X \mid \mathcal{E}) = \underline{E}\Big(\underline{E}(X \mid \mathcal{F}) \,\Big|\, \mathcal{E}\Big)$.

~~Def~~ <u>Filteration</u>: X is a process.

the <u>Filteration</u> generated by X is the family of σ-algebras.

$\{\mathcal{F}_t^X \mid t \geq 0\}$, where

$$\mathcal{F}_t^X \overset{\text{def}}{=\!=} \sigma\Big(\bigcup_{s \le t} \sigma(X(s))\Big).$$ $\leftarrow$ all information generated by observing X up to time t.

① Note if $s \le t$, $\mathcal{F}_s^X \subseteq \mathcal{F}_t^X$.

<u>Def:</u> We say a family of σ-alg $\{\mathcal{F}_t \mid t \geq 0\}$ is a FILTERATION if

① Each $\mathcal{F}_t$ is a σ-alg

& ② If $s \le t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

<u>Def:</u> A martingale is a process M such that for every $s \le t$, $E(M_t \mid \mathcal{F}_s) = M_s$.