Last time: ① ~~G~~ σ-alg: Ω (Sample space).

$\mathcal{G}$ = σ-alg: $\mathcal{G}$ non empty, $\mathcal{G} \subseteq$ power set (Ω).

① $A \in \mathcal{G} \Longrightarrow A^c \in \mathcal{G}$

& ② $A_1, A_2, \dots \in \mathcal{G} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Probability measure: $P: \mathcal{G} \longrightarrow [0,1]$. (i.e. If $A \in \mathcal{G}$, $P(A) \in [0,1]$.)

Assume P is ctably additive:

$$A_1, A_2, \dots \in \mathcal{G} \Rightarrow P\left(\bigcup_1^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

provided $A_i \cap A_j = \phi$ when $i \neq j$.

Random variables: We say X is a random variable

if: ① $X : \Omega \longrightarrow \mathbb{R}$

& ② for every $\alpha \in \mathbb{R}$, $\{\omega \mid X(\omega) \le \alpha\} \in \mathcal{G}$.

(Notation: X is also called a measurable function.

or a $\mathcal{G}$-measurable function).

Fix $\alpha \in \mathbb{R}$, $A = \{\underline{\omega} \mid X(\underline{\omega}) \le \alpha\} = \{X \le \alpha\}$.

Note $P(X \le \alpha) = P(\{\omega \mid X(\omega) \le \alpha\})$.

makes sense.

Q: $P(X > \alpha)$: $\{X > \alpha\} = \{X \le \alpha\}^c \in \mathcal{G} \Rightarrow P(X > \alpha)$ makes sense

<u>You check:</u> For every $a, b \in \mathbb{R}$,

$\cancel{R(X}$ $\{X \in (a,b)\} \in \mathcal{G}$, also $\{X = a\} \in \mathcal{G}$.

$\{X \in [a,b]\} \in \mathcal{G}$

$\{X \in (a,b]\} \in \mathcal{G}$ etc.

$\Rightarrow P(X \in (a,b))$, $P(X \in [a,b])$, etc ~~make sense~~ are defined.

<u>Prop:</u> If X, Y are R.V's $\Rightarrow X+Y$, XY etc are all RV's.

$|X|$, $X^+ = \max\{X, 0\}$, X^-, etc

If $f: \mathbb{R} \to \mathbb{R}$ is any "nice" fn, then $f(X)$ is also

Expectations:

Eg: Say $A_1, A_2, \dots A_N \in \mathcal{G}$,

$a_1, a_2, \dots a_N \in \mathbb{R}$.

"Indicator fn".

$$\boxed{X = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}}$$

~~Ex.~~ Notation: $A \in \mathcal{G}$, $A \subseteq \Omega$,

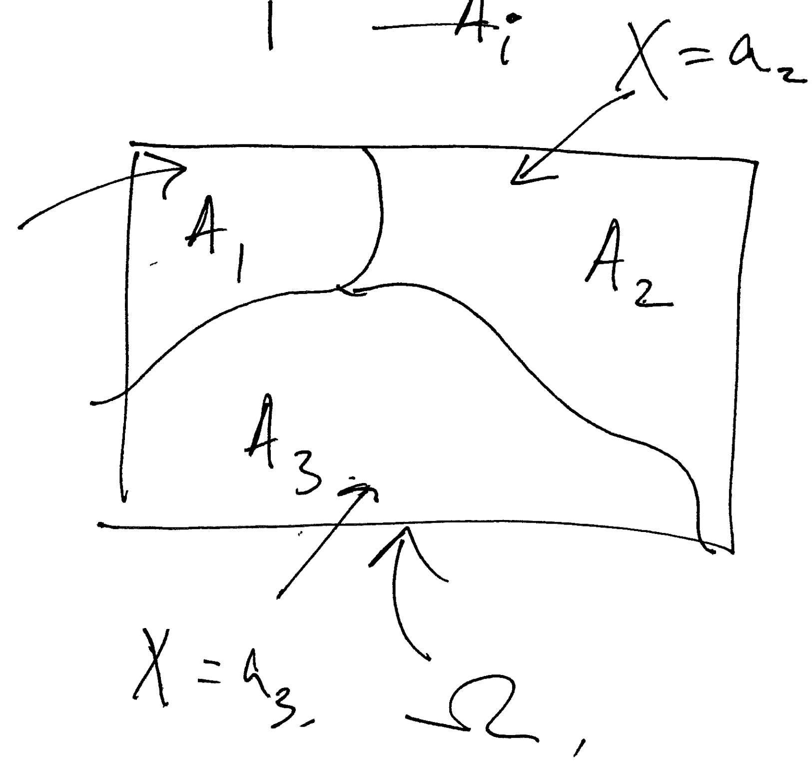
define $\mathbb{1}_A : \Omega \longrightarrow \mathbb{R}$,

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

Note $\mathbb{1}_A$ is a $\mathcal{G}$ measurble RV $\iff A \in \mathcal{G}$.

(Check: $\{\mathbb{1}_A \le \alpha\} = \begin{cases} \Omega & \alpha \geq 1 \\ A^c & \alpha \in [0,1) \\ \phi & \alpha < 0 \end{cases}$)

__Def:__ X is a simple R.V. if $X = \sum_{1}^{N} a_i \mathbb{1}_{A_i}$



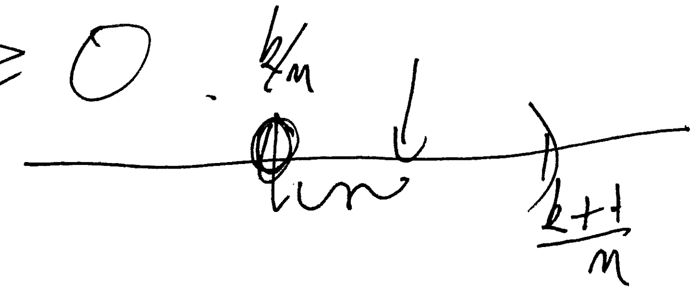
$$EX = \sum P(X = a_i) \cdot a_i .$$

$$= \sum P(A_i)\, a_i$$

__Def:__ If $X = \sum_{1}^{N} a_i \mathbb{1}_{A_i}$, then define

$$EX = \sum_{1}^{N} a_i P(A_i).$$

① Say first X is a R.V & $X \geq 0$.



Let $$X_n = \sum_{k=0}^{n^2} \frac{k}{n} \mathbb{1}_{\{\frac{k}{n} \leq X < \frac{k+1}{n}\}}.$$

(Note ~~if~~ if $|X| \leq n$, $|X_n - X| \leq \frac{1}{n}$.

<u>Define</u> $$EX = \lim_{n \to \infty} EX_n = \lim_{n \to \infty} \sum_{k=0}^{n^2} \frac{k}{n} P\left(X \in \left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$$

② If X is any R.V., define $EX = EX^+ - EX^-$

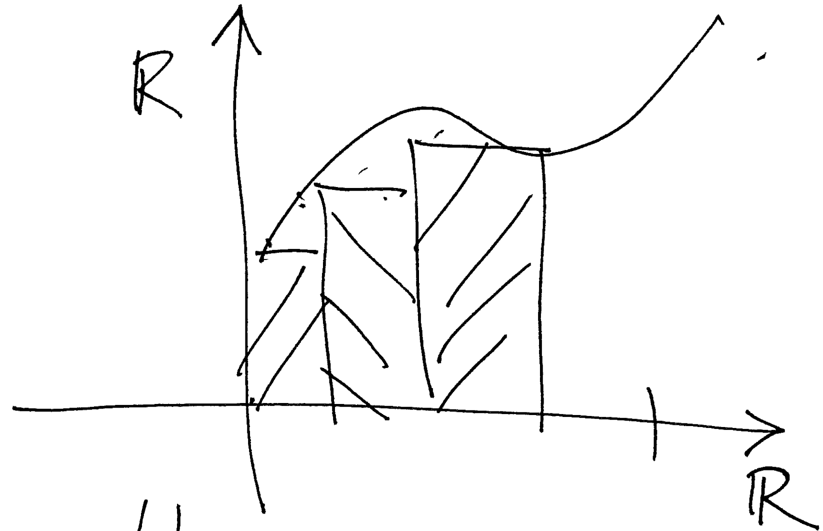
where $X^+ = \max\{X, 0\}$
$X^- = -\min\{X, 0\}$

$E \longrightarrow$ Integral.

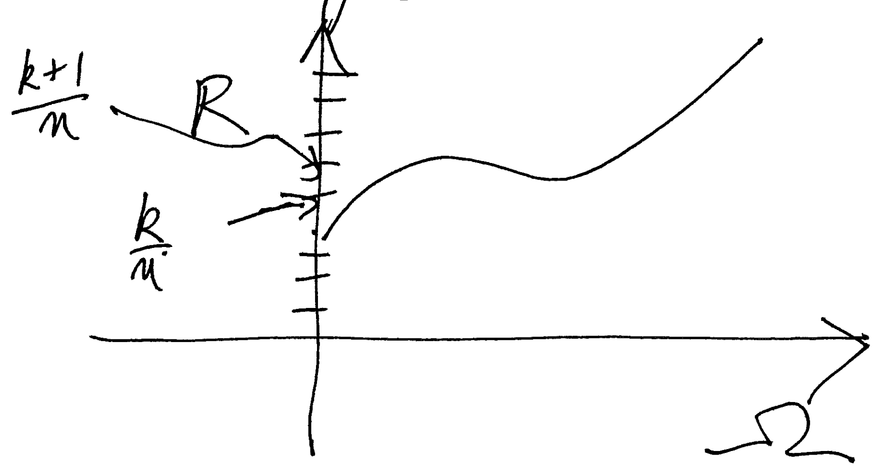
(Notation $EX = \int_\Omega X \, dP$

called the Lebesgue integral).

Riemann.



Lebesgue.



Properties of E:

① Linearity: If X, Y are RV, $\boxed{\alpha \in \mathbb{R}}$, constant.

$$E(X+\alpha Y) = EX + \alpha EY$$

& ② Positivity: If $X \leq Y \Rightarrow EX \leq EY$.

(If $X < Y \Rightarrow EX < EY$).

Brownian Motion:

Def: ① A Brownian Motion is a continuous process that has stationary, independent increments.

Def ②: (Equivalent to ①).

A B.M. is a cts process W such that

① W has independent increments.

② for $s < t$, $W(t) - W(s) \sim N(0, \sigma^2 (t-s))$.

(Here $\cancel{\mu \in \mathbb{R}}$, $\sigma > 0$ is a constant).

Rmk: A standard BM is a BM for which $W(0) = 0$ & $W(t) - W(s) \sim N(0, t-s)$.

① Cts process:

ⓐ Def: A process (a.k.a, stochastic process) is a fn.

$$X : \Omega \times [0, \infty) \longrightarrow \mathbb{R}.$$

such that for every t, ~~All~~ $\omega \longmapsto X(\omega, t)$ is a RV.

Notation: ~~suppress~~ ω. $X(t) \rightarrow$ is a RV.

i.e. For every t, $X(t)$ is a RV.

ⓑ Def ~~a~~ continuous process is a process such that for every ω, the function $t \longmapsto X(\omega, t)$ is continuous.

($X(t)$ is continuous as a function of t).

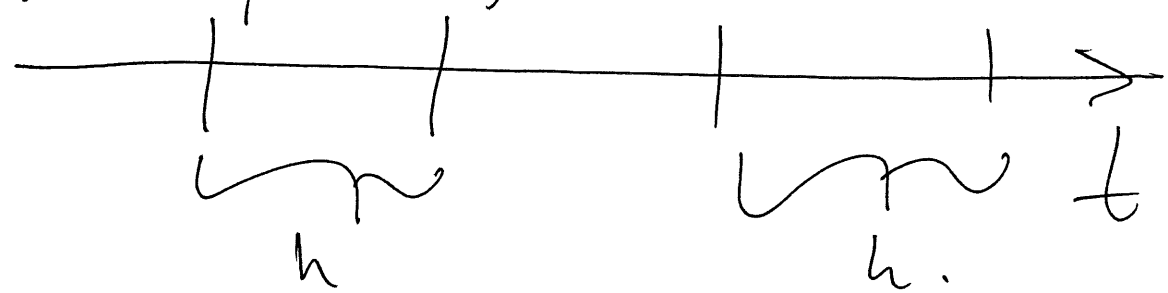
Stationary Increments:

A process X has stationary increments if.

the distribution of ~~X_{t+h}~~ ~~X_{t+h}~~ $X(t+h) - X(t)$

is independent of t.

(Note the dist can of course depend on h).



Independent Increments: A process has independent increments if

~~if~~ for every sequence of times ~~$t_0 < t_1$~~

$$t_0 < t_1 < t_2 < \cdots < t_N.$$

The random variables:

$$X(t_0), \quad X(t_1) - X(t_0), \quad X(t_2) - X(t_1), \ldots$$

$$X(t_N) - X(t_{N-1})$$

are all independent

Intuition: BM $\longrightarrow$ cts, stationary, ind inc.

Say W is a B.M., $W(0)=0$.

$$
\begin{aligned}
W(1) &= W(1) - W(0) \\
&= \left(W(1) - W(\tfrac{1}{2})\right) + \left(W(\tfrac{1}{2}) - W(0)\right). \\
&= \left(W(1) - W(\tfrac{3}{4})\right) + \left(W(\tfrac{3}{4}) - W(\tfrac{1}{2})\right) + \left(W(\tfrac{1}{2}) - W(\tfrac{1}{4})\right) + \\
&\qquad + W(\tfrac{1}{4}) - W(0) \\
&=
\end{aligned}
$$

Independence of R.V's:

① Independence of events: $A, B \in \mathcal{G}$, A, B are indep. if

$$\left(P(A|B) = P(A) \right) \quad P(A \cap B) = P(A)P(B).$$

Def: Say X is a RV.

Define $\sigma(X) = \sigma$ alg generated by X.

$= \sigma$ alg generated by the events $\{X \leq \alpha\}$ for $\alpha \in \mathbb{R}$.

Intuition: $\sigma(X) =$ all info you can deduce by observing X

Eg: $\Omega = \{1, \dots 52\}$ (cards).

$P(1) = P(2) \cdots = \frac{1}{52}$. $\mathcal{F}$ = Power set.

$$X = \begin{cases} 1 & \omega \le 26 \quad \text{(clubs / spades)} \\ -1 & \omega > 26 \quad \text{(hearts / diamonds)}. \end{cases}$$

$$\sigma(X) = \{\phi, \{1, \dots 26\}, \{27, \dots 52\}, \Omega\},$$

Def: X & Y are two RV's.

We say X & Y are independent if for any

$A \in \sigma(X)$ & $B \in \sigma(Y)$, A & B are ind.

$\left(\Leftrightarrow P(A \cap B) = P(A) P(B) \right) \Leftrightarrow P(A|B) = P(A)P(B)$),

Note: $X \in [a, b) \in \sigma(X)$.

~~$Y \in [a, b)$~~

$Y \in [c, d) \in \sigma(Y)$.

X, Y ind $\Rightarrow P(X \in [a,b)$ & $Y \in [c,d)) = P(X \in [a,b)) P(Y \in [c,d))$.

$\Leftrightarrow P(X \le \alpha, Y \le \beta) = P(X \le \alpha) P(Y \le \beta)$