Recitation 1

Administrative Stuff.

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↳ These notes will be scanned and uploaded:

Today: - σ- Algebras + Examples.
- probality measure examples.
- ~~RV~~ Random Variables
- Expectation.

~~Ex2~~

Recall: $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra if:

i). $\emptyset \in \mathcal{G}$.

ii). $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$

iii) $\{A_n\}_{n=1}^{\infty} \in \mathcal{G} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$.

Ex 1 a). T or F It $\mathcal{G}$ is a σ-algebra and $\mathcal{H}$ is a σ-algebra is $\mathcal{G} \cap \mathcal{H}$ a σ-algebra.

i) $\emptyset \in \mathcal{G}$ and $\emptyset \in \mathcal{H}$. $\Rightarrow \emptyset \in \mathcal{G} \cap \mathcal{H}$.

ii) suppose $A \in \mathcal{G} \cap \mathcal{H} \Rightarrow A \in \mathcal{G}$ and $A \in \mathcal{H} \xRightarrow{\sigma\text{-algebra}} A^c \in \mathcal{G}, A^c \in \mathcal{H}$.

$\Rightarrow A^c \in \mathcal{G} \cap \mathcal{H}$

iii) $\{A_n\}_{n=1}^{\infty} \in \mathcal{G} \cap \mathcal{H} \Rightarrow \{A_n\}_{n=1}^{\infty} \in \mathcal{G}$, $\{A_n\} \in \mathcal{H} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{G} \cap \mathcal{H}$

6). $\mathcal{G}, \mathcal{H}$ σ-algebras. IS $\mathcal{G} \cup \mathcal{H}$ a σ-algebra?

~~Ex~~ NOT TRUE IN GENERAL

Ex $\Omega = \{(HH), (HT), (TH), (TT)\}$ i.e. 2 coin tosses

$$\left.\begin{aligned} \mathcal{G} &= \{\emptyset, \Omega, \{(HH)\}, \{(HT), (TH), (TT)\}\} \\ \mathcal{H} &= \{\emptyset, \Omega, \{(TT)\}, \{(HH), (TH), (HT)\}\} \end{aligned}\right\} \rightarrow \text{check these are } \sigma\text{-algebras.}$$

$$\mathcal{G} \cup \mathcal{H} = \{\emptyset, \Omega, \{HH\}, \{(HT), (TH), (TT)\}, \{(TT)\}, \{(HH), (TH), (HT)\}\}$$

so we see that $\{(HH)\}, \{(TT)\} \in \mathcal{G} \cup \mathcal{H}$ but $\underbrace{(HH) \cup (TT)}_{\notin \mathcal{G} \cup \mathcal{H}} = \{(HH), (TT)\}$ $\notin \mathcal{G} \cup \mathcal{H}$.

How can we make this into a σ-algebra? Exercise.

In general $\sigma(\mathcal{G} \cup \mathcal{H}) \supseteq \mathcal{G} \cup \mathcal{H}$.

property that fails is prop (iii).

$\mathcal{B} \rightsquigarrow$ Borel Sigma-algebra.

$\mathcal{B}$ is the smallest σ-algebra containing $\{(-\infty, a) : a \in \mathbb{R}\}$.

What does smallest mean?

If $\mathcal{G}$ is another sigma algebra containing $\{(-\infty, a) ; a \in \mathbb{R}\}$

then $\mathcal{B} \subseteq \mathcal{G}$.

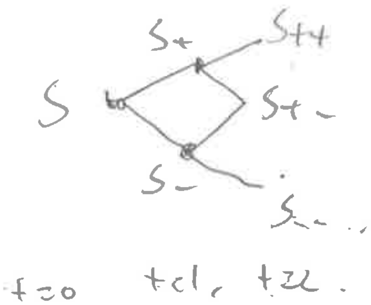
Ex Show that $[a,b] \in \mathcal{B} \quad \forall \; a < b$.

$$\forall n \in \mathbb{N}.\ (-\infty, b + \tfrac{1}{n}) \in \mathcal{B} \Rightarrow \bigcap_{n=1}^{\infty} (-\infty, b + \tfrac{1}{n}) = \underbrace{(-\infty, b]}_{A_2} \in \mathcal{B}.$$

we also know that $(-\infty, a) \in \mathcal{B} \Rightarrow (-\infty, a)^c = \underbrace{[a, \infty)}_{A_1} \in \mathcal{B}$.

$$A_1 \cap A_2 = [a,b] \in \mathcal{B}.$$

Ex Binomial tree Example.



$\Omega = \{$all possible stock price paths$\}$.

$\Omega = \{(S_+, S_{++}), (S_+, S_{+-}), (S_-, S_{+-}), (S_-, S_{--})\}$ ← Notice this is coin toss space.

$\mathcal{G}_1 \Leftrightarrow$ the sets we "see" at time $t=1$.

$\mathcal{G}_1 = \{\emptyset, \Omega, \{(S_+, S_{++}), (S_+, S_{+-})\}, \{(S_-, S_{+-}), (S_-, S_{--})\}\}$.

Recall: A probability measure $\mathbb{P}: \mathcal{G} \longrightarrow [0,1]$ s.t.

i) $P(\Omega) = 1$

ii) $\{A_n\}_{n=1}^{\infty}$ s.t. A_n are pairwise disjoint then:

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

a). $A_1 \subset A_2 \subset \ldots$ show that $P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

$B_1 := A_1$, $B_2 = A_2 \setminus A_1$, ..., $B_n = A_n \setminus A_{n-1}$ $\longrightarrow \bigcup_{i=1}^{K} B_i = A_1 \cup A_2 \setminus A_1 \cup \ldots \cup A_K \setminus A_{K-1}$

$\{x : x \in A_1 \text{ or } x \in A_2 \setminus A_1\}$ $\parallel$ $A_1 \cup A_2 = A_2$ $= A_K = \bigcup_{n=1}^{K} A_n$

$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ ~~becau~~, Also then B_n's are disjoint.

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) \overset{ii)}{=} \sum_{n=1}^{\infty} P(B_n) = P(A_1) + \sum_{n=2}^{\infty} P(A_n \setminus A_{n-1})$$

$$= P(A_1) + \lim_{n\to\infty} \sum_{i=2}^{n} P(A_i) - P(A_{i-1}) = \cancel{P(A_1)} + \lim_{n\to\infty} (P(A_n) - \cancel{P(A_1)})$$

$$= \lim_{n\to\infty} P(A_n)$$

6). $A_1 \supset A_2 \supset \ldots$ Want to show $\lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n)$.

$B_1 = A_1 \setminus A_1 = \emptyset$, $B_2 = A_1 \setminus A_2$, ..., $B_n = A_1 \setminus A_n$

Now B_n is an increasing sequence i.e. $B_n \subseteq B_{n+1}$.

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} (A_n). \qquad \bigcup_{i=1}^{k} B_i = \emptyset \cup A_1 \setminus A_2 \cup \ldots A_1 \setminus A_k = \bigcup_{i=1}^{k} (A_1 \setminus A_i)$$
$$= A_1 \setminus (\bigcap_{i=1}^{k} A_i)$$

$$P(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(A_1 \setminus A_n) = \lim_{n\to\infty} (\cancel{P(A_1)} - P(A_n)).$$

$$\|$$

$$P(A_1 \setminus \bigcap_{n=1}^{\infty} A_n) = P(A_1) - P(\bigcap_{n=1}^{\infty} A_n) \Rightarrow \lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n). \square$$

3. Random Variables: $(\Omega, \mathcal{G}, \mathbb{P})$. If $X: \Omega \to \mathbb{R}$ satisfies

Def

$$\{\omega \in \Omega : X(\omega) < a\} = X^{-1}(-\infty, a) \in \mathcal{G} \quad \forall a \in \mathbb{R}$$ then X is a random variable.

Notice i) A r.v. depends on the σ-algebra.

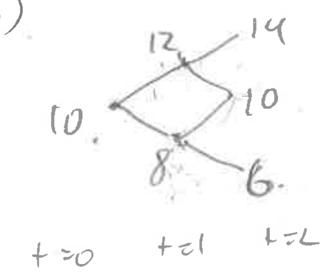
ii). ~~does~~ definition of a r.v. does not depend on $\mathbb{P}$.

Ex 1) Ω = {all outcomes of m coin tosses}, $\mathcal{G} = \mathcal{P}(\Omega)$, X = # of heads.

This is a r.v. → Check.

2). $\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$ $\mathcal{G} = \mathcal{P}(\Omega)$ then X = sum of die is a r.v.

3)



$\Omega = \{\text{all possible stock paths}\}.$

$= \{(12,14), (12,10), (8,10), (8,6)\}$

$\mathcal{G} = \mathcal{G}_1 = \{\emptyset, \Omega, \{(12,14), (12,10)\}, \{(8,10), (8,6)\}\}.$

$\mathcal{G}_2 = \mathcal{P}(\Omega).$

a) Is $X = \frac{S_1 - 10}{10}$ where S_1 is the actual stock price at time 1, a r.v. wrt $\mathcal{G}_1$.

Let's check: $X = \begin{cases} 1/5 & S_1 = 12 \\ -1/5 & S_1 = 8. \end{cases}$

for $a \le -\frac{1}{5}$. $X^{-1}(-\infty, a) = \emptyset \in \mathcal{G}_1$.

for $-\frac{1}{5} < a \le \frac{1}{5}$ $X^{-1}(-\infty, a) = \{(8,6), (8,10)\} \in \mathcal{G}_1$.

for $a > \frac{1}{5}$: $X^{-1}(-\infty, a) = \Omega \in \mathcal{G}_1$. e.g $a = 1$. $X^{-1}(-\infty, 1) = \Omega$.

So X is a r.v. wrt $\mathcal{G}_1$.

Is X a r.v. wrt $\mathcal{G}_2$?.

Yes $\forall a$ we checked that $X^{-1}(-\infty, a) \in \mathcal{G}_1 \subseteq \mathcal{G}_2$.

In general if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and X is a r.v. wrt $\mathcal{G}_1$

then X is a r.v. wrt $\mathcal{G}_2$.

6) Let $Y = \dfrac{S_2 - 10}{10}$. IS Y a r.v. wrt $\mathcal{G}_1$?

$$Y = \begin{cases} 2/5 & \text{if } S_2 = 14. \\ 0. & \text{if } S_2 = 10 \\ -2/5 & \text{if } S_2 = 6. \end{cases}$$

$a = \frac{1}{5}$. then $Y^{-1}(-\infty, -\frac{1}{5}) = \{w \in \Omega;\ Y(w) = -\frac{2}{5}\} = \{(8,6)\} \notin \mathcal{G}_1$.

BUT Y is a r.v. wrt to $\mathcal{G}_2$. (check it!)

①. ↳. σ- algebra's represent information flow.

as t increases. $\mathcal{G}_0 = \{\phi, \Omega\} \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2$.

$\mathcal{G}_t \subseteq \mathcal{G}_{t+1}$

4. $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{G} = \mathcal{P}(\Omega)$, $X = \#$ of heads.

$P(HH) = P(TT) = P(HT) = P(TH) = \frac{1}{4}$.

Notation: We drop the w's.

$P(X=0) = P(X(w)=0) = P(TT) = \frac{1}{4}$.

$P(X=1) = P(HT) + P(TH) = \frac{1}{2}$

$P(X=2) = P(HH) = \frac{1}{4}$.

if we define for $0 \leq p \leq 1$ $P(HH) = p^2$, $P(TT) = (1-p)^2$.

$P(HT) = P(TH) = p(1-p) \longrightarrow Bin(2, p)$.

Def Given RV X, $(\Omega, \mathcal{G}, \mathbb{P})$. the expectation of X is

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) \longrightarrow \text{Lebesgue Integral.}$$

if X is a discrete r.v. with pmf P

then $\mathbb{E}[X] = \sum_{n=1}^{\infty} x_n P(X = x_n)$

If X is a continuous r.v. with pdf f.

then $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$.

Ex if $X \geq 0$ show that $\mathbb{E}[X] = \int_0^\infty P(X > x)\, dx$.

$$\int_0^\infty P(X > x)\, dx = \int_0^\infty \mathbb{E}[\mathbb{I}[X > x]]\, dx. \qquad \mathbb{I}[X > x] = \begin{cases} 1 & X(\omega) > x \\ 0 & \text{o/w.} \end{cases}$$

$$= \int_0^\infty \int_\Omega \mathbb{I}[X(\omega) > x]\, dP(\omega)\, dx = \int_\Omega \int_0^\infty \mathbb{I}[X(\omega) > x]\, dx\, d\mathbb{P}(\omega)$$

$$= \int_\Omega \int_0^{X(\omega)} 1\, dx\, dP(\omega) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}[X]$$