Logistics: I HATE CANVAS.

HW: → Due Wed:
- → 2:45PM → 100%. Full.
- → 24 hours → 10% penalty
- → 24 hours. → 25% penalty

→ Drop lowest HW score.

on Canvas.

Midterm: Nov 20th (in class).

Final: Dec 16th (Sat).

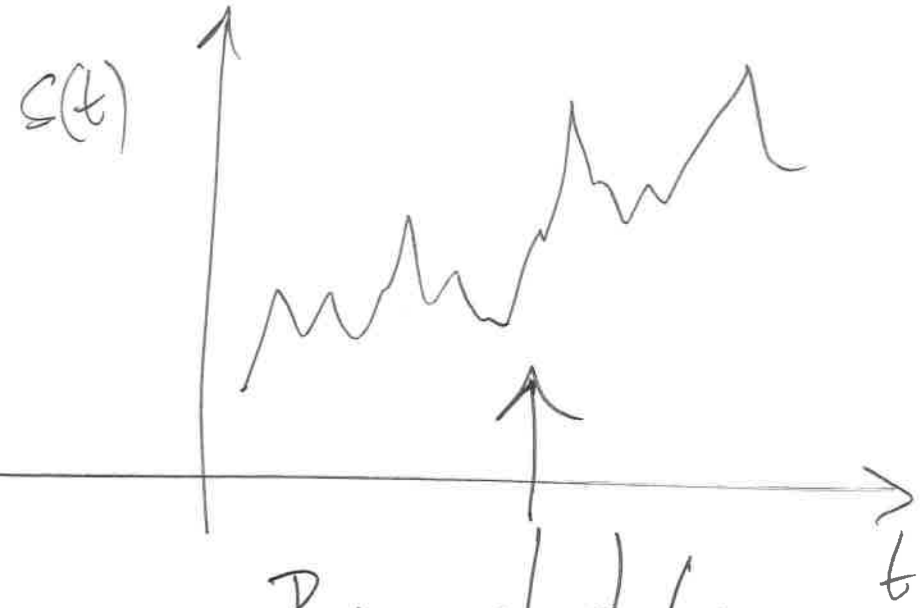
Office Hours: (

Grade:
- ① 20% HW 80% Exams.
- $\boxed{\text{② 100\% Exams}}$

LOWER

① Motivation:

Diff fns.



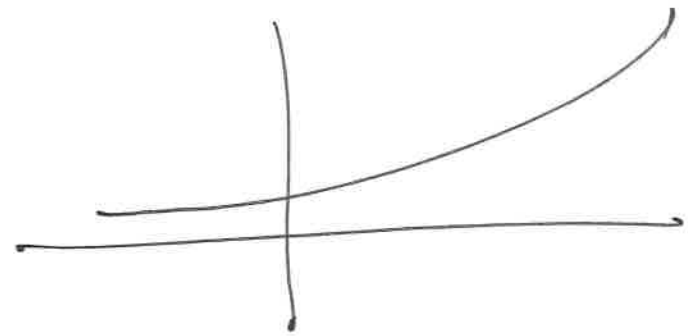
Price of stocks.
(not diff).

Model stock prices → "Geometric Brownian Motion"

$$\hookrightarrow \quad dS(t) = \alpha S(t)\, dt + \underbrace{\sigma S(t)\, dW}_{\text{"noisy fluctuation"}}$$

Say $dS(t) = \alpha S(t)\, dt + 0$

$\Rightarrow \partial_t S = \alpha S(t) \Rightarrow S(t) = S(0)e^{\alpha t}$



Brownian Motion

Ito formula etc.

Black Scholes formula:

Price Options: $S(t)$

European call option, strike K, maturity T.

→ Pay off: at time T. Option worth $\begin{cases} (S(T)-K)^+ \\ 0 \end{cases}$

Say $S(t) = x$, what is the "fair price" of this option?

$c(t, x)$.
$\cancel{c(\ldots)}$ = option price. (~~...~~ also depends on K, T, α, σ, interest rate r).

$$c(t, x) = x\, N\left(d_+(T-t, x)\right) - K e^{-r(T-t)} N\left(d_-(T-t, x)\right)$$

$N \longrightarrow$ cdf of normal.

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right).$$

Finally : $\boxed{\text{Risk Neutral Measures}}$. & Fundamental $\boxed{\text{theorems}}$ of asset pricing.

$\boxed{\text{Brownian Motion:}}$

$$\boxed{dS = \alpha S\,dt + \underbrace{\sigma S\,d\boxed{W}}_{\text{noisy fluctuation.}}}$$

$W \longrightarrow$ Brownian Motion.

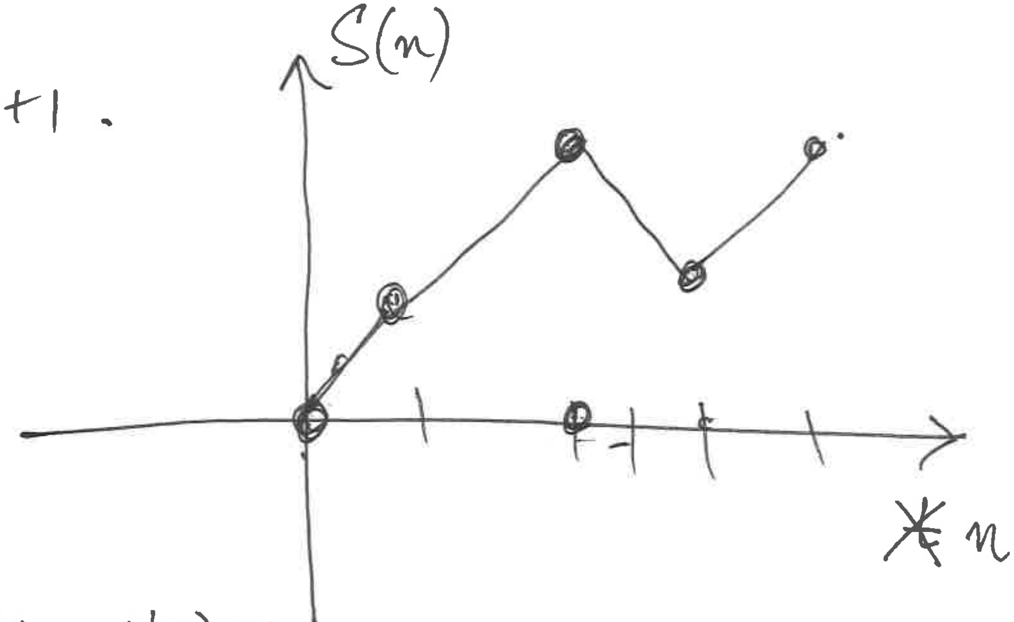
"Continuous time Random Walk".

Discrete Random Walk:

$X_1, X_2, \ldots$ i.i.d. Random variables $\begin{pmatrix} \text{mean } 0 \\ \text{variance } 1 \end{pmatrix}$.

$$S(n+1) = S(n) + X_{n+1}.$$

$$S(n) = \sum_{1}^{n} X_i$$



Define $S(t) = S(\lfloor t \rfloor) + (t - \lfloor t \rfloor) X_{\lfloor t \rfloor + 1}$

$n = \lfloor t \rfloor$: $\quad S(t) = S(n) + (t-n) X_{n+1} \qquad (t \in [n, n+1))$.

Let $\varepsilon > 0$ Let $S^\varepsilon(t) = S\left(\frac{t}{\varepsilon}\right)$.

Try $\lim_{\varepsilon \to 0} S^\varepsilon =$ nonsense.

Compute $\mathrm{Var}(S^\varepsilon(t)) = \mathrm{Var}\left(S\left(\frac{t}{\varepsilon}\right)\right)$.

$$= \left\lfloor \frac{t}{\varepsilon} \right\rfloor + \left(\frac{t}{\varepsilon} - \left\lfloor \frac{t}{\varepsilon} \right\rfloor\right)^2 \xrightarrow[\varepsilon \to 0]{} +\infty.$$

Rescale to keep variance constant.

Try #2: Let $S^{\varepsilon}(t) = \alpha_{\varepsilon} S\left(\frac{t}{\varepsilon}\right)$.

Choose α_{ε} + $\mathrm{Var}(S^{\varepsilon}(t))$ remains bdd as $\varepsilon \longrightarrow 0$.

Compute $\mathrm{Var}(S^{\varepsilon}(t)) = \alpha_{\varepsilon}^{2}\left(\left\lfloor \frac{t}{\varepsilon} \right\rfloor + \left(\frac{t}{\varepsilon} - \left\lfloor \frac{t}{\varepsilon}\right\rfloor\right)^{2}\right)$.

Choose $\alpha_{\varepsilon} = \sqrt{\varepsilon}$: $\mathrm{Var}(S^{\varepsilon}(t)) \xrightarrow[\varepsilon \to 0]{} t$

Theorem: If $\alpha_{\varepsilon} = \sqrt{\varepsilon}$, then the process S^{ε} converge as $\varepsilon \to 0$.

Let $W(t) = \lim_{\varepsilon \to 0} S^{\varepsilon}(t)$. [$W$ is a Brownian motion].

"Crash course" on $\boxed{\text{measure theoretic}}$ Prob:

Def: A *sample space* is a non-empty set.

Def: (σ-alg): A σ-algebra a non-empty set $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ (power set of Ω).

such that: (1) If $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$

and (2) If $A_1, A_2, \dots \in \mathcal{G} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Intuition: σ-alg = Information.

σ-alg = sets you can decide the probability of.

Q: ~~If $A_1, \ldots \in \mathcal{G}$ must $A_1 \cap A_2 \ldots \in \mathcal{G}$.~~

$A \in \mathcal{G}$ & $B \in \mathcal{G}$, must $A \cap B \in \mathcal{G}$?

Yes: Reason: $A \cap B = (A^c \cup B^c)^c$.

$A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$ (by 1).

$B \in \mathcal{G}$ " " "

$A^c \in \mathcal{G}$ & $B^c \in \mathcal{G} \Rightarrow A^c \cup B^c \in \mathcal{G} \Rightarrow$

$(A^c \cup B^c)^c \in \mathcal{G}$ by 1.

Eg 1: $\mathcal{G} = \{\phi, \Omega\}$ is a σ-alg.

Probability measure:

A probability measure on $(\Omega, \mathcal{G})$ is

"a countably additive" fn $\boxed{P: \mathcal{G} \longrightarrow [0,1]}$ such that $P(\Omega)=1$.

I.e: ① for any $A \in \mathcal{G}$, $P(A) \in [0,1]$.

② $P(\Omega) = 1$.

③ If $A_1, A_2, \dots \in \mathcal{G}$ are disjoint (i.e. $A_i \cap A_j = \phi$ whenever $i \neq j$).

then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

Properties:

① $P(\phi) = 0$

$$\left(\underbrace{P(\phi \cup \phi)}_{\| \atop P(\phi)} = P(\phi) + P(\phi) \right\} \Rightarrow P(\phi) = 0 \Big)$$

② If $A, B \in \mathcal{G}$, & $A \subseteq B \Rightarrow P(B - A) = P(B) - P(A)$

③ $P(A^c) = 1 - P(A)$.

④ Say $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$.

$A_i \in \mathcal{G}$: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} P(A_i)$

(5) Say $A_1 \supseteq A_2 \supseteq A_3 \cdots$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim P(A_i).$$