46-944 Stochastic Calculus for Finance I: Midterm.

2017-02-08

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 90 minutes. The exam has a total of 5 questions and 30 points.*
- *The questions are roughly ordered by difficulty. Good luck* ☺.

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \,|\, t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

1. Let X be the Itô process defined by $X(t) = \displaystyle\int_0^t W(r)\, dW(r)$, and let $s, t \in \mathbb{R}$ with $0 \leqslant s < t$.

[4] (a) Compute $\boldsymbol{E}(X(t) \,|\, \mathcal{F}_s)$. [Your answer can involve, W, X, t and s, but not any expectations or integrals.]

[4] (b) Compute $\boldsymbol{E}X(t)^2$. [Your answer should only involve t, and not any expectations or integrals.]

[5] 2. Let $X(t) = te^{3W(t)}$. Explicitly find adapted processes b, σ such that

$$X(t) = X(0) + \int_0^t b(r)\, dr + \int_0^t \sigma(r)\, dW(r)\,.$$

[5] 3. Suppose X and Y are processes such that

$$X(t) = \int_0^t s\, dW(s) + 2\int_0^t W(s)\, ds \quad \text{and} \quad dY(t) = \Big(\big(2W(t) + 2t^2\big)e^{2tW(t)}\Big)\, dt + \Big(2te^{2tW(t)}\Big)dW(t)\,.$$

Let $Z(t) = X(t) + Y(t)$. Find the quadratic variation of Z. Express your answer in the form $\int_0^t f(s, W(s))\, ds$, for some function f that you explicitly compute a formula for.

[6] 4. If $0 \leqslant s < t$, compute $\boldsymbol{E}(W(s)^2 \,|\, W(t))$. [Recall $\boldsymbol{E}(W(s)^2 \,|\, W(t))$ is defined to be $\boldsymbol{E}[W(s)^2 \,|\, \sigma(W(t))]$. To solve this problem I recommend you first find $\alpha \in \mathbb{R}$ such that $W(s) - \alpha W(t)$ is independent of $W(t)$.]

[6] 5. Let $X(t) = \displaystyle\int_0^t W(r)\, dW(r)$. Given $0 \leqslant s < t$, compute $\boldsymbol{E}(X(t)^2 \,|\, \mathcal{F}_s)$. [Your answer can involve, W, X, t and s, but not any expectations or integrals.]