46-944 Stochastic Calculus for Finance I: Final.

2017-03-04

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 7 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck* ☺.

Unless otherwise stated, W denotes a standard (one dimensional) Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

[5] 1. If $0 \leqslant s < t$ compute

$$\boldsymbol{E}\Big(W(s)^3 \int_0^t (r + W(r))^2 \, dW(r) \Bigm| \mathcal{F}_s\Big).$$

Your answer may involve W and Itô integrals, but not any expectations or conditional expectations.

[5] 2. Let $X(t) = e^{3t} W(t)^2$. Explicitly find adapted processes b, σ such that

$$X(t) = X(0) + \int_0^t b(r) \, dr + \int_0^t \sigma(r) \, dW(r).$$

[6] 3. Let W_1 and W_2 be two independent standard one-dimensional Brownian motions. Find an adapted process σ such that the process B defined by

$$B(t) = \int_0^t \frac{1}{1 + W_1(s)^2} \, dW_1(s) + \int_0^t \sigma(s, W_1(s), W_2(s)) \, dW_2(s)$$

is also a standard one-dimensional Brownian motion.

[6] 4. Compute

$$\boldsymbol{E}\Big(W(t) \int_0^t e^{3W(s)} \, dW(s)\Big).$$

You may leave your answer as a *Riemann* integral of a function that does not involve W or expectations.

[6] 5. Consider the statement *"The replicating portfolio of an European put is always long on cash."* We interpret this statement mathematically as follows: Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Consider a European put with strike K and maturity T. Let $p(t, x)$ be the price of this option at time t given that the stock price is x. In order to price this option, we construct a replicating portfolio that at time t holds $\Delta(t)$ shares of the stock and has $\Gamma(t)$ invested in the money market account. Write down a formula for $\Gamma(t)$ and use your formula to determine whether $\Gamma(t) \geqslant 0$ for all $t < T$ or not.

[You do NOT have to re-derive the Black-Scholes formula, and or any of the Greeks, and may use whatever you know here. In order to get full credit you only need to produce a (correct) formula for $\Gamma(t)$, and use this formula to determine whether $\Gamma(t) \geqslant 0$ or not. If you have either an incorrect formula with no explanation, or an incorrect explanation (even with a correct formula) you will get no partial credit whatsoever. If you have a correct explanation with a slightly incorrect formula, you might get some partial credit.]

[6] 6. Compute

$$\boldsymbol{E}\Big[\Big|\int_0^t W(s) \, ds\Big|^{1/2}\Big].$$

Your answer may involve t and Riemann integrals, but *may not* involve W or expectations.

[6] 7. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $T > 0$ and consider a derivative security that pays

$$V(T) \stackrel{\text{def}}{=} \Big(\ln\Big(\frac{S(T)}{S(0)}\Big)\Big)^+ = \max\Big\{0, \ln\Big(\frac{S(T)}{S(0)}\Big)\Big\},$$

at maturity T. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your answer may involve α, σ, r, t, $S(t)$, T and or Riemann integrals. However, your answer **should not** involve W, $\tilde{W}$, expectations or conditional expectations.