Brief notes on Multidimensional Calculus

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Preface

These are really brief notes on the material covered in class for the Fall 2015 section of *21-268: Multidimensional Calculus*. The main focus of these notes *is not* the evaluation of complicated integrals over the region enclosed by various oddly shaped surfaces. Most of these integrals can, for instance, be computed much more efficiently by a computer or a sufficiently trained monkey. This course instead focuses on the concepts and underlying intuition instead.

A student taking this course is assumed to have a through knowledge of *one variable calculus* and *elementary linear algebra*. This includes:

- Standard theorems about limits (e.g. sums, products, etc.)
- Theorems about one derivatives that are similar to the one variable case (e.g. product and quotient rules).
- How to differentiate and integrate various complicated trigonometric functions.

In accordance with this, the material concerning limits and differentiation (Chapters 1 and 2) is *extremely brief*. The proofs of most results in these chapters can be done rigorously with very little background, and some are presented here.

These notes are a little more detailed regarding implicit functions, multiple integrals and vector calculus (Chapters 3–6). However, the proofs of most results in these chapters require either a stronger analysis background (e.g. the knowledge of compactness), or more technicality than is appropriate in a first multi-variable calculus course. If the reader accepts the results concerning inverse functions and multiple integrals without proof, then enough material is provided here for the reader to be able to prove the important theorems concerning line and surface integrals (e.g. Greens, Stokes and the divergence theorems).

Contributing

These notes are open-source. If you want to fill in a few details, or add a section on something related *I welcome your help*. The LaTeX source for these notes is currently publicly hosted at GitLab: `https://gitlab.com/gi1242/cmu-math-268`. Please contact me (via email) if you are interested in contributing.

Contents

CHAPTER 1

Limits and Continuity

1. Open sets in $\mathbb{R}^d$.

We begin by defining *open sets*. As we will see shortly, in order to study limits and derivatives of functions, we need their domains of definition to be open sets.

DEFINITION 1.1. A set $U \subset \mathbb{R}^d$ is open if for every $a \in U$ there exists $r > 0$ such that $B(a, r) \subset U$.

Recall, $B(a, r) \subseteq \mathbb{R}^d$ is defined by

$$B(a, r) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid |x - a| < r\},$$

is the *open ball* with center a and radius r. Throughout these notes we use the convention that a vector $x \in \mathbb{R}^d$ has coordinates $(x_1, x_2, \ldots, x_d)$. Recall $|x| = (\sum x_i^2)^{1/2}$ is the length of the vector x.

EXAMPLE 1.2. The empty set and $\mathbb{R}^d$ are both open.

EXAMPLE 1.3. The ball $B(a, r)$ is open.

PROOF. Let $x \in B(a, r)$, and choose $\varepsilon = r - |x - a|$. (Notice $\varepsilon > 0$, since $|x - a| < r$ by definition of $B(a, r)$.) Now by the triangle inequality

$$|y - a| \leqslant |y - x| + |x - a| < \varepsilon + |x - a| = r,$$

and hence $y \in B(a, r)$. This shows $B(x, \varepsilon) \subseteq B(a, r)$ and hence $B(a, r)$ is open. □

For now, the above should be all that's needed. But here are a few related concepts that will come up later on in the semester.

DEFINITION 1.4. The *boundary* of a set $A \subseteq \mathbb{R}^d$, denoted by ∂A, is defined by

$$\partial A \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid \forall \varepsilon > 0,\ B(x, \varepsilon) \cap A \neq \emptyset \text{ and } B(x, \varepsilon) \cap A^c \neq \emptyset\}.$$

EXAMPLE 1.5. The boundary of a ball $B(a, r) \subseteq \mathbb{R}^d$ is the $d - 1$ dimensional sphere $\{x \in \mathbb{R}^d \mid |x - a| = r\}$.

One of the reasons the notion of boundary arises is because many fundamental theorems in calculus relate integrals of functions on domains to integrals over the boundary of of this domain.

Another notion that will be important later is that of connectedness. Domains of functions are usually open *connected* sets. Connectedness, however, is a bit harder to define precisely at this stage. Here is the "official" definition:

DEFINITION 1.6. An open set $U \subset \mathbb{R}^d$ is *connected* if it can not be expressed as the union of two non-empty, disjoint open sets.

While this definition is the "official" one, it is a little harder to grasp (e.g. try proving $\mathbb{R}^d$ is connected)! Instead we will use the circular, but intuitive definition to work with instead.

DEFINITION 1.7. A set $U \subset \mathbb{R}^d$ is connected if for any x and y in U, there exists a *continuous* path that connects x and y that stays entirely within the set U.

This definition isn't quite legal because we have not yet defined what a continuous path is.

DEFINITION 1.8. A domain (sometimes called an open domain) is an open connected set.

Most functions we study will have an open connected set as their domain of definition. Moreover, the boundary of this domain will usually be a (piecewise) smooth curve or surface.

2. Limits

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function. Intuitively, we say $\lim_{x\to a} f(x) = \ell$ if by making x close enough to a, we can make $f(x)$ arbitrarily close to ℓ.[1] We make this precise mathematically as follows.

DEFINITION 2.1. We say $\lim_{x\to a} f(x) = l$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \varepsilon$.

REMARK 2.2. If f is only defined on an open set U, then we also insist $x, a \in U$ above.

The standard theorems about limits (sums, products, quotients) from one variable calculus still hold in this context.

PROPOSITION 2.3. *Let $U \subseteq \mathbb{R}^d$ be open, and $f, g\colon U \to \mathbb{R}^n$ be two functions. Suppose for some $a \in U$, $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$.*

(1) For any $\alpha \in \mathbb{R}$, $\lim_{x\to a}(f(x) + \alpha g(x)) = \ell + \alpha m$.
(2) $\lim_{x\to a} f(x) \cdot g(x) = \ell \cdot m$.
(3) If instead $f\colon U \to \mathbb{R}$, then $\lim_{x\to a} f(x)g(x) = lm$.
(4) If $n = 1$ and $m \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = \ell/m$.

PROOF. The proofs of these are almost identical to the one dimensional analogues which we assume the reader is familiar with. For brevity we only prove the first assertion. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = \ell$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|f(x) - \ell| < \varepsilon/2$. Also, $\lim_{x\to a} g(x) = \ell$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_1$ implies $|g(x) - \ell| < \varepsilon/(2(|\alpha| + 1))$.

Now choose $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - a| < \delta$ we have

$$|f(x) + \alpha g(x) - (\ell + \alpha m)| \leqslant |f(x) - \ell| + |\alpha||g(x) - m| \leqslant \frac{\varepsilon}{2} + \frac{|\alpha|\varepsilon}{2|\alpha| + 1} < \varepsilon\,. \quad \square$$

Since limits of functions of one variable have been studied previously, we now attempt to reduce limits of functions of several variables to limits of functions of one variable.

[1] Note, often people say $\lim_{x\to a} f(x) = \ell$ if as x gets closer to a, $f(x)$ gets closer to ℓ. This is wrong! If you draw a graph of $f(x) = x\sin(1/x)$, for instance, you will see that as $x \to 0$, $f(x) \to 0$. But as x gets closer to 0, $f(x)$ certainly *doesn't* get closer to 0.

PROPOSITION 2.4. *If* $\lim_{x\to a} f(x) = l$ *then for every* $v \in \mathbb{R}^d$ *with* $v \neq 0$, *we must have* $\lim_{t\to 0} f(a+tv) = l$.

PROOF. Pick $\varepsilon > 0$. We know $\exists \delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - l| < \varepsilon$. Choose $\delta_1 = \delta/|v|$. Now it immediately follows that $0 < t < \delta_1$ implies $|f(a+tv) - l| < \varepsilon$. □

The converse (surprisingly) is false!

EXAMPLE 2.5. Let $f(x) = 1$ if $0 < x_2 < x_1^2$ and $f(x) = 0$ otherwise. Then $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{t\to 0} f(tv) = 0$ for all $v \in \mathbb{R}^2 - \{0\}$.

EXAMPLE 2.6. Let $f(x) = x_1^2 x_2/(x_1^4 + x_2^2)$, and $f(0) = 0$. Then $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{t\to 0} f(tv) = 0$ for all $v \in \mathbb{R}^2 - \{0\}$.

Proposition 2.4 can be used to show that various limits don't exist.

EXAMPLE 2.7. Show that $\lim\limits_{x\to 0} \dfrac{x_1 x_2}{|x|^2}$ does not exist.

PROOF. Choosing $v_1 = (1,1)$ and $v_2 = (1,0)$ we see

$$\lim_{t\to 0} f(tv_1) = \frac{1}{2} \quad \text{and} \quad \lim_{t\to 0} f(tv_2) = 0 \neq \frac{1}{2}.$$

So by Proposition 2.4, $\lim_{x\to 0} x_1 x_2/|x|^2$ can not exist. □

3. Continuity

Intuitively, a continuous function is one which sends close by points to close by points. To define continuity precisely, one needs to use limits (or the ε-δ definition directly).

DEFINITION 3.1. Let $U \subset \mathbb{R}^m$ be a domain, and $f : U \to R^d$ be a function. We say f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

DEFINITION 3.2. If f is continuous at every point $a \in U$, then we say f is continuous on U (or sometimes simply f is continuous).

Again the standard results on continuity from one variable calculus hold. Sums, products, quotients (with a non-zero denominator) and composites of continuous functions will all yield continuous functions.

The notion of continuity gives us a generalization of Proposition 2.4 that is useful is computing the limits along arbitrary curves instead.

PROPOSITION 3.3. *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a function, and* $a \in \mathbb{R}^d$. *Let* $\gamma : [0,1] \to \mathbb{R}^d$ *be a* any continuous function *with* $\gamma(0) = a$, *and* $\gamma(t) \neq a$ *for all* $t > 0$. *If* $\lim_{x\to a} f(x) = \ell$, *then we must have* $\lim_{t\to 0} f(\gamma(t)) = \ell$.

PROOF. The proof of this is very similar to the fact that the composition of continuous functions is again continuous. Let $\varepsilon > 0$ be arbitrary. Since $f(x) \to \ell$ as $x \to a$, there exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \text{ whenever } 0 < |x - a| < \delta\,. \tag{3.1}$$

Now since $\gamma(t) \to a$ as $t \to 0$, there exists δ_1 such that $|\gamma(t) - a| < \delta$ whenever $|t - 0| < \delta_1$. Since $\gamma(t) \neq a$ when $t \neq 0$, we can use (3.1) to guarantee that

$$|f(\gamma(t)) - \ell| < \varepsilon, \text{ whenever } |t - 0| < \delta_1\,.$$

□

COROLLARY 3.4. *If there exists two continuous functions* $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^d$ *such that for* $i \in 1,2$ *we have* $\gamma_i(0) = a$ *and* $\gamma_i(t) \neq a$ *for all* $t > 0$. *If* $\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$ *then* $\lim_{x\to a} f(x)$ *can not exist.*

One can use Corollary 3.4 to quickly check that the limit of the function in Example 2.6 does not exist.

CHAPTER 2

Differentiation

1. Directional and Partial Derivatives

DEFINITION 1.1. Let $U \subset \mathbb{R}^d$ be a domain, $f : U \to \mathbb{R}$ be a function, and $v \in \mathbb{R}^d - \{0\}$ be a vector. We define the *directional derivative* of f in the direction v at the point a by

$$D_v f(a) \stackrel{\text{def}}{=} \frac{d}{dt} f(a + tv)\Big|_{t=0}$$

EXAMPLE 1.2. If $f(x) = |x|^2$, then $D_v f(x) = 2x \cdot v$.

REMARK 1.3. Be aware that some authors define $D_v f$ by additionally dividing by the length of v. We will never do that!

DEFINITION 1.4. We define the i^{th} partial derivative of f (denoted by $\partial_i f$) to be the directional derivative of f in direction e_i (where e_i is the i^{th} elementary basis vector).

Practically, to compute the i^{th} partial derivative of f differentiate it with respect to x_i treating all the other coordinates as constant.

EXAMPLE 1.5. For $x \neq 0$ we have $\partial_i |x| = x_i/|x|$.

2. Derivatives

DEFINITION 2.1. Let $U \subseteq \mathbb{R}^d$ be a domain, $f : \mathbb{R}^d \to \mathbb{R}$ be a function, and $a \in U$. We say f is *differentiable at* a if there exists a linear transformation $T : \mathbb{R}^d \to \mathbb{R}$ and a function e such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Th}{|h|} = 0\,.$$

In this case, the linear transformation T is called *the derivative of f at a*, and denoted by Df_a.

PROPOSITION 2.2. *Let $U \subseteq \mathbb{R}^d$ be a domain, $f : \mathbb{R}^d \to \mathbb{R}$ be a function, and $a \in U$. The function f is differentiable at a* if and only if *there exists a linear transformation $T : \mathbb{R}^d \to \mathbb{R}$ and a function e such that*

(1) $f(a+h) = f(a) + Th + e(h)$
(2) and $\lim_{h \to 0} |e(h)|/|h| = 0$.

PROPOSITION 2.3. *If f is differentiable at a, then all the directional derivatives $D_v f(a)$ exist. Further,*

$$Df_a = \begin{pmatrix} \partial_1 f(a) & \partial_2 f(a) & \cdots & \partial_d f(a) \end{pmatrix}$$

and

$$D_v f(a) = Df_a v = \sum_{i=1}^{d} v_i \partial_i f(a).$$

REMARK 2.4. This shows that the linear transformation appearing in the definition of f is unique!

The converse of Proposition 2.3 is (surprisingly?) false. All directional derivatives can exist, however, the function need not be differentiable (or even continuous!)

EXAMPLE 2.5. Let $f(x,y) = x^2y/(x^4+y^2)$. Then for every $v \in \mathbb{R}^2 - \{0\}$, $D_v f(0)$ exists, but f is not differentiable (or even continuous) at 0.

The converse of Proposition 2.3 is true under the additional assumption that the partial derivatives are continuous.

THEOREM 2.6. *If all partial derivatives of f exist in a neighbourhood of a,* and are continuous at a, *then f is differentiable at a.*

PROOF. For simplicity we assume $d = 2$. By the mean value theorem

$$\begin{aligned} f(a+h) - f(a) &= f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2) + f(a_1+h_1, a_2) - f(a_1, a_2) \\ &= h_2 \partial_2 f(a_1+h_1, a_2+\xi_2) + h_1 \partial_1 f(a_1+\xi_1, a_2) \end{aligned}$$

for some ξ_1, ξ_2 such that ξ_i lies between 0 and h_i. Now let T be the matrix $(\partial_1 f(a)\ \partial_2 f(a))$ and observe

$$f(a+h) = f(a) + Th + e(h),$$

where

$$e(h) = h_2(\partial_2 f(a_1+h_1, a_2+\xi_2) - \partial_2 f(a)) + h_1(\partial_1 f(a_1+\xi_1, a_2) - \partial_1 f(a)).$$

Clearly

$$\frac{|e(h)|}{|h|} \leqslant |\partial_2 f(a_1+h_1, a_2+\xi_2) - \partial_2 f(a)| + |\partial_1 f(a_1+\xi_1, a_2) - \partial_1 f(a)|,$$

which converges to 0 as $h \to 0$. □

Note, however, it is possible for a function to be differentiable, and for the partial derivatives to exist and be discontinuous.

EXAMPLE 2.7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = |x|^2 \sin(1/|x|)$ when $x \neq 0$, and $f(0) = 0$. Then f is differentiable on all of $\mathbb{R}^2$ (including $x = 0$), and hence all partial derivatives of f exist at all points in $\mathbb{R}^2$. However, $\partial_1 f$ and $\partial_2 f$ are not continuous at $x = 0$.

DEFINITION 2.8. Let $U \subset \mathbb{R}^m$ be a domain, and $a \in U$. We say a function $U \to \mathbb{R}^n$ is differentiable if if there exists a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ and a function e such that

(1) $f(a+h) = f(a) + Th + e(h)$
(2) *and* $\lim_{h\to 0} |e(h)|/|h| = 0$.

Note this is almost *the same* as Definition 2.1. The only change is that the linear transformation T is now a map from $\mathbb{R}^m$ to $\mathbb{R}^n$ instead.

PROPOSITION 2.9. *Let $f = (f_1, \dots, f_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ and $a \in \mathbb{R}^m$. The function f is differentiable at a if and only if each coordinate function f_i is differentiable at a. Further, the derivative Df is a $n \times m$ matrix given by*

$$Df_a = \begin{pmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \cdots & \partial_m f_1(a) \\ \partial_1 f_2(a) & \partial_2 f_2(a) & \cdots & \partial_m f_2(a) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_n(a) & \partial_2 f_n(a) & \cdots & \partial_m f_n(a) \,. \end{pmatrix}$$

As before, the derivative Df is also called the *Jacobian Matrix*.

3. Tangent planes and Level Sets

Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be differentiable.

DEFINITION 3.1. The *graph* of f is the set $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^n$ defined by

$$\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}^d\}.$$

Given a point $(a, f(a)) \in \Gamma$ we define the *tangent plane* of f at the point a by the equation

$$y = f(a) + Df_a(x - a)$$

Note that the tangent plane is a d-dimensional hyper-plane in $\mathbb{R}^d \times \mathbb{R}^n$. It is the *best linear approximation* to the graph Γ at the point a. Projecting the tangent plane into 2 dimensions (by freezing other coordinates) gives you a tangent line.

DEFINITION 3.2. The *tangent space* to the graph Γ at the point $(a, f(a))$, denoted by $T\Gamma_{(a,f(a))}$ is defined by

$$T\Gamma_{(a,f(a))} = \{(x, y) \mid y = Df_a x,\ x \in \mathbb{R}^d\}\,.$$

Namely, the tangent space is the space of all vectors parallel to the tangent plane, and passing through the origin.

DEFINITION 3.3. Given $c \in \mathbb{R}$ we define the *level set of f* to be the set $\{x \in \mathbb{R}^d \mid f(x) = c\}$.

If $d = 2$, then level sets are typically curves. If $d = 3$, then level sets are typically surfaces. In higher dimensions (for "nice functions") level sets of f are typically $d - 1$-dimensional hyper-surfaces.

EXAMPLE 3.4. Let $d = 3$ and $f(x) = |x|^2$. Then $\{f(x) = c\}$ is the sphere of radius $\sqrt{c}$ for $c > 0$, a point for $c = 0$ and the empty set for $c < 0$.

Level sets are very useful in plotting, and are often used to produce *contour plots*. We will see later that if v is tangent to a level set of f, then $D_v f = 0$. Moreover if $f \colon \mathbb{R}^d \to \mathbb{R}$, then ∇f (defined to be $(Df)^T$) is orthogonal to level sets.

4. Chain rule

The one variable calculus rules for differentiation of sums, products and quotients (when they make sense) are still valid in higher dimensions.

PROPOSITION 4.1. *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be two differentiable functions.*

- *$f + g$ is differentiable and $D(f + g) = Df + Dg$.*
- *fg is differentiable and $D(fg) = fDg + gDf$.*

- *At points where $g \neq 0$, f/g is also differentiable and*

$$D\Big(\frac{f}{g}\Big) = \frac{gDf - fDg}{g^2}$$

These follow in a manner very similar to the one variable analogues, and are left for you to verify. The one rule that is a little different in this context is the differentiation of composites.

THEOREM 4.2 (Chain Rule). *Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ be domains, $g : U \to V$, $f : V \to \mathbb{R}^d$ be two differentiable functions. Then $f \circ g : U \to \mathbb{R}^d$ is also differentiable and*

$$D(f \circ g)_a = (Df_{g(a)})(Dg_a)$$

Note Df_g and Dg are both matrices, and the product above is the *matrix product* of Df and Dg.

PROOF. The basic intuition is as follows. Since f, g are differentiable we know there exist functions e_1 and e_2 such that

$$g(a+h) = g(a) + Dg_a + e_2(h) \quad \text{and} \quad f(g(a)+h) = f(g(a)) + Df_{g(a)} + e_1(h)\,,$$

with $\lim_{h\to 0} e_i(h)/|h| = 0$. Consequently,

$$\begin{aligned} f(g(a+h)) &= f\big(g(a) + Dg_a h + e_2(h)\big) \\ &= f(g(a)) + Df_{g(a)}\big(Dg_a h + e_2(h)\big) + e_1\big(Dg_a h + e_2(h)\big) \\ &= f(g(a)) + \big(Df_{g(a)}\, Dg_a\big)h + e_3(h)\,, \end{aligned}$$

where

$$e_3(h) = Df_{g(a)} e_2(h) + e_1\big(Dg_a h + e_2(h)\big)\,.$$

Now to finish the proof one only needs to show $\lim_{h\to 0} e_3(h)/|h| = 0$, which can be done directly from the ε-δ definition. □

REMARK 4.3. An extremely useful arises when $d = 1$ and we need to compute $\partial_i(f \circ g)$. In this case, by the chain rule

$$\partial_i(f \circ g) = (Df_g)(Dg)e_i = \sum_{j=1}^{n} \partial_j f\Big|_g \partial_i g_j\,.$$

While this can be derived immediately by multiplying the matrices Df_g and Dg, it arises often enough that it is worth directly remembering.

Another version of the chain rule that often shows up in problems is as follows.

PROPOSITION 4.4. *Suppose z is a function of x and y, and x and y are in turn functions of s and t. Then*

$$\partial_s z = \partial_x z \partial_s x + \partial_y z \partial_s y\,, \qquad \text{and} \qquad \partial_t z = \partial_x z \partial_t x + \partial_y z \partial_t y\,. \tag{4.1}$$

PROOF. Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ for some functions f, g, h. Define $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\psi = (g, h)$. Now equation (4.1) follows immediately by realizing $z = f \circ \psi$ and using the chain rule. □

EXAMPLE 4.5. Compute $\frac{d}{dx}x^x$ and $\frac{d}{dt}\int_0^t e^{-(t-s)^2}\,ds$.

PROPOSITION 4.6. *Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be differentiable. Then $fg\colon \mathbb{R}^d \to \mathbb{R}$ is differentiable and $D(fg) = f(Dg) + g(Df)$.*

PROOF. Let $F(x,y) = xy$ and $G(x) = (f(x), g(x))$. Then observe $fg = F \circ G$, and the conclusion follows from the chain rule. □

REMARK 4.7. A similar trick can be used to prove the quotient rule.

As a consequence, here is a "proof" that directional derivatives in directions tangent to level sets vanish.

PROPOSITION 4.8. *Let $\Gamma = \{x \mid f(x) = c\}$ be a level set of a differentiable function f. Let $\gamma : [-1,1] \to \Gamma$ be a differentiable function, $v = D\gamma(0)$, and $a = \gamma(0)$. Then $D_v f(a) = 0$.*

Think of $\gamma(t)$ as the position of a particle at time t. If for all t, $\gamma(t)$ belongs to the curve Γ, then the velocity $D\gamma$ should be tangent to the curve γ, and thus thus the vector v above should be tangent to Γ. (When we can define this rigorously, we will revisit it and prove it.)

PROOF. Note $f \circ \gamma = c$ (since $\gamma(t) \in \Gamma$ for all t). By the chain rule $D(f \circ \gamma) = Df_\gamma D\gamma$. At $t = 0$ this gives $Df_{\gamma(0)} v = 0 \implies D_v f(\gamma(0)) = 0$ as desired. □

DEFINITION 4.9. If $f : \mathbb{R}^d \to R$ is differentiable, define the *gradient* of f (denoted by ∇f) to be the transpose of the derivative of f.

We've seen above that if v is tangent to a level set of f at a, then $D_v f(a) = 0$. This is equivalent to saying $\nabla f(a) \cdot v = 0$, or that *the gradient of f is perpendicular to level sets of f*. Note, in directions tangent to level sets, f is changing the *least*. One would expect that in the perpendicular direction (given by ∇f), the function f is changing the most. This is shown by the following proposition.

PROPOSITION 4.10. *If $v \in \mathbb{R}^d$ with $|v| = 1$. Then $D_v f(a)$ is maximised when $v = \nabla f(a)$ and $D_v f(a)$ is minimised when $v = -\nabla f(a)$.*

REMARK 4.11. This fact is often used when numerically finding minima of functions, and is known as *the method of gradient descent*. Namely, start with a guess for the minimum x_0. Now choose successive approximations to move directly against ∇f. That is, define

$$x_{n+1} = x_n + \gamma_n \nabla f(x_n)\,,$$

for some small γ_n. Usually, the standard numerical algorithms suggest using

$$\gamma_n = \frac{(x_n - x_{n-1}) \cdot (\nabla f(x_n) - \nabla f(x_{n-1}))}{|\nabla f(x_n) - \nabla f(x_{n-1})|^2}\,,$$

which guarantees converge to a local minimum, under certain assumptions on f.

5. Higher order derivatives

Given a function f, treat $\partial_i f$ as a function. If $\partial_i f$ is itself a differentiable function, we can differentiate it again. The second derivative (denoted by $\partial_j \partial_i f$) is called a second order partial of f. These can further be differentiated to obtain third order partials.

THEOREM 5.1 (Clairaut). *If $\partial_i \partial_j f$ and $\partial_j \partial_i f$ both exist in a neighbourhood of a, and are continuous at a then they must be equal.*

If the mixed second order partials are not continuous, however, they need not be equal.

EXAMPLE 5.2. Let $f(x,y) = x^3y/(x^2+y^2)$ for $(x,y) \neq 0$ and $f(0,0) = 0$. Then $\partial_x\partial_y f(0,0) = 1$ but $\partial_y\partial_x f(0,0) = 0$.

PROOF OF CLAIRAUT'S THEOREM. Here's the idea in 2D (the same works in higher dimensions). For simplicity assume $a = 0$.

- Let R be the rectangle with corners $(0,0)$, $(h,0)$, $(0,k)$, (h,k).
- Using the mean value theorem, show $f(h,k) - f(h,0) - f(0,k) + f(0,0) = hk\partial_x\partial_y f(\alpha)$ for some point $\alpha \in R$.
- Observe $f(h,k) - f(h,0) - f(0,k) + f(0,0) = f(h,k) - f(0,k) - f(h,0) + f(0,0)$ and so using the mean value theorem show $f(h,k) - f(h,0) - f(0,k) + f(0,0) = hk\partial_y\partial_x f(\beta)$ for some point $\beta \in R$.
- Note that as $(h,k) \to 0$, we have $\alpha, \beta \to 0$. Consequently, if $\partial_x\partial_y f$ and $\partial_y\partial_x f$ are both continuous at 0 we must have

$$\partial_x\partial_y f(0,0) = \lim_{(h,k)\to 0} \frac{f(h,k) - f(h,0) - f(0,k) + f(0,0)}{hk} = \partial_y\partial_x f(0,0),$$

proving equality as desired. □

DEFINITION 5.3. A function is said to be of class C^k if all its k^{th}-order partial derivatives exist and are continuous.

By Clairaut's theorem, we know that mixed partials are equal for C^k functions.

6. Maxima and Minima

DEFINITION 6.1. A function f has a local maximum at a if $\exists \varepsilon > 0$ such that whenever $|x - a| < \varepsilon$ we have $f(x) \leqslant f(a)$.

Our aim is now to understand what having a local maximum / minimum translates to in terms of derivatives of f. For this we do a simple calculation: Observe that if f has a local maximum at a, then for all $v \in \mathbb{R}^d - \{0\}$ the function $f(a+tv)$ must have a local maximum at $t = 0$. Hence we must have $\partial_t f(a+tv)|_{t=0} = 0$ and $\partial_t^2 f(a+tv)|_{t=0} \leqslant 0$. Using the chain rule, we compute

$$\partial_t f(a+tv) = \sum_{i=1}^d \partial_i f(a+tv)v_i \quad \text{and} \quad \partial_t^2 f(a+tv) = \sum_{i,j=1}^d \partial_i\partial_j f(a+tv)v_iv_j$$

Thus at a local maximum we must have

$$\sum_{i=1}^d \partial_i f(a)v_i = 0 \quad \text{and} \quad \sum_{i,j=1}^d \partial_i\partial_j f(a)v_iv_j \leqslant 0$$

for every $v \in \mathbb{R}^d$. This translates to the following proposition.

PROPOSITION 6.2. *If f is a C^2 function which has a local maximum at a, then*

(1) The first derivative Df must vanish at a (i.e. $Df_a = 0$). $Df_a = 0$
(2) The Hessian *Hf is negative semi-definite at a.*

For a local minimum, we replace negative semi-definite *above with* positive semi-definite.

DEFINITION 6.3. The *Hessian* of a C^2 function (denoted by Hf) is defined to be the matrix

$$Hf = \begin{pmatrix} \partial_1\partial_1 f & \partial_2\partial_1 f & \cdots & \partial_d\partial_1 f \\ \partial_1\partial_2 f & \partial_2\partial_2 f & \cdots & \partial_d\partial_2 f \\ \vdots & \vdots & & \vdots \\ \partial_1\partial_d f & \partial_2\partial_d f & \cdots & \partial_d\partial_d f \end{pmatrix}$$

Note if $f \in C^2$, Hf is symmetric.

DEFINITION 6.4. Let A be a $d \times d$ *symmetric* matrix.

- If $(Av) \cdot v \leqslant 0$ for all $v \in \mathbb{R}^d$, then A is called *negative semi-definite*.
- If $(Av) \cdot v < 0$ for all $v \in \mathbb{R}^d$, then A is called *negative definite*.
- If $(Av) \cdot v \geqslant 0$ for all $v \in \mathbb{R}^d$, then A is called *positive semi-definite*.
- If $(Av) \cdot v > 0$ for all $v \in \mathbb{R}^d$, then A is called *positive definite*.

Recall a symmetric matrix is positive semi-definite if all the eigenvalues are non-negative. In 2D this simplifies to the following:

PROPOSITION 6.5. *Let A be the symmetric 2×2 matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$.*

(1) A is positive definite if and only if $a > 0$ and $ac - b^2 > 0$.
(2) A is negative definite if and only if $a < 0$ and $ac - b^2 > 0$.
(3) A is positive semi-definite if and only if $a, c \geqslant 0$ and $ac - b^2 \geqslant 0$.
(4) A is negative semi-definite if and only if $a, c \leqslant 0$ and $ac - b^2 \geqslant 0$.

Finally, we address the converse: Namely, we look for a condition on the derivatives of f that guarantees that f attains a local maximum or minimum at a.

THEOREM 6.6. *Let f be a C^2 function.*

(1) If $Df_a = 0$ and further Hf_a is positive definite, then f attains a local minimum at a.
(2) If $Df_a = 0$ and further Hf_a is negative definite, then f attains a local maximum at a.

The proof uses Taylor's theorem, and we will prove it in Section 7, below

DEFINITION 6.7. We say a is a *local saddle* of f if there exist two linearly independent vectors v_1 and v_2 such that f has a strict local minimum in direction v_1 and a strict local maximum in direction v_2.

PROPOSITION 6.8. *If f is C^2, $Df_a = 0$ and Hf_a has at least one strictly positive and one strictly negative eigenvalue, then a is a local saddle of f.*

This corresponds to points where f has a local maximum in one direction and a local minimum in the other.

EXAMPLE 6.9. The function $|x|^2$ has a local minimum at 0. The function $-|x|^2$ has a local maximum at 0. The function $x_1^2 - x_2^2$ has a saddle at 0.

EXAMPLE 6.10. Let $f\colon \mathbb{R}^d \to \mathbb{R}$, and let $\Gamma \subseteq \mathbb{R}^{d+1}$ be the graph of f (i.e. $\Gamma = \{(x, y) \mid x \in \mathbb{R}^d,\ y = f(x)\}$). Fix $(x', y') \in \mathbb{R}^{d+1}$, and let $(a, f(a))$ be the point on Γ which is closest to (x', y'). Then $x' - a$ is parallel to $\nabla f(a)$ and $(x' - a, y' - f(a))$ is normal to the tangent plane at $(a, f(a))$.

PROOF. Let $d(x) = |x - x'|^2 + (f(x) - y')^2$. At a max $\nabla d = 0$, and hence

$$2(a - x') + 2(f(a) - y')\nabla f(a) = 0\,. \tag{6.1}$$

This shows $x' - a$ is parallel to $\nabla f(a)$.

For the second assertion recall that the tangent plane is defined by

$$y = f(a) + \nabla f(a) \cdot (x - a)\,.$$

This is equivalent to saying $(\nabla f(a), -1) \cdot (x - a, y - f(a)) = 0$, and hence $(\nabla f(a), -1)$ is normal to the tangent plane. But from (6.1) we immediately see

$$\begin{pmatrix} x' - a \\ y' - f(a) \end{pmatrix} = -(y' - f(a)) \begin{pmatrix} \nabla f(a) \\ -1 \end{pmatrix},$$

and hence $(x' - a, y' - f(a))$ is also normal to the tangent plane at a. □

7. Taylors theorem

THEOREM 7.1. *If $f \in C^2$, then*

$$f(a + h) = f(a) + Df_a h + \frac{1}{2} h \cdot Hf_a h + R_2(h), \tag{7.1}$$

where $R_2(h)$ is some function such that

$$\lim_{h \to 0} \frac{R_2(h)}{|h|^2} \to 0.$$

In coordinates equation (7.1) is

$$f(a + h) = f(a) + \sum_i \partial_i f(a) h_i + \frac{1}{2} \sum_{i,j} \partial_i \partial_j f(a) h_i h_j + R_2(h).$$

PROOF. Let $g(t) = f(a + th)$. Using the 1D Taylors theorem we have

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(\xi)$$

for some $\xi \in (0, 1)$. Writing this in terms of f finishes the proof. □

The same technique can show the following mean value theorem:

THEOREM 7.2 (Mean value theorem). *If f is differentiable on the entire line joining a and b,*

$$f(b) = f(a) + (b - a) \cdot \nabla f(\xi)$$

for some point ξ on the line segment joining a and b.

Taylor's theorem allows us to prove Theorem 6.6.

PROOF OF THEOREM 6.6. Suppose $Df_a = 0$ and Hf_a is positive definite. Let λ_0 be the smallest eigenvalue of Hf_a. Expanding in terms of an orthonormal basis of eigenfunctions of Hf_a we see $Hh \cdot h \geqslant \lambda_0 |h|^2$.

Now choose $\delta > 0$ so that $|R_2(h)| < \lambda_0 |h|^2/2$ for $h < \delta$, and note $f(a + h) \geqslant f(a) + \frac{|h|^2}{2} \geqslant f(a)$, showing f has a local min at a. □

A higher order version of Taylor's theorem is also true. It is usually stated using the multi-index notation, collecting all mixed partials that are equal.

DEFINITION 7.3. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, with $\alpha_i \in \mathbb{N} \cup \{0\}$. If $h \in \mathbb{R}^d$ define

$$h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \text{and} \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_d!.$$

Given a $C^{|\alpha|}$ function f, define

$$D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f,$$

with the convention that $\partial_i^0 f = f$.

THEOREM 7.4. *If f is a C^n function on $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ we have*

$$f(a+h) = \sum_{|\alpha|<n} \frac{1}{\alpha!} D^\alpha f(a) + R_n(h),$$

for some function R_n such that

$$\lim_{h \to 0} \frac{R_n(h)}{|h|^n} = 0.$$

The proof follows from the one variable Taylor's theorem in exactly the same as our second order version does, and collecting all mixed partials that are equal puts it in the above form.

CHAPTER 3

Inverse and Implicit functions

1. Inverse Functions and Coordinate Changes

Let $U \subseteq \mathbb{R}^d$ be a domain.

THEOREM 1.1 (Inverse function theorem). *If $\varphi : U \to \mathbb{R}^d$ is differentiable at a and $D\varphi_a$ is invertible, then there exists a domains U', V' such that $a \in U' \subseteq U$, $\varphi(a) \in V'$ and $\varphi : U' \to V'$ is bijective. Further, the inverse function $\psi : V' \to U'$ is differentiable.*

The proof requires compactness and is beyond the scope of this course.

REMARK 1.2. The condition $D\varphi_a$ is necessary: If φ has a differentiable inverse in a neighbourhood of a, then $D\varphi_a$ must be invertible. (Proof: Chain rule.)

This is often used to ensure the existence (and differentiability of local coordinates).

DEFINITION 1.3. A function $\varphi : U \to V$ is called a (differentiable) coordinate change if φ is differentiable and bijective and $D\varphi$ is invertible at every point.

Practically, let φ be a coordinate change function, and set $(u, v) = \varphi(x, y)$. Let $\psi = \varphi^{-1}$, and we write $(x, y) = \psi(u, v)$. Given a function $f : U \to \mathbb{R}$, we treat it as a function of x and y. Now using ψ, we treat (x, y) as functions of (u, v).

Thus we can treat f as a function of u and v, and it is often useful to compute $\partial_u f$ etc. in terms of $\partial_x f$ and $\partial_y f$ and the coordinate change functions. By the chain rule:

$$\partial_u f = \partial_x f \partial_u x + \partial_y f \partial_u y,$$

and we compute $\partial_u x$, $\partial_u y$ etc. either by directly finding the inverse function and expressing x, y in terms of u, v; or implicitly using the chain rule:

$$I = D\psi_\varphi \, D\varphi = \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \implies \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}^{-1}.$$

EXAMPLE 1.4. Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Let $\varphi(x, y) = (u, v)$. For any $a \neq 0 \in \mathbb{R}^2$, there exists a small neighbourhood of a in which φ has a differentiable inverse.

The above tells us that *locally* x, y can be expressed as functions of u, v. This might not be true globally. In the above case we can explicitly solve and find x, y:

$$x = \Big(\frac{\sqrt{u^2 + v^2} + u}{2}\Big)^{1/2} \quad \text{and} \quad y = \Big(\frac{\sqrt{u^2 + v^2} - u}{2}\Big)^{1/2} \tag{1.1}$$

is one solution. (Negating both, gives another solution.)

Regardless, even without using the formulae, we can implicitly differentiate and find $\partial_u x$. Consequently,

$$\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}^{-1} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}^{-1} = \frac{1}{2(x^2+y^2)} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

It is instructive to differentiate (1.1) directly and double check that the answers match.

Polar coordinates is another example, and has been done extensively your homework.

2. Implicit functions

Let $U \subseteq \mathbb{R}^{d+1}$ be a domain and $f : U \to \mathbb{R}$ be a differentiable function. If $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$, we'll concatenate the two vectors and write $(x, y) \in \mathbb{R}^{d+1}$.

THEOREM 2.1 (Implicit function theorem). *Suppose $c = f(a, b)$ and $\partial_y f(a, b) \neq 0$. Then, there exists a domain $U' \ni a$ and differentiable function $g : U' \to \mathbb{R}$ such that $g(a) = b$ and $f(x, g(x)) = c$ for all $x \in U'$. Further, there exists a domain $V' \ni b$ such that $\{(x, y) \mid x \in U', y \in V', f(x, y) = c\} = \{(x, g(x)) \mid x \in U'\}$. (In other words, for all $x \in U'$ the equation $f(x, y) = c$ has a unique solution in V' and is given by $y = g(x)$.)*

REMARK 2.2. To see why $\partial_y f \neq 0$ is needed, let $f(x, y) = \alpha x + \beta y$ and consider the equation $f(x, y) = c$. To express y as a function of x we need $\beta \neq 0$ which in this case is equivalent to $\partial_y f \neq 0$.

REMARK 2.3. If $d = 1$, one expects $f(x, y) = c$ to some curve in $\mathbb{R}^2$. To write this curve in the form $y = g(x)$ using a differentiable function g, one needs the curve to never be vertical. Since ∇f is perpendicular to the curve, this translates to ∇f never being horizontal, or equivalently $\partial_y f \neq 0$ as assumed in the theorem.

REMARK 2.4. For simplicity we chose y to be the last coordinate above. It could have been any other, just as long as the corresponding partial was non-zero. Namely if $\partial_i f(a) \neq 0$, then one can locally solve the equation $f(x) = f(a)$ (uniquely) for the variable x_i and express it as a differentiable function of the remaining variables.

EXAMPLE 2.5. $f(x, y) = x^2 + y^2$ with $c = 1$.

PROOF OF THE IMPLICIT FUNCTION THEOREM. Let $\varphi(x, y) = (x, f(x, y))$, and observe $D\varphi_{(a,b)} \neq 0$. By the inverse function theorem φ has a unique local inverse ψ. Note ψ must be of the form $\psi(x, y) = (x, g(x, y))$. Also $\varphi \circ \psi = \mathrm{Id}$ implies $(x, y) = \varphi(x, g(x, y)) = (x, f(x, g(x, y)))$. Hence $y = g(x, c)$ uniquely solves $f(x, y) = c$ in a small neighborhood of (a, b). □

Instead of $y \in \mathbb{R}$ above, we could have been fancier and allowed $y \in \mathbb{R}^n$. In this case f needs to be an $\mathbb{R}^n$ valued function, and we need to replace $\partial_y f \neq 0$ with the assumption that the $n \times n$ minor in Df (corresponding to the coordinate positions of y) is invertible. This is the general version of the implicit function theorem.

THEOREM 2.6 (Implicit function theorem, general case). *Let $U \subseteq \mathbb{R}^{d+n}$ be a domain, $f : \mathbb{R}^{d+n} \to \mathbb{R}^n$ be a differentiable function, $a \in U$ and M be the $n \times n$ matrix obtained by taking the i_1^{th}, i_2^{th}, $\ldots i_n^{th}$ columns from Df_a. If M is invertible, then one can locally solve the equation $f(x) = f(a)$ (uniquely) for the variables x_{i_1}, $\ldots$, x_{i_n} and express them as a differentiable function of the remaining d variables.*

To avoid too many technicalities, we only state a more precise version of the above in the special case where the $n \times n$ matrix obtained by taking the last n columns of Df is invertible. Let's use the notation $(x, y) \in \mathbb{R}^{d+n}$ when $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$. Now the precise statement is almost identical to Theorem 2.1:

THEOREM 2.7 (Implicit function theorem, precise statement in a special case). *Suppose $c = f(a, b)$ and the $n \times n$ matrix obtained by taking the last n columns of $Df_{a,b}$ is invertible. Then, there exists a domain $U' \subseteq \mathbb{R}^d$ containing a and differentiable function $g : U' \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = c$ for all $x \in U'$. Further, there exists a domain $V' \subseteq \mathbb{R}^n$ containing b such that $\{(x, y) \mid x \in U', y \in V', f(x, y) = c\} = \{(x, g(x)) \mid x \in U'\}$. (In other words, for all $x \in U'$ the equation $f(x, y) = c$ has a unique solution in V' and is given by $y = g(x)$.)*

EXAMPLE 2.8. Consider the equations

$$(x-1)^2 + y^2 + z^2 = 5 \quad \text{and} \quad (x+1)^2 + y^2 + z^2 = 5$$

for which $x = 0$, $y = 0$, $z = 2$ is one solution. For all other solutions close enough to this point, determine which of variables x, y, z can be expressed as differentiable functions of the others.

SOLUTION. Let $a = (0, 0, 1)$ and

$$F(x, y, z) = \begin{pmatrix} (x-1)^2 + y^2 + z^2 \\ (x+1)^2 + y^2 + z^2 \end{pmatrix}$$

Observe

$$DF_a = \begin{pmatrix} -2 & 0 & 4 \\ 2 & 0 & 4 \end{pmatrix},$$

and the 2×2 minor using the first and last column is invertible. By the implicit function theorem this means that in a small neighbourhood of a, x and z can be (uniquely) expressed in terms of y. □

REMARK 2.9. In the above example, one can of course solve explicitly and obtain

$$x = 0 \quad \text{and} \quad z = \sqrt{4 - y^2},$$

but in general we won't get so lucky.

3. Tangent planes and spaces

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and consider the implicitly defined curve $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$. (Note this is some level set of f.) Pick $(a, b) \in \Gamma$, and suppose $\partial_y f(a, b) \neq 0$. By the implicit function theorem, we know that the y-coordinate of this curve can locally be expressed as a differentiable function of x. In this case the tangent line through (a, b) has slope $\frac{dy}{dx}$.

Directly differentiating $f(x, y) = c$ with respect to x (and treating y as a function of x) gives

$$\partial_x f + \partial_y f \frac{dy}{dx} = 0 \iff \frac{dy}{dx} = \frac{-\partial_x f(a, b)}{\partial_y f(a, b)}.$$

Further, note that the normal vector at the point (a, b) has direction $(-\frac{dy}{dx}, 1)$. Substituting for $\frac{dy}{dx}$ using the above shows that the normal vector is parallel to ∇f.

REMARK 3.1. Geometrically, this means that ∇f is *perpendicular* to level sets of f. This is the direction along which f is changing "the most". (Consequently, the directional derivative of f along directions tangent to level sets is 0.)

The same is true in higher dimensions, which we study next. Consider the surface $z = f(x, y)$, and a point (x_0, y_0, z_0) on this surface. Projecting it to the x-z plane, this becomes the curve $z = f(x, y_0)$ which has slope $\partial_x f$. Projecting it onto the y-z plane, this becomes the curve with slope $\partial_y f$. The *tangent plane* at the point (x_0, y_0, z_0) is defined to be the unique plane passing through (x_0, y_0, z_0) which projects to a line with slope $\partial_x f(x_0, y_0)$ in the x-z plane and projects to a line with slope $\partial_y f(x_0, y_0)$ in the y-z plane. Explicitly, the equation for the tangent plane is

$$z - z_0 = (x - x_0)\partial_x f(x_0, y_0) + (y - y_0)\partial_y f(x_0, y_0).$$

REMARK 3.2. Consider a curve Γ in $\mathbb{R}^2$ and $a \in \Gamma$. The usual scenario is that Γ "touches" the tangent line at a and the continues (briefly) on the same side of the tangent line. The exception is of course inflection points, where Γ passes through its tangent line. In a generic curve, inflection points are usually isolated and this doesn't happen too often.

In $2D$ however, the picture is quite different. A surface will "touch" and locally stay on the same side of the tangent plane if the Hessian is either positive definite or negative definite. If the Hessian has both a strictly positive and a strictly negative eigenvalue, then the curve will necessarily "pass through" the tangent plane at the point of contact. Further, it is possible to construct surfaces where this happens at *every single point*. One such example is the surface $z = x^2 - y^2$.

DEFINITION 3.3. The *tangent space* to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is defined to be the subspace

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = x\partial_x f(x_0, y_0) + y\partial_x f(x_0, y_0) = Df_{(x_0,y_0)} \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

Elements of the tangent space are said to be *tangent vectors* at the point (x_0, y_0, z_0).

REMARK 3.4. The tangent space is *parallel* to the tangent plane, but shifted so that is passes through the origin (and hence is also a vector subspace).

REMARK 3.5. Clearly the vector $(\partial_x f, \partial_y f, -1)$ is *normal* to the tangent space of the surface $z = f(x, y)$.

Now let $g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable, $c \in \mathbb{R}$ and consider the *implicitly defined* surface $\Sigma = \{(x, y, z) \mid g(x, y, z) = c\}$. Note again, this is a level set of g. Suppose (x_0, y_0, z_0) is a point on this surface and $\partial_z g(x_0, y_0, z_0) \neq 0$. Then using the implicit function theorem, the z-coordinate of this surface can locally be expressed as a differentiable function of x and y (say $z = f(x, y)$). In terms of f we know how to compute the tangent plane and space of Σ. Our aim is to write this directly in terms of g.

PROPOSITION 3.6. *Let* $a = (x_0, y_0, z_0) \in \Sigma$.

- *The* tangent space *at* a *(denoted by* $T\Sigma_a$*) is exactly* $\ker(Dg_a)$.
- *The* tangent plane *at* a *is* $\{x \in \mathbb{R}^3 \mid Dg_a(x - a) = 0\}$.

Recall elements of the tangent space are called *tangent vectors*. If $v \in T\Sigma_a$ then $Dg_a(v) = 0$, and hence the directional derivative of g in the direction v must be

0. Note further ∇g is *normal* to the surface Σ. Both these statements were made earlier, but not explored in detail as we didn't have the implicit function theorem at our disposal.

PROOF OF PROPOSITION 3.6. Substituting $z = f(x, y)$ in $g(x, y, z) = c$ and differentiating with respect to x and y gives

$$\partial_x g + \partial_z g \partial_x f = 0 \quad \text{and} \quad \partial_y g + \partial_z g \partial_y f = 0$$

Thus the tangent plane to the surface $g(x, y, z) = c$ at the point (x_0, y_0, z_0) is given by

$$z - z_0 = Df_{(x_0,y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \iff Dg_{(x_0,y_0,z_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$$

The tangent space is given by

$$T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \nabla g_{(x_0,y_0,z_0)} = 0 \right\}. \qquad \square$$

These generalizes in higher dimensions. Without being too precise about the definitions, here is the bottom line:

PROPOSITION 3.7. *Let $g : \mathbb{R}^{n+d} \to \mathbb{R}^n$ be a differentiable function, $c \in \mathbb{R}^n$ and let $M = \{x \in \mathbb{R}^{n+d} \mid g(x) = c\}$. Suppose the implicit function theorem applies at all points in M. Then M is a d-dimensional "surface" (called a d-dimensional manifold). At any point $a \in M$, the tangent space is exactly* $\ker Dg_a$. *Consequently, $D_v g(a) = 0$ for all tangent vectors v, and $\nabla g_1, \ldots \nabla g_n$ are n linearly independent vectors that are orthogonal to the tangent space.*

4. Parametric curves.

DEFINITION 4.1. Let $\Gamma \subseteq \mathbb{R}^d$ be a (differentiable) closed curve. We say γ is a (differentiable) parametrization of Γ if $\gamma : [a, b] \to \Gamma$ is differentiable, $D\gamma \neq 0$, $\gamma : [a, b) \to \Gamma$ is bijective, $\gamma(b) = \gamma(a)$ and $\gamma'(a) = \gamma'(b)$. A curve with a parametrization is called a parametric curve.

EXAMPLE 4.2. The curve $x^2 + y^2 = 1$ can be parametrized by $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$

REMARK 4.3. A curve can have many parametrizations. For example, $\delta(t) = (\cos t, \sin(-t))$ also parametrizes the unit circle, but runs clockwise instead of counter clockwise. Choosing a parametrization requires choosing the direction of traversal through the curve.

REMARK 4.4. If γ is a curve with endpoints, then we require $\{\gamma(a), \gamma(b)\}$ to be the endpoints of the curve (instead of $\gamma(b) = \gamma(a)$).

REMARK 4.5. If γ is an open curve, then we only require γ to be defined (and bijective) on (a, b).

REMARK 4.6. While curves can not self-intersect, we usually allow parametric curves to self-intersect. This is done by replacing the requirement that γ is injective with the requirement that if for $x, y \in (a, b)$ we have $\gamma(x) = \gamma(y)$ then $D\gamma_x$ and $D\gamma_y$ are linearly independent. Sometimes, one also allows parametric curves loop back on themselves (e.g. $\gamma(t) = (\cos(t), \sin(t))$ for $t \in \mathbb{R}$.

DEFINITION 4.7. If γ represents a differentiable parametric curve, we define $\gamma' = D\gamma$.

REMARK 4.8. For any t, $\gamma'(t)$ is a vector in $\mathbb{R}^d$. Think of $\gamma(t)$ representing the position of a particle, and γ' to represent the velocity.

PROPOSITION 4.9. *Let Γ be a curve and γ be a parametrization, $a = \gamma(t_0) \in \Gamma$. Then*

$$T\Gamma_a = \operatorname{span}\{\gamma'(t_0)\}.$$

Consequently, tangent line through a is $\{\gamma(t_0) + t\gamma'(t_0) \mid t \in \mathbb{R}\}$.

If we think of $\gamma(t)$ as the position of a particle at time t, then the above says that the tangent space is spanned by the *velocity* of the particle. That is, the velocity of the particle is always tangent to the curve it traces out. However, the acceleration of the particle (defined to be γ'') *need not* be tangent to the curve! In fact if the magnitude of the velocity $|\gamma'|$ is constant, then the acceleration will be *perpendicular* to the curve!

PROOF OF PROPOSITION 4.9. We only do the proof in 3D. Write $\Gamma = \{f = 0\}$ where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a differentiable function such that $\operatorname{rank}(Df_a) = 2$. In this case $\Gamma = S^{(1)} \cap S^{(2)}$ where $S^{(i)}$ is the surface $\{f_i = 0\}$. Since $\Gamma \subseteq S^{(i)}$, $f_i \circ \gamma = 0$ and hence (by the chain rule) $\gamma'(t) \in \ker(Df_i(a))$. By dimension counting this forces

$$T\Gamma_a = TS^{(1)}_a \cap TS^{(2)}_a = \ker(Df_1(a)) \cap \ker(Df_2(a)) = \operatorname{span}\{\gamma'(t)\}. \qquad \square$$

5. Curves, surfaces, and manifolds

In the previous sections we talked about tangents to curves and surfaces. However, we haven't ever precisely defined what a curve or surface is. For the curious, the definitions are here. The main result (i.e. that ∇g is orthogonal to level sets, and that $\ker(Dg)$ is the tangent space) is still true in arbitrary dimensions.

DEFINITION 5.1. We say $\Gamma \subseteq \mathbb{R}^n$ is a (differentiable) *curve* if for every $a \in \Gamma$ there exists domains $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}$ and a differentiable function $\varphi : V \to U$ such that $D\varphi \neq 0$ in V and $U \cap \Gamma = \varphi(V)$.

REMARK 5.2. Many authors insist $V = (0, 1)$ or $V = \mathbb{R}$. This is equivalent to what we have.

EXAMPLE 5.3. If $f : \mathbb{R} \to \mathbb{R}^n$ is a differentiable function, then the graph $\Gamma \subseteq \mathbb{R}^{n+1}$ defined by $\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\}$ is a differentiable curve.

PROPOSITION 5.4. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is differentiable, $c \in \mathbb{R}^n$ and $\Gamma = \{x \in \mathbb{R}^{n+1} \mid f(x) = c\}$ be the level set of f. If at every point in Γ, the matrix Df has rank n then Γ is a curve.*

PROOF. Let $a \in \Gamma$. Since $\operatorname{rank}(Df_a) = d$, there must be d linearly independent columns. For simplicity assume these are the first d ones. The implicit function theorem applies and guarantees that the equation $f(x) = c$ can be solved for $x_1, \ldots, x_n$, and each x_i can be expressed as a differentiable function of x_{n+1} (close to a). That is, there exist open sets $U' \subseteq \mathbb{R}^n$, $V' \subseteq \mathbb{R}$ and a differentiable function g such that $a \in U' \times V'$ and $\Gamma \cap (U' \times V') = \{(g(x_{n+1}), x_{n+1}) \mid x_{n+1} \in V'\}$. $\square$

Surfaces in higher dimensions are defined similarly.

DEFINITION 5.5. We say $\Sigma \subseteq \mathbb{R}^n$ is a (differentiable) *surface* if for every $a \in \Sigma$ there exists domains $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^2$ and a differentiable function $\varphi : V \to U$ such that $\operatorname{rank}(D\varphi) = 2$ at every point in V and $U \cap \Sigma = \varphi(V)$.

The difference from a curve is that now $V \subseteq \mathbb{R}^2$ and not $\mathbb{R}$.

DEFINITION 5.6. We say $M \subseteq \mathbb{R}^n$ is a d-dimensional (differentiable) *manifold* if for every $a \in M$ there exists domains $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$ and a differentiable function $\varphi : V \to U$ such that $\operatorname{rank}(D\varphi) = d$ at every point in V and $U \cap M = \varphi(V)$.

REMARK 5.7. For $d = 1$ this is just a curve, and for $d = 2$ this is a surface.

REMARK 5.8. If $d = 1$ and Γ is a connected, then there exists an interval U and an injective differentiable function $\gamma : U \to \mathbb{R}^n$ such that $D\gamma \neq 0$ on U and $\gamma(U) = \Gamma$. If $d > 1$ this is no longer true: even though near every point the surface is a differentiable image of a rectangle, the entire surface need not be one.

As before d-dimensional manifolds can be obtained as level sets of functions $f : \mathbb{R}^{n+d} \to \mathbb{R}^d$ provided we have $\operatorname{rank}(Df) = d$ on the entire level set.

PROPOSITION 5.9. *Let $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ is differentiable, $c \in \mathbb{R}^n$ and $\Gamma = \{x \in \mathbb{R}^{n+1} \mid f(x) = c\}$ be the level set of f. If at every point in Γ, the matrix Df has rank d then Γ is a d-dimensional manifold.*

The results from the previous section about tangent spaces of implicitly defined manifolds generalize naturally in this context.

DEFINITION 5.10. Let $U \subseteq \mathbb{R}^d$, $f : U \to R$ be a differentiable function, and $M = \{(x, f(x)) \in \mathbb{R}^{d+1} \mid x \in U\}$ be the graph of f. (Note M is a d-dimensional manifold in $\mathbb{R}^{d+1}$.) Let $(a, f(a)) \in M$.

- The *tangent "plane"* at the point $(a, f(a))$ is defined by
$$\{(x, y) \in \mathbb{R}^{d+1} \mid y = f(a) + Df_a(x - a)\}$$
- The *tangent space* at the point $(a, f(a))$ (denoted by $TM_{(a,f(a))}$) is the subspace defined by
$$TM_{(a,f(a))} = \{(x, y) \in \mathbb{R}^{d+1} \mid y = Df_a x\}.$$

REMARK 5.11. When $d = 2$ the tangent plane is really a plane. For $d = 1$ it is a line (the tangent line), and for other values it is a d-dimensional hyper-plane.

PROPOSITION 5.12. *Suppose $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ is differentiable, and the level set $\Gamma = \{x \mid f(x) = c\}$ is a d-dimensional manifold. Suppose further that Df_a has rank n for all $a \in \Gamma$. Then the tangent space at a is precisely the kernel of Df_a, and the vectors $\nabla f_1, \ldots \nabla f_n$ are n linearly independent vectors that are normal to the tangent space.*

6. Constrained optimization.

Consider an implicitly defined surface $S = \{g = c\}$, for some $g : \mathbb{R}^3 \to \mathbb{R}$. Our aim is to maximise or minimise a function f *on this surface*.

DEFINITION 6.1. We say a function f attains a local maximum at a on the surface S, if there exists $\varepsilon > 0$ such that $|x - a| < \varepsilon$ and $x \in S$ imply $f(a) \geqslant f(x)$.

REMARK 6.2. This is sometimes called constrained local maximum, or local maximum subject to the constraint $g = c$.

PROPOSITION 6.3. *Suppose $\nabla g \neq 0$ at every point on S. If f attains a constrained local maximum (or minimum) at a on the surface S, then $\exists \lambda \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a)$.*

INTUITION. If $\nabla f(a) \neq 0$, then $S' \stackrel{\text{def}}{=} \{f = f(a)\}$ is a surface. If f attains a constrained maximum at a then S' must be tangent to S at the point a. This forces $\nabla f(a)$ and $\nabla g(a)$ to be parallel. □

PROPOSITION 6.4 (Multiple constraints). *Let $f, g_1, \ldots, g_n : \mathbb{R}^d \to \mathbb{R}$ be $: \mathbb{R}^d \to \mathbb{R}$ be differentiable. If f attains a local maximum (or minimum) at a subject to the constraints $g_1 = c_1$, $g_2 = c_2$, $\ldots g_n = c_n$, and $\nabla g_1(a), \ldots, \nabla g_n(a)$ are linearly independent, then $\exists \lambda_1, \ldots \lambda_n \in \mathbb{R}$ such that $\nabla f(a) = \sum_1^n \lambda_i \nabla g_i(a)$.*

PROOF IN A SPECIAL CASE. Let's assume $n = 1$ and $d = 3$. The proof in this case generalizes without much difficulty to the general case, but the notation is much more cumbersome. For notational convenience we will also write $g = g_1$.

By assumption $\nabla g(a) \neq 0$ and hence $\partial_i g(a) \neq 0$ for some i. For notational simplicity, we assume $i = 3$. Now, by the implicit function theorem there exists a function h defined in a small neighborhood of (a_1, a_2) such that for all x close to a, we have $g(x) = c$ if and only if $x_3 = h(x_1, x_2)$. In particular, this means that close to a, all points on the surface $\{g = c\}$ are of the form $(x_1, x_2, h(x_1, x_2))$. Thus, f attains a constrained local extremum at a if and only if the function $(x_1, x_2) \mapsto f(x_1, x_2, h(x_1, x_2))$ attains an *unconstrained* local extremum at (a_1, a_2).

Define

$$\varphi(x_1, x_2) \stackrel{\text{def}}{=} f(x_1, x_2, h(x_1, x_2)) .$$

By the above argument we know that φ attains an *unconstrained* local extremum at (a_1, a_2). Now we know that unconstrained local extrema are only attained at critical points. Thus to finish the proof, all we need to do is compute the critical points of φ and check that it yields $\nabla f(a) = \lambda \nabla g(a)$ for some $\lambda \in \mathbb{R}$.

To do this, define

$$\psi(x_1, x_2) = (x_1, x_2, h(x_1, x_2)) ,$$

and observe $\varphi = f \circ \psi$. By the chain rule, we know $D\varphi = Df_\psi D\psi$. Transposing this, and using the fact that $D\varphi_{(a_1,a_2)} = 0$, gives

$$(D\varphi_a)^T = (D\psi_{(a_1,a_2)})^T \nabla f(a) = 0 \quad \Longrightarrow \quad \nabla f(a) \in \ker\big((D\psi_{(a_1,a_2)})^T\big) . \tag{6.1}$$

Moreover, by construction of h, we know $g \circ \psi(x_1, x_2) = c$, and hence $D(g \circ \psi) = 0$. Transposing this and evaluating at (a_1, a_2) gives

$$(D\psi_{(a_1,a_2)})^T \nabla g(a) = 0 \quad \Longrightarrow \quad \nabla g(a) \in \ker\big(D\psi^T_{(a_1,a_2)}\big) . \tag{6.2}$$

Observe now that

$$(D\psi)^T = \begin{pmatrix} 1 & 0 & \partial_1 h \\ 0 & 1 & \partial_2 h \end{pmatrix} ,$$

and hence $\operatorname{rank}(D\psi)^T = 2$. Thus by the rank nullity theorem, $\dim \ker((D\psi)^T) = 1$. Since $\nabla g(a) \neq 0$, by assumption and $\nabla g(a) \in \ker(D\psi_{(a_1,a_2)})$ (from (6.2)), we must have $\ker((D\psi_{(a_1,a_2)})^T) = \operatorname{span}\{\nabla g(a)\}$. But since $\nabla f(a) \in \ker((D\psi_{(a_1,a_2)})^T)$ (by (6.1)), this implies there exists $\lambda \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a)$. □

REMARK 6.5 (Proof when $n \neq 1$). The main difference in this case is to note that $\nabla g_1, \ldots, \nabla g_n$ form a basis of $\ker((D\psi)^T)$, and use this to conclude the existence of $\lambda_1, \ldots, \lambda_n$.

To explicitly find constrained local maxima in $\mathbb{R}^d$ with n constraints we do the following:

- Simultaneously solve the system of equations
$$\nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots \lambda_n \nabla g_n(x)$$
$$g_1(x) = c_1,$$
$$\cdots$$
$$g_n(x) = c_n.$$
- The unknowns are the d-coordinates of x, and the Lagrange multipliers $\lambda_1, \ldots, \lambda_n$. This is $n + d$ variables.
- The first equation above is a vector equation where both sides have d coordinates. The remaining are scalar equations. So the above system is a system of $n + d$ equations with $n + d$ variables.
- The typical situation will yield a finite number of solutions.
- There is a test involving the *bordered Hessian* for whether these points are constrained local minima / maxima or neither. These are quite complicated, and are usually more trouble than they are worth, so one usually uses some ad-hoc method to decide whether the solution you found is a local maximum or not.

EXAMPLE 6.6. Find necessary conditions for $f(x, y) = y$ to attain a local maxima/minima of subject to the constraint $y = g(x)$.

Of course, from one variable calculus, we know that the local maxima / minima must occur at points where $g' = 0$. Let's revisit it using the constrained optimization technique above.

SOLUTION. Note our constraint is of the form $y - g(x) = 0$. So at a local maximum we must have
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \nabla f = \lambda \nabla (y - g(x)) = \begin{pmatrix} -g'(x) \\ 1 \end{pmatrix} \quad \text{and} \quad y = g(x).$$
This forces $\lambda = 1$ and hence $g'(x) = 0$, as expected. □

EXAMPLE 6.7. Maximise xy subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION. At a local maximum,
$$\begin{pmatrix} y \\ x \end{pmatrix} = \nabla(xy) = \lambda \nabla \Big(\frac{x^2}{a^2} + \frac{y^2}{b^2} \Big) = \lambda \begin{pmatrix} 2x/a^2 \\ 2y/b^2. \end{pmatrix}$$
which forces $y^2 = x^2 b^2 / a^2$. Substituting this in the constraint gives $x = \pm a/\sqrt{2}$ and $y = \pm b/\sqrt{2}$. This gives four possibilities for xy to attain a maximum. Directly checking shows that the points $(a/\sqrt{2}, b/\sqrt{2})$ and $(-a/\sqrt{2}, -b/\sqrt{2})$ both correspond to a local maximum, and the maximum value is $ab/2$. □

PROPOSITION 6.8 (Cauchy-Schwartz). *If $x, y \in \mathbb{R}^n$ then $|x \cdot y| \leqslant |x||y|$.*

PROOF. Maximise $x \cdot y$ subject to the constraint $|x| = a$ and $|y| = b$. □

PROPOSITION 6.9 (Inequality of the means). *If* $x_i \geqslant 0$, *then*

$$\frac{1}{n}\sum_1^n x_i \geqslant \Big(\prod_1^n x_i\Big)^{1/n}.$$

PROPOSITION 6.10 (Young's inequality). *If* $p, q > 1$ *and* $1/p + 1/q = 1$ *then*

$$|xy| \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

CHAPTER 4

Multiple Integrals

1. Double Integrals

Let $R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ be a rectangle, and $f : R \to \mathbb{R}$ be continuous. Let $P = \{x_0, \dots, x_M, y_0, \dots, y_M\}$ where $a = x_0 < x_1 < \cdots < x_M = b$ and $c = y_0 < y_1 < \cdots < y_M = d$. The set P determines a partition of R into a grid of (non-overlapping) rectangles $R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $0 \leqslant i < M$ and $0 \leqslant j < N$. Given P, choose a collection of points $\Xi = \{\xi_{i,j}\}$ so that $\xi_{i,j} \in R_{i,j}$ for all i, j.

Definition 1.1. The *Riemann sum* of f with respect to the partition P and points Ξ is defined by

$$\mathcal{R}(f, P, \Xi) \stackrel{\text{def}}{=} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f(\xi_{i,j}) \operatorname{area}(R_{i,j}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f(\xi_{i,j})(x_{i+1} - x_i)(y_{j+1} - y_j)$$

Definition 1.2. The *mesh size* of a partition P is defined by

$$\|P\| = \max\{x_{i+1} - x_i \mid 0 \leqslant i < M\} \cup \{y_{j+1} - y_j \mid 0 \leqslant j \leqslant N\}.$$

Definition 1.3. The *Riemann integral* of f over the rectangle R is defined by

$$\int_R f(x,y)\, dx\, dy \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \mathcal{R}(f, P, \Xi),$$

provided the limit exists and is independent of the choice of the points Ξ. A function is said to be *Riemann integrable* over R if the Riemann integral exists and is finite.

Remark 1.4. A few other popular notation conventions used to denote the integral are

$$\iint_R f\, dA, \quad \iint_R f\, dx\, dy, \quad \iint_R f\, dx_1\, dx_2, \quad \text{and} \quad \iint_R f.$$

Remark 1.5. The double integral represents the volume of the region under the graph of f. Alternately, if $f(x,y)$ is the density of a planar body at point (x,y), the double integral is the total mass.

Theorem 1.6. *Any bounded continuous function is Riemann integrable on a bounded rectangle.*

Remark 1.7. Most bounded functions we will encounter will be Riemann integrable. Bounded functions with reasonable discontinuities (e.g. finitely many jumps) are usually Riemann integrable on bounded rectangle. An example of a "badly discontinuous" function that is not Riemann integrable is the function $f(x,y) = 1$ if $x, y \in \mathbb{Q}$ and 0 otherwise.

Now suppose $U \subseteq \mathbb{R}^2$ is an nice bounded[1] domain, and $f : U \to \mathbb{R}$ is a function. Find a bounded rectangle $R \supseteq U$, and as before let P be a partition of R into a grid of rectangles. Now we define the Riemann sum by only summing over all rectangles $R_{i,j}$ that are completely contained inside U. Explicitly, let

$$\chi_{i,j} = \begin{cases} 1 & R_{i,j} \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

and define

$$\mathcal{R}(f, P, \Xi, U) \stackrel{\text{def}}{=} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \chi_{i,j} f(\xi_{i,j})(x_{i+1} - x_i)(y_{j+1} - y_j).$$

DEFINITION 1.8. The *Riemann integral* of f over the *domain* U is defined by

$$\int_U f(x, y)\, dx\, dy \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \mathcal{R}(f, P, \Xi, U),$$

provided the limit exists and is independent of the choice of the points Ξ. A function is said to be *Riemann integrable* over R if the Riemann integral exists and is finite.

THEOREM 1.9. *Any bounded continuous function is Riemann integrable on a bounded region.*

REMARK 1.10. As before, most reasonable *bounded* functions we will encounter will be Riemann integrable.

To deal with unbounded functions over unbounded domains, we use a limiting process.

DEFINITION 1.11. Let $U \subseteq \mathbb{R}^2$ be a domain (which is not necessarily bounded) and $f : U \to \mathbb{R}$ be a (not necessarily bounded) function. We say f is integrable if

$$\lim_{R \to \infty} \int_{U \cap B(0,R)} |f|\, dA$$

exists and is finite.

PROPOSITION 1.12. *If f is integrable on the domain U, then*

$$\lim_{R \to \infty} \int_{U \cap B(0,R)} f\, dA$$

exists and is finite.

REMARK 1.13. If f is integrable, then the above limit is independent of how you expand your domain. Namely, you can take the limit of the integral over $U \cap [-R, R]^2$ instead, and you will still get the same answer.

DEFINITION 1.14. If f is integrable we define

$$\int_U f\, dx\, dy = \lim_{R \to \infty} \int_{U \cap B(0,R)} f\, dA$$

[1]We will subsequently always assume U is "nice". Namely, U is open, connected and the boundary of U is a piecewise differentiable curve. More precisely, we need to assume that the "area" occupied by the boundary of U is 0. While you might suspect this should be true for all open sets, it isn't! There exist open sets of *finite area* whose boundary occupies an infinite area!

2. Iterated integrals and Fubini's theorem

Let $U \subseteq \mathbb{R}^2$ be a domain.

DEFINITION 2.1. For $x \in \mathbb{R}$, define

$$S_x U = \{y \mid (x,y) \in U\} \quad \text{and} \quad T_y U = \{x \mid (x,y) \in U\}$$

EXAMPLE 2.2. If $U = [a,b] \times [c,d]$ then

$$S_x U = \begin{cases} [c,d] & x \in [a,b] \\ \emptyset & x \notin [a,b] \end{cases} \quad \text{and} \quad T_y U = \begin{cases} [a,b] & y \in [c,d] \\ \emptyset & y \notin [c,d]. \end{cases}$$

For domains we will consider, $S_x U$ and $T_y U$ will typically be an interval (or a finite union of intervals).

DEFINITION 2.3. Given a function $f : U \to \mathbb{R}$, we define the two *iterated* integrals by

$$\int_{x\in\mathbb{R}} \Big(\int_{y \in S_x U} f(x,y)\, dy \Big)\, dx \quad \text{and} \quad \int_{y\in\mathbb{R}} \Big(\int_{x \in T_y U} f(x,y)\, dx \Big)\, dy,$$

with the convention that an integral over the empty set is 0. (We included the parenthesis above for clarity; and will drop them as we become more familiar with iterated integrals.)

Suppose $f(x,y)$ represents the density of a planar body at point (x,y). For any $x \in \mathbb{R}$,

$$\int_{y \in S_x U} f(x,y)\, dy$$

represents the mass of the body contained in the vertical line through the point $(x,0)$. It's only natural to expect that if we integrate this with respect to y, we will get the total mass, which is the double integral. By the same argument, we should get the same answer if we had sliced it horizontally first and then vertically. Consequently, we expect both iterated integrals to be equal to the double integral. This is true, under a finiteness assumption.

THEOREM 2.4 (Fubini's theorem). *Suppose $f : U \to \mathbb{R}$ is a function such that either*

(2.1)
$$\int_{x\in\mathbb{R}} \Big(\int_{y \in S_x U} |f(x,y)|\, dy \Big)\, dx < \infty \quad \text{or} \quad \int_{y\in\mathbb{R}} \Big(\int_{x \in T_y U} |f(x,y)|\, dx \Big)\, dy < \infty,$$

then f is integrable over U and

$$\int_U f\, dA = \int_{x\in\mathbb{R}} \Big(\int_{y \in S_x U} f(x,y)\, dy \Big)\, dx = \int_{y\in\mathbb{R}} \Big(\int_{x \in T_y U} f(x,y)\, dx \Big)\, dy.$$

Without the assumption (2.1) the iterated integrals need not be equal, even though both may exist and be finite.

EXAMPLE 2.5. Define

$$f(x,y) = -\partial_x \partial_y \tan^{-1}\big(\frac{y}{x}\big) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Then

$$\int_{x=0}^{1}\int_{y=0}^{1} f(x,y)\,dy\,dx = \frac{\pi}{4} \quad \text{and} \quad \int_{y=0}^{1}\int_{x=0}^{1} f(x,y)\,dx\,dy = -\frac{\pi}{4}$$

EXAMPLE 2.6. Let $f(x,y) = (x-y)/(x+y)^3$ if $x, y > 0$ and 0 otherwise, and $U = (0,1)^2$. The iterated integrals of f over U both exist, but are not equal.

EXAMPLE 2.7. Define

$$f(x,y) = \begin{cases} 1 & y \in (x, x+1) \text{ and } x \geqslant 0 \\ -1 & y \in (x-1, x) \text{ and } x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the iterated integrals of f both exist and are not equal.

EXAMPLE 2.8. Compute the area of a parallelogram.

EXAMPLE 2.9. Let U be the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$ and $f(x,y) = e^{-y^2}$. Compute $\int_U f\,dA$.

3. Triple integrals

Triple integrals are just like double integrals, except we integrate over regions in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. Let C be the cuboid $C = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and $f : C \to \mathbb{R}$ be a function. As before, define the Riemann sum

$$\mathcal{R}(f, P, \Xi) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} f(\xi_{i,j,k})(x_{i+1} - x_i)(y_{i+1} - y_i)(z_{i+1} - z_i)$$

define the Riemann integral of f by taking the limit of Riemann sums:

$$\int_U f\,dV = \lim_{\|P\| \to 0} \mathcal{R}(f, P, \Xi).$$

Here we use dV (or sometimes $dx\,dy\,dz$ to denote that the integral is a volume (or triple) integral.

When dealing with unbounded functions over unbounded domains,[2] we use the same limiting procedure. If

$$\lim_{R\to\infty} \int_{U \cap B(0,R)} |f|\,dV$$

exists and is finite then we define

$$\int_U f\,dV = \lim_{R\to\infty} \int_{U\cap B(0,R)} f\,dV$$

We can break a volume integral into three iterated integrals, and Fubini's theorem is still true. Rather than restate everything, we do a few examples.

EXAMPLE 3.1. Let $U = B(0,R) \subseteq \mathbb{R}^3$. Compute $\int_U 1\,dV$ and derive a formula for the volume.

[2] As before, we make the "niceness" assumption that the boundary of U is a differentiable surface.

SOLUTION. Note

$$\begin{aligned}\int_U 1\,dV &= \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{z=-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx\\ &= 2\int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}dy\,dx\\ &= 2\int_{x=-R}^{R}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}(R^2-x^2)\cos^2\theta\,d\theta\,dx = \pi\Big[R^2x-\frac{x^3}{3}\Big]_{-R}^{R} = \frac{4}{3}\pi R^3 \quad \square\end{aligned}$$

EXAMPLE 3.2. Compute the volume of a cylinder.

EXAMPLE 3.3. Compute the volume of the pyramid bounded by the intersection of the planes $x+y+z=1$ and the three coordinate planes.

4. Coordinate transformations

Let f be a function of (x,y) defined on the domain U. Let

$$\begin{pmatrix}x\\y\end{pmatrix} = \varphi(u,v)$$

for some coordinate change function $\varphi : U \to V$. We claim

$$\int_U f(x,y)\,dx\,dy = \int_V f\circ\varphi(u,v)\,|\det D\varphi|\,du\,dv. \tag{4.1}$$

Namely, the integral remains unchanged if we replace U with V, make the substitution $(x,y)=\varphi(u,v)$, and replace $dx\,dy$ with $|\det D\varphi|\,du\,dv$. This is the change of variables theorem.

THEOREM 4.1 (Change of Variables). *Let $U, V \subseteq \mathbb{R}^2$ be two domains, $\varphi : V \to U$ be a coordinate change map. If $f : U \to \mathbb{R}$ is integrable, then*

$$\int_U f\,dA = \int_V f\circ\varphi\,|\det D\varphi|\,dA.$$

REMARK 4.2. Recall, a coordinate change transformation is a function φ which is *bijective* and differentiable for which $D\varphi$ is invertible at all points in the domain.

REMARK 4.3. Theorem 4.1 is still true under the following relaxed assumptions on φ: Let $U' \subseteq U$ and $V' \subseteq V$ be obtained by removing *finitely many* differentiable curves or points from U and V respectively. If $\varphi : V' \to U'$ is differentiable, bijective and $D\varphi$ is invertible on all of V', then Theorem 4.1 still holds. In typical situations $D\varphi$ will be invertible except for a few isolated points, so checking this won't be the bottle neck. Bijectivity, however, can fail in subtle ways and needs to be explicitly checked.

REMARK 4.4. The same result is true for triple integrals. Further, for the relaxed assumptions on φ, U' and V' can be obtained by additionally removing finitely many differentiable surfaces from U and V respectively.

The intuition behind Theorem 4.1 is as follows: First if R is any rectangle, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then we know that

$$\text{area}(R) = |\det T|$$

Now, divide V into many small non-overlapping regions $R_{i,j}$ and set $R'_{i,j} = \varphi(R_{i,j})$. Since φ is bijective, the regions $R'_{i,j}$ must also be non-overlapping and cover all of U. If $R_{i,j}$ are small enough, φ can be approximated by an affine function (using $D\varphi$) and hence we expect

$$\operatorname{area}(R'_{i,j}) \approx \operatorname{area}(R_{i,j})|\det D\varphi_{\xi_{i,j}}|$$

where $\xi_{i,j} \in R_{i,j}$. Multiplying by f, summing and taking limits suggests the formula (4.1) as claimed.

EXAMPLE 4.5 (Polar Coordinates). Let $(x, y) = \varphi(r, \theta) = (r\cos\theta, r\sin\theta)$. Then $|\det D\varphi| = r$ and hence when transforming area integrals to polar coordinates, we replace $dx\,dy$ with $r\,dr\,d\theta$.

EXAMPLE 4.6. Compute the area of a circle of radius r.

EXAMPLE 4.7. Show $\int_{x^2+y^2>1} \frac{1}{(x^2+y^2)^{p/2}}\,dA < \infty$ if and only if $p > 2$.

EXAMPLE 4.8. Show $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

EXAMPLE 4.9 (Spherical coordinates). Let

$$(x, y, z) = \varphi(r, \theta, \phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi).$$

From homework we know $|\det D\varphi| = r^2\sin\phi$, and hence when transforming volume integrals into spherical coordinates we replace dV with $r^2\sin\phi\,dr\,d\phi\,d\theta$.

EXAMPLE 4.10. Compute the volume of a solid sphere.

EXAMPLE 4.11. Show $\int_{x^2+y^2+z^2>1} \frac{1}{(x^2+y^2+z^2)^{p/2}}\,dV < \infty$ if and only if $p > 3$.

EXAMPLE 4.12. Compute the volume of a cylinder.

EXAMPLE 4.13. Compute the volume of a cone with an oddly shaped base.

REMARK 4.14 (Failure of bijectivity). Here is an example where the failure of bijectivity gives a "weird" result. Consider the coordinate change

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(u, v) = \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix},$$

for which $|\det\varphi| = 4(u^2 + v^2) = 4(x^2 + y^2)^{1/2}$. By making x, y arbitrarily large (or small) we can do the same for u and v. So one might hastily write

$$\int_{\mathbb{R}^2} f(x, y)\,dx\,dy = 4\int_{\mathbb{R}^2} f\circ\varphi(u, v)\,(u^2 + v^2)\,du\,dv.$$

This doesn't yield the right answer though! Indeed, choosing

$$f(x, y) = \frac{\exp(-\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} = \frac{\exp(-(u^2 + v^2))}{u^2 + v^2}$$

yields

$$\int_{\mathbb{R}^2} f(x, y)\,dx\,dy = 2\pi \quad \text{and} \quad 4\int_{\mathbb{R}^2} f\circ\varphi(u, v)\,(u^2 + v^2)\,du\,dv = 4\pi.$$

What failed here is exactly bijectivity. You can explicitly solve and check that for every $(x, y) \neq (0, 0)$ there exist exactly two values of (u, v) for which $\varphi(u, v) = (x, y)$. Indeed, choosing

$$H = \{(u, v) \mid u > 0\} \quad \text{and} V = \{(x, y) \mid y \neq 0 \text{ or } x > 0\},$$

we see now that $\varphi : H \to V$ is a coordinate transformation. Of course V is simply $\mathbb{R}^2$ with a half line removed, but H is "half" of $\mathbb{R}^2$. With these domains, we of course have the identity

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = 4\int_H f\circ\varphi(u,v)\,(u^2+v^2)\,du\,dv,$$

and you can explicitly verify this for the specific choice of f above.

CHAPTER 5

Line Integrals

1. Line integrals

DEFINITION 1.1. If a force F acting on a body produces an instantaneous displacement v, then the work done by the force is $F \cdot v$.

Let $\Gamma \subseteq \mathbb{R}^3$ be a curve, with a given direction of traversal, and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a (vector) function. Here F represents the force that acts on a body and pushes it along the curve Γ. The work done by the force can be approximated by

$$W = \sum_{i=0}^{N-1} F(x_i) \cdot (x_{i+1} - x_i)$$

where $x_0, x_1, \ldots, x_{N-1}$ are N points on Γ, *chosen along the direction of traversal*. The limit as the largest distance between neighbours approaches 0 is defined to be the line integral.

DEFINITION 1.2. Let $\Gamma \subseteq \mathbb{R}^d$ be a curve (with a given direction of traversal), and $F : \Gamma \to \mathbb{R}^d$ be a (vector) function. The *line integral* of F over Γ is defined to be

$$\int_\Gamma F \cdot d\ell = \lim_{\|P\| \to 0} \sum_{i=0}^{N-1} F(x_i) \cdot (x_{i+1} - x_i).$$

Here $P = \{x_0, x_1, \ldots, x_{N-1}\}$, the points x_i are chosen along the direction of traversal, and $\|P\| = \max|x_{i+1} - x_i|$.

REMARK 1.3. If $F = (F_1, \ldots, F_d)^T$, where $F_i : \Gamma \to \mathbb{R}$ are functions, then one often writes the line integral in the *differential form* notation as

$$\int_\Gamma F \cdot d\ell = \int_\Gamma F_1\, dx_1 + \cdots + F_d\, dx_d = \int_\Gamma \sum_{i=1}^{d} F_i\, dx_i.$$

PROPOSITION 1.4. *If $\gamma : [a, b] \to \mathbb{R}^d$ is a parametrization of Γ (in the direction of traversal), then*

$$\int_\Gamma F \cdot d\ell = \int_a^b F \circ \gamma(t) \cdot \gamma'(t)\, dt \tag{1.1}$$

In the differential form notation (when $d = 2$) say

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $f, g : \Gamma \to \mathbb{R}$ are functions. Then Proposition 1.4 says

$$\int_\Gamma F \cdot d\ell = \int_\Gamma f\, dx + g\, dy = \int_\Gamma \big(f(x(t), y(t))\, x'(t) + g(x(t), y(t))\, y'(t)\big)\, dt$$

REMARK 1.5. Sometimes (1.1) is used as the definition of the line integral. In this case, one needs to verify that this definition is *independent* of the parametrization. Since this is a good exercise, we'll do it anyway a little later.

EXAMPLE 1.6. Suppose a body of mass M is placed at the origin. The force experienced by a body of mass m at the point $x \in \mathbb{R}^3$ is given by $F(x) = \frac{-GMx}{|x|^3}$, where G is the *gravitational constant*. Compute the work done when the body is moved from a to b along a straight line.

SOLUTION. Let Γ be the straight line joining a and b. Clearly $\gamma : [0,1] \to \Gamma$ defined by $\gamma(t) = a + t(b-a)$ is a parametrization of Γ. Now

$$W = \int_\Gamma F \cdot d\ell = -GMm \int_0^1 \frac{\gamma(t)}{|\gamma(t)|^3} \cdot \gamma'(t)\, dt = \frac{GMm}{|b|} - \frac{GMm}{|a|}. \qquad \square$$

REMARK 1.7. If the line joining through a and b passes through the origin, then some care has to be taken when doing the above computation. We will see later that gravity is a *conservative force*, and that the above line integral only depends on the endpoints and not the actual path taken.

2. Parametrization invariance and arc length

So far we have always insisted all curves and parametrizations are differentiable or C^1. We now relax this requirement and subsequently only assume that all curves (and parametrizations) are *piecewise differentiable*, or *piecewise* C^1.

DEFINITION 2.1. A function $f : [a,b] \to \mathbb{R}^d$ is called *piecewise* C^1 if there exists a finite set $F \subseteq [a,b]$ such that f is C^1 on $[a,b] - F$, and further both left and right limits of f and f' exist at all points in F.

DEFINITION 2.2. A (connected) curve Γ is *piecewise* C^1 if it has a parametrization which is continuous *and* piecewise C^1.

REMARK 2.3. A piecewise C^1 function need not be continuous. But curves are always assumed to be at least continuous; so for notational convenience, we define a piecewise C^1 curve to be one which has a parametrization which is both continuous and piecewise C^1.

EXAMPLE 2.4. The boundary of a square is a piecewise C^1 curve, but not a differentiable curve.

PROPOSITION 2.5 (Parametrization invariance). *If $\gamma_1 : [a_1, b_1] \to \Gamma$ and $\gamma_2 : [a_2, b_2] \to \Gamma$ are two parametrizations of Γ that traverse it in the same direction, then*

$$\int_{a_1}^{b_1} F \circ \gamma_1(t) \cdot \gamma_1'(t)\, dt = \int_{a_2}^{b_2} F \circ \gamma_2(t) \cdot \gamma_2'(t)\, dt.$$

PROOF. Let $\varphi : [a_1, b_1] \to [a_2, b_2]$ be defined by $\varphi = \gamma_2^{-1} \circ \gamma_1$. Since γ_1 and γ_2 traverse the curve in the same direction, φ must be increasing. One can also show (using the inverse function theorem) that φ is continuous and piecewise C^1. Now

$$\int_{a_2}^{b_2} F \circ \gamma_2(t) \cdot \gamma_2'(t)\, dt = \int_{a_2}^{b_2} F(\gamma_1(\varphi(t))) \cdot \gamma_1'(\varphi(t))\varphi'(t)\, dt.$$

Making the substitution $s = \varphi(t)$ finishes the proof. $\square$

DEFINITION 2.6. If $\Gamma \subseteq \mathbb{R}^d$ is a piecewise C^1 curve, then

$$\operatorname{arc\,len}(\Gamma) = \lim_{\|P\|\to 0} \sum_{i=0}^{N} |x_{i+1} - x_i|,$$

where as before $P = \{x_0, \dots, x_{N-1}\}$. More generally, if $f : \Gamma \to \mathbb{R}$ is any scalar function, we define[1]

$$\int_\Gamma f\,|d\ell| \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{i=0}^{N} f(x_i)\,|x_{i+1} - x_i|,$$

The *arc length* of a curve can be computed by taking the line integral of the unit tangent vector.

PROPOSITION 2.7. *Let $\Gamma \subseteq \mathbb{R}^d$ be a piecewise C^1 curve, $\gamma : [a, b] \to \mathbb{R}$ be any parametrization (in the given direction of traversal), $f : \Gamma \to \mathbb{R}$ be a (scalar) function, and $\tau : \Gamma \to \mathbb{R}^d$ is the unit tangent vector (i.e. $|\tau| \equiv 1$ and τ is always tangent to Γ) along the direction of traversal. Then*

$$\int_\Gamma f\,|d\ell| = \int_\Gamma f\tau \cdot d\ell = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt,$$

and consequently

$$\operatorname{arc\,len}(\Gamma) = \int_\Gamma 1\,|d\ell| = \int_a^b |\gamma'(t)|\,dt.$$

EXAMPLE 2.8. Compute the circumference of a circle of radius r.

REMARK 2.9. A very useful way to describe curves is to parametrize them by arc length. Namely, let $\gamma(s) \in \Gamma$ be the unique point so that the portion of Γ traversed up to the point $\gamma(s)$ has arc length exactly s.

3. The fundamental theorem

THEOREM 3.1 (Fundamental theorem for line integrals). *Suppose $U \subseteq \mathbb{R}^d$ is a domain, $\varphi : U \to \mathbb{R}$ is C^1 and $\Gamma \subseteq \mathbb{R}^d$ is any differentiable curve that starts at a, ends at b and is completely contained in U. Then*

$$\int_\Gamma \nabla\varphi \cdot d\ell = \varphi(b) - \varphi(a).$$

PROOF. Let $\gamma : [0, 1] \to \Gamma$ be a parametrization of Γ. Note

$$\int_\Gamma \nabla\varphi \cdot d\ell = \int_0^1 \nabla\varphi(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^1 \frac{d}{dt}\varphi(\gamma(t))\,dt = \varphi(b) - \varphi(a). \qquad \square$$

DEFINITION 3.2. A *closed curve* is a curve that starts and ends at the same point. A *simple closed curve* is a closed curve that never crosses itself. (More precisely, a simple closed curve is a compact 1-dimensional manifold with no boundary.)

If Γ is a closed curve, then line integrals over Γ are denoted by

$$\oint_\Gamma F \cdot d\ell.$$

[1] Unfortunately $\int_\Gamma f\,|d\ell|$ is also called the line integral. To avoid confusion, we will call this the *line integral with respect to arc-length* instead.

COROLLARY 3.3. *If $\Gamma \subseteq \mathbb{R}^d$ is a closed curve, and $\varphi : \Gamma \to \mathbb{R}$ is C^1, then*

$$\oint_\Gamma \nabla\varphi \cdot d\ell = 0.$$

DEFINITION 3.4. Let $U \subseteq \mathbb{R}^d$, and $F : U \to \mathbb{R}^d$ be a vector function. We say F is a *conservative force* (or *conservative vector field*) if

$$\oint F \cdot d\ell = 0,$$

for all closed curves Γ which are completely contained inside U.

Clearly if $F = -\nabla V$ for some C^1 function $V : U \to \mathbb{R}$, then F is conservative. The converse is also true provided U is *simply connected*, which we'll return to later.

EXAMPLE 3.5. If φ fails to be C^1 even at one point, the above can fail quite badly. Let $\varphi(x, y) = \tan^{-1}(y/x)$, extended to $\mathbb{R}^2 - \{(x, y) \mid x \leqslant 0\}$ in the usual way. Then

$$\nabla\varphi = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

which is defined on $\mathbb{R}^2 - (0, 0)$. In particular, if $\Gamma = \{(x, y) \mid x^2 + y^2 = 1\}$, then $\nabla\varphi$ is defined on all of Γ. However, you can easily compute

$$\oint_\Gamma \nabla\varphi \cdot d\ell = 2\pi \neq 0.$$

The reason this doesn't contradict the previous corollary is that Corollary 3.3 requires φ itself to be defined on all of Γ, and not just $\nabla\varphi$! This example leads into something called the *winding number* which we will return to later.

4. Greens theorem

THEOREM 4.1 (Greens Theorem). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain whose exterior boundary is a piecewise C^1 curve Γ. If Ω has holes, let $\Gamma_1, \ldots, \Gamma_N$ be the interior boundaries. If $F : \bar{\Omega} \to \mathbb{R}^2$ is C^1, then*

$$\int_\Omega \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA = \oint_\Gamma F \cdot d\ell + \sum_{i=1}^{N} \oint_{\Gamma_i} F \cdot d\ell,$$

where all line integrals above are computed by traversing the exterior boundary counter clockwise*, and every interior boundary* clockwise*.*

REMARK 4.2. A common convention is to denote the *boundary* of Ω by $\partial\Omega$ and write

$$\partial\Omega = \Gamma \cup \big(\bigcup_{i=1}^{N} \Gamma_i\big).$$

Then Theorem 4.1 becomes

$$\int_\Omega \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA = \oint_{\partial\Omega} F \cdot d\ell,$$

where again the exterior boundary is oriented *counter clockwise* and the interior boundaries are all oriented *clockwise*.

REMARK 4.3. In the differential form notation, Greens theorem is stated as

$$\int_\Omega \big(\partial_x Q - \partial_y P\big)\, dA = \int_{\partial\Omega} P\, dx + Q\, dy,$$

$P, Q : \bar{\Omega} \to \mathbb{R}$ are C^1 functions. (We use the same assumptions as before on the domain Ω, and orientations of the line integrals on the boundary.)

REMARK 4.4. Note, Greens theorem requires that Ω is bounded and F (or P and Q) is C^1 on *all* of Ω. If this fails at even one point, Greens theorem need not apply anymore!

PROOF. The full proof is a little cumbersome. But the main idea can be seen by first proving it when Ω is a square, and then applying a coordinate transformation. Indeed, suppose first $\Omega = (0,1)^2$. Then the fundamental theorem of calculus gives

$$\int_\Omega \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA = \int_{y=0}^1 \big(F_2(1,y) - F_2(0,y)\big)\, dy - \int_{x=0}^1 \big(F_1(x,1) - F_1(x,0)\big)\, dx$$

The first integral is the line integral of F on the two vertical sides of the square, and the second one is line integral of F on the two horizontal sides of the square. This proves Theorem 4.1 in the case when Ω is a square.

Now let U be an arbitrary region for which there exists a C^2 coordinate transformation $\varphi : \Omega \to U$, where Ω is the unit square. We assume that φ also maps $\partial\Omega$ to ∂U and *preserves the orientation of the boundaries*. (One can show that this will imply $\det D\varphi > 0$ in U.) Now, using Greens theorem on the square,

$$\oint_{\partial U} F \cdot d\ell = \oint_{\partial\Omega} (D\varphi)^T F \circ \varphi \cdot d\ell = \int_\Omega (\partial_1 G_2 - \partial_2 G_1)\, dA,$$

where

$$G = (D\varphi)^T F \circ \varphi = \sum_{i,j} \partial_i \varphi_j F_j \circ \varphi\, e_i$$

Now we compute using the chain rule

$$\partial_1 G_2 - \partial_2 G_1 = \sum_{i,j} \partial_2\varphi_j\, \partial_i F_j\big|_\varphi\, \partial_1\varphi_i - \partial_1\varphi_j\, \partial_i F_j\big|_\varphi\, \partial_2\varphi_i = \big(\partial_1 F_2 - \partial_2 F_1\big)\circ\varphi\, \det(D\varphi).$$

Thus, by the change of variable theorem,

$$\int_\Omega \big(\partial_1 G_2 - \partial_2 G_1\big)\, dA = \int_\Omega \big(\partial_1 F_2 - \partial_2 F_1\big) \circ \varphi\, \det(D\varphi)\, dA = \int_U \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA,$$

finishing the proof. □

REMARK 4.5. The above strategy will only work if the domain has no holes. In the presence of holes, you can make one or more cuts and then find a coordinate transformation $\varphi : \Omega \to U$ as above. The only difference is now part of the boundary of Ω will be mapped to the cut you just made. The boundary integral over this piece, however, will cancel since it will now be traversed twice in opposite directions.

COROLLARY 4.6. *If $\Omega \subseteq \mathbb{R}^2$ is bounded with a C^1 boundary, then*

$$\text{area}(\Omega) = \frac{1}{2}\int_{\partial\Omega} \big(-y\, dx + x\, dy\big) = \int_{\partial\Omega} -y\, dx = \int_{\partial\Omega} x\, dy$$

REMARK 4.7. A *planimeter* is a measuring instrument used to determine the area of an arbitrary two-dimensional shape. The operational principle of the planimeter can be proved using the previous corollary.

COROLLARY 4.8 (Surveyor's Formula). *Let $P \subseteq \mathbb{R}^2$ be a (not necessarily convex) polygon whose vertices, ordered counter clockwise, are (x_1, y_1), ..., (x_N, y_N). Then*

$$\text{area}(P) = \frac{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_Ny_1 - x_1y_N)}{2}.$$

CHAPTER 6

Surface Integrals

1. Surface integrals

Suppose a curved metal plate (or soap film) lies along the surface $\Sigma \subseteq \mathbb{R}^3$, and $f : \Sigma \to \mathbb{R}$ is the density of the plate. If we divide the surface Σ into many small regions R_i, then the mass of the plate can be approximated by

$$M = \sum_i f(\xi_i) \operatorname{area}(R_i),$$

where $\xi_i \in R_i$ is some point.

DEFINITION 1.1. Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, and $f : \Sigma \to \mathbb{R}$ be a function. Define

$$\int_\Sigma f \, dS = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(\xi_i) \operatorname{area}(R_i),$$

where P is a partition of Σ into the regions $R_1, \ldots, R_N$, and $\|P\|$ is the diameter of the largest region R_i.

REMARK 1.2. Other common notation for the surface integral is

$$\int_\Sigma f \, dS = \iint_\Sigma f \, dS = \int_\Sigma f \, d\sigma = \int_\Sigma f \, dA$$

As with line integrals, we obtained a formula in terms of a parametrization. We follow the same approach for surfaces, but there are a few subtle points that need to be addressed.

DEFINITION 1.3. Let $\Sigma \subseteq \mathbb{R}^3$ be a surface. We say φ is a (C^1) parametrization of Σ if there exists a domain $U \subseteq \mathbb{R}^2$ such that $\varphi : U \to \Sigma$ is C^1, bijective, and $\operatorname{rank}(D\varphi) = 2$ at all points in U.

EXAMPLE 1.4. Let $U \subseteq \mathbb{R}^2$, $f : U \to \mathbb{R}$ is a function, and Σ be the graph of f. Then $\varphi(x, y) = (x, y, f(x, y))$ is a parametrization of Σ.

EXAMPLE 1.5. Define

$$\varphi(\theta, \phi) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

for $\theta \in (-\pi, \pi)$ and $\phi \in (0, \pi)$. Then φ parametrizes the surface

$$\Sigma \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid |x|^2 = 1 \;\&\; x_3 \neq \pm 1\},$$

which is the unit sphere with an arc joining the north and south poles removed.

REMARK 1.6. Typically $U = (0, 1)^2$ is the unit square.

REMARK 1.7. While every curve has a parametrization, not every surface has a parametrization! The torus, for instance, can not be parametrized. Further, surfaces like the Klein-Bottle, can not even be visualised in three dimensions (but can in four dimensions).

PROPOSITION 1.8. *If $\varphi : U \to \Sigma$ is a parametrization of the surface Σ, then*

$$\int_\Sigma f\, dS = \int_U f \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA.$$

Here $\partial_i \varphi = D\varphi\, e_i = (\partial_i \varphi_1, \partial_i \varphi_2, \partial_i \varphi_3)^T$.

REMARK 1.9. If the surface can not be parametrized, the integral can be computed by breaking up Σ into finitely many pieces which *can* be parametrized. The formula above will yield an answer that is independent of the chosen parametrization and how you break up the surface (if necessary).

While a rigorous proof is beyond the scope of this course, we provide some intuition here. First, we know that if $R \subseteq \mathbb{R}^3$ is a parallelogram whose sides are the vectors $u, v \in \mathbb{R}^3$, then

$$\text{area}(R) = |u||v| \sin\theta = |u \times v| = \left| \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_3 \end{pmatrix} \right|$$

Now let $R_{i,j} \subseteq U$ be a small rectangle, and $R'_{i,j} = \varphi(R_{i,j})$. Let R be one of these rectangles, and a be the bottom left corner, and $a + h$ be the top right corner. Now $R' = \varphi(R)$ is approximately the parallelogram with sides $\partial_1 \varphi(a) h_1$ and $\partial_2 \varphi(a) h_2$, and so

$$\text{area}(R') \approx |\partial_1 \varphi \times \partial_2 \varphi| h_1 h_2. = |\partial_1 \varphi \times \partial_2 \varphi| \, \text{area}(R).$$

Thus

$$\int_\Sigma f\, dS \xleftarrow{\|P\| \to 0} \sum_{i,j} f(\xi_{i,j}) \, \text{area}(R'_{i,j})$$

$$\approx \sum_{i,j} f(\varphi(\eta_{i,j})) |\partial_1 \varphi \times \partial_2 \varphi| \, \text{area}(R_{i,j}) \xrightarrow{\|P\| \to 0} \int_U f \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA.$$

EXAMPLE 1.10. Compute the surface area of a sphere of radius R.

2. Surface integrals of vector functions

DEFINITION 2.1. We say $(\Sigma, \hat{n})$ is an *oriented surface* if $\Sigma \subseteq \mathbb{R}^3$ is a C^1 surface, $\hat{n} : \Sigma \to \mathbb{R}^3$ is a continuous function such that for every $x \in \Sigma$, the vector $\hat{n}(x)$ is normal to the surface Σ at the point x, and $|\hat{n}(x)| = 1$.

EXAMPLE 2.2. Let $\Sigma = \{x \in \mathbb{R}^3 \mid |x| = 1\}$, and choose $\hat{n}(x) = x/|x|$.

REMARK 2.3. At any point $x \in \Sigma$ there are exactly two possible choices of $\hat{n}(x)$. An oriented surface simply provides a consistent choice of one of these *in a continuous way* on the *entire surface*. Surprisingly this isn't always possible! If Σ is the surface of a Möbius strip, for instance, can not be oriented.

EXAMPLE 2.4. If Σ is the graph of a function, we orient Σ by chosing $\hat{n}$ to always be the unit normal vector with a positive z coordinate.

EXAMPLE 2.5. If Σ is a closed surface, then we will typically orient Σ by letting $\hat{n}$ to be the *outward pointing* normal vector.

DEFINITION 2.6. Let $U \subseteq \mathbb{R}^3$ be a domain. We say $u : U \to \mathbb{R}^3$ is a *vector field*.

Typical examples of vector fields are electric and magnetic fields, or the velocity field of a fluid. Let u be a vector field representing the velocity of a fluid (i.e. for $x \in \mathbb{R}^3$, $u(x) \in \mathbb{R}^3$ is the velocity of the fluid at the point x.). If Σ is some oriented surface with unit normal $\hat{n}$, then the amount of fluid flowing through Σ per unit time is exactly

$$\int_\Sigma u \cdot \hat{n}\, dS.$$

Note, both u and $\hat{n}$ above are *vector functions*, and $u \cdot \hat{n} : \Sigma \to \mathbb{R}$ is a scalar function. The surface integral of this was defined in the previous section.

DEFINITION 2.7. Let $(\Sigma, \hat{n})$ be an oriented surface, and $u : \Sigma \to \mathbb{R}^3$ be a vector field. The *surface integral* of u over Σ is defined to be

$$\int_\Sigma u \cdot \hat{n}\, dS.$$

REMARK 2.8. Other common notation for the surface integral is

$$\int_\Sigma u \cdot \hat{n}\, dS = \iint_\Sigma u \cdot dS = \int_\Sigma u \cdot dS = \int_\Sigma u \cdot d\sigma$$

PROPOSITION 2.9. *Let $\varphi : U \to \Sigma$ be a parametrization of the oriented surface $(\Sigma, \hat{n})$. Then either*

$$\hat{n} \circ \varphi = \frac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|} \tag{2.1}$$

on all of Σ, or

$$\hat{n} \circ \varphi = -\frac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|} \tag{2.2}$$

on all of Σ. Consequently, in the case (2.1) holds, we have

$$\int_\Sigma u \cdot \hat{n}\, dS = \int_U (u \circ \varphi) \cdot (\partial_1\varphi \times \partial_2\varphi)\, dA. \tag{2.3}$$

PROOF. Clearly (2.3) follows from (2.1) and Proposition 1.8. To prove (2.1), observe first that the curve $\gamma(t) = \varphi(a + te_i)$ is contained in the surface Σ. Consequently $\gamma' = \partial_i\varphi$ must be tangent to Σ for $i \in \{1, 2\}$. This forces $\partial_1\varphi \times \partial_2\varphi$ to be *normal* to Σ and hence parallel to $\hat{n}$. Thus

$$s \overset{\text{def}}{=} \hat{n} \cdot \frac{\partial_1\varphi \times \partial_2\varphi}{|\partial_1\varphi \times \partial_2\varphi|}$$

must be a function that only takes on the values ± 1. Since s is also continuous, it must either be identically 1 or identically -1, finishing the proof. □

EXAMPLE 2.10. Gauss's law sates that the total charge enclosed by a surface Σ is given by

$$Q = \varepsilon_0 \int_\Sigma E \cdot dS,$$

where ε_0 the permittivity of free space, and E is the electric field. By convention, the normal vector is chosen to be pointing outward.

If $E(x) = e_3$, compute the charge enclosed by the top half of the hemisphere bounded by $|x| = 1$ and $x_3 = 0$.

3. Stokes theorem

DEFINITION 3.1. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, we define the *curl of F*

$$\nabla \times F \stackrel{\text{def}}{=} \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}.$$

This is sometimes also denoted by $\operatorname{curl}(F)$.

REMARK 3.2. A mnemonic to remember this formula is to write

$$\nabla \times F = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix},$$

and compute the cross product treating both terms as 3-dimensional vectors.

REMARK 3.3. Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field representing the velocity of a fluid. The quantity $\nabla \times u$ measures the infinitesimal circulation of the fluid. Namely, if a small ball is placed in the fluid, then due to friction between the fluid and the ball's surface, the ball will start rotating. Indeed, a counter clockwise rotation about the x_3-axis will be produced if u_2 is smaller on the left of the ball than the right, or if u_1 is larger in the front of the ball than at the back. This velocity differential is captured by $\partial_1 u_2 - \partial_2 u_1$, which is exactly the third component of $\nabla \times u$. A more precise calculation can be used to show that the rotation axis of the ball (according to the right hand rule), will be parallel to $\nabla \times u$, and the angular speed will be exactly $|\nabla \times u|/2$.

EXAMPLE 3.4. If $F(x) = x/|x|^3$, then $\nabla \times F = 0$.

REMARK 3.5. In the example above, F is proportional to a gravitational force exerted by a body at the origin. We know from experience that when a ball is pulled towards the earth by gravity alone, it doesn't start to rotate; which is consistent with our computation $\nabla \times F = 0$.

EXAMPLE 3.6. If $v(x, y, z) = (\sin z, 0, 0)$, then $\nabla \times v = (0, \cos z, 0)$.

REMARK 3.7. Think of v above as the velocity field of a fluid between two plates placed at $z = 0$ and $z = \pi$. A small ball placed closer to the bottom plate experiences a higher velocity near the top than it does at the bottom, and so should start rotating counter clockwise along the y-axis. This is consistent with our calculation of $\nabla \times v$.

REMARK 3.8. Formally if F is a vector field, then so is $\nabla \times F$. However, structurally $\nabla \times F$ is a *2-form*, and not a vector field! It is usually identified with a vector field using Hodge duality. Since the discussion of differential forms is beyond the scope of this course, we will gloss over this point and simply treat the curl of a vector field as a vector field.

THEOREM 3.9 (Stokes Theorem). *Let $U \subseteq \mathbb{R}^3$ be a domain, $(\Sigma, \hat{n}) \subseteq U$ be a bounded, oriented, piecewise C^1, surface whose boundary is the (piecewise C^1) curve Γ. If $F : U \to \mathbb{R}^3$ be a C^1 vector field, then*

$$\int_\Sigma \nabla \times F \cdot \hat{n}\, dS = \oint_\Gamma F \cdot d\ell.$$

Here Γ is traversed in the counter clockwise direction when viewed by an observer standing with his feet on the surface and head in the direction of the normal vector.

REMARK 3.10. The rule determining the direction of traversal of Γ is often called the *right hand rule*. Namely, if you put your right hand on the surface with thumb aligned with $\hat{n}$, then Γ is traversed in the pointed to by your index finger.

REMARK 3.11. If the surface Σ has holes in it, then (as we did with Greens theorem) we orient each of the holes clockwise, and the exterior boundary counter clockwise following the right hand rule. Now Stokes theorem becomes

$$\int_\Sigma \nabla \times F \cdot \hat{n}\, dS = \int_{\partial\Sigma} F \cdot d\ell,$$

where the line integral over $\partial\Sigma$ is defined to be the sum of the line integrals over each component of the boundary.

REMARK 3.12. If Σ is contained in the x, y plane and is oriented by choosing $\hat{n} = e_3$, then Stokes theorem reduces to Greens theorem.

Stokes theorem allows us to quickly see how the curl of a vector field measures the infinitesimal circulation.

PROPOSITION 3.13. *Suppose a small, rigid paddle wheel of radius a is placed in a fluid with center at x_0 and rotation axis parallel to $\hat{n}$. Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field describing the velocity of the ambient fluid. If ω the angular speed of rotation of the paddle wheel about the axis $\hat{n}$, then*

$$\lim_{a\to 0} \omega = \frac{\nabla \times v(x_0) \cdot \hat{n}}{2}.$$

PROOF. Let Σ be the surface of a disk with center x_0, radius a, and face perpendicular to $\hat{n}$, and $\Gamma = \partial\Sigma$. (Here Σ represents the face of the paddle wheel, and Γ the boundary.) The angular speed ω will be such that

$$\oint_\Gamma (v - a\omega\hat{\tau}) \cdot d\ell = 0,$$

where $\hat{\tau}$ is a unit vector tangent to Γ, pointing in the direction of traversal. Consequently

$$\omega = \frac{1}{2\pi a^2} \oint_\Gamma v \cdot d\ell = \frac{1}{2\pi a^2} \int_\Sigma \nabla \times v \cdot \hat{n}\, dS \xrightarrow{a\to 0} \frac{\nabla \times v(x_0) \cdot \hat{n}}{2}. \qquad \square$$

REMARK 3.14. If the axis of the paddle wheel is chosen to maximise the angular velocity, we see that $\hat{n}$ must be parallel to $\nabla \times v$, and the maximum angular velocity is exactly $|\nabla \times v|/2$. Treating a small sphere as a combination of paddle wheels will prove the rotation formula claimed in Remark 3.3.

PROOF OF STOKES THEOREM. In the case that Σ admits a C^2 parametrization, we can quickly deduce Stokes theorem from Greens theorem as follows. Let $\varphi : U \to \Sigma$ be a C^1 parametrization of Σ such that $\hat{n} \cdot (\partial_1\varphi \times \partial_2\varphi) > 0$. Now

$$\int_\Sigma \nabla \times F \cdot \hat{n}\, dS = \int_U (\nabla \times F) \circ \varphi \cdot (\partial_1\varphi \times \partial_2\varphi)\, dA.$$

If we define $G : U \to \mathbb{R}^2$ by

$$G = (D\varphi)^T (F \circ \varphi),$$

then a direct calculation using the chain and product rules shows

$$\partial_1 G_2 - \partial_2 G_1 = (\nabla \times F) \circ \varphi \cdot (\partial_1\varphi \times \partial_2\varphi).$$

Consequently

$$\int_\Sigma \nabla \times F \cdot \hat{n}\, dS = \int_U (\partial_1 G_2 - \partial_2 G_1)\, dA = \oint_{\partial U} G\, d\ell.$$

Parametrising the curve ∂U the same calculation we did in the proof of Greens theorem shows

$$\oint_{\partial U} G\, d\ell = \oint_\Gamma F \cdot d\ell,$$

finishing the proof. □

4. Conservative and Potential Forces.

DEFINITION 4.1. Let $U \subseteq \mathbb{R}^3$, and $F : U \to \mathbb{R}^3$ be a C^1 vector field.

- We say F is a *conservative force* if
$$\oint_\Gamma F \cdot d\ell = 0,$$
for all closed curves Γ which are completely contained inside U.
- We say F is a *potential force* there exists a C^2 function $V : U \to \mathbb{R}$ such that $F = -\nabla V$. (The function V is called the potential.)

DEFINITION 4.2. A domain $U \subseteq \mathbb{R}^3$ is called *simply connected* if for every simple closed curve $\Gamma \subseteq U$, there exists a surface $\Sigma \subseteq U$ whose boundary is exactly the curve Γ.

We've seen before that any potential force must be conservative. We address the converse here.

THEOREM 4.3. *Let $U \subseteq \mathbb{R}^3$ be a simply connected domain, and $F : U \to \mathbb{R}^3$ be a C^1 vector field. Then F is a conservative force, if and only if F is a potential force, if and only if $\nabla \times F = 0$.*

The physics of conservative and potential forces aside, this result has interesting mathematical content: One can easily check that

$$\nabla \times (\nabla V) = 0,$$

for any C^2 function V. Is the converse true? Namely if $\nabla \times F = 0$, must $F = \nabla V$ for some function $V : U \to \mathbb{R}$? Theorem 4.3 says yes, provided the domain of F is simply connected.

PROOF OF THEOREM 4.3. Clearly, if F is a potential force, equality of mixed partials shows $\nabla \times F = 0$. Suppose now $\nabla \times F = 0$. By Stokes theorem

$$\oint_\Gamma F \cdot d\ell = \int_\Sigma \nabla \times F \cdot \hat{n}\, dS = 0,$$

and so F is conservative. Thus to finish the proof of the theorem, we only need to show that a conservative force is a potential force. We do this next.

Suppose F is a conservative force. Fix $x_0 \in U$ and define

$$V(x) = -\int_\Gamma F \cdot d\ell,$$

where Γ is *any* path joining x_0 and x that is completely contained in U. Since F is conservative, we seen before that the line integral above *will not* depend on the path itself but only on the endpoints.

Now let $h > 0$, and let Γ be a path that joins x_0 to a, and is a straight line between a and $a + he_1$. Then

$$-\partial_1 V(a) = \lim_{h\to 0} \frac{1}{h} \int_{a_1}^{a_1+h} F_1(a + te_1)\, dt = F_1(a).$$

The other partials can be computed similarly to obtain $F = -\nabla V$ concluding the proof. □

REMARK 4.4. Let $U = \mathbb{R}^3 - \{te_3 \mid t \in \mathbb{R}\}$, and define $F : U \to \mathbb{R}^3$ by

$$F(x) = \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

It's easy to check that $F \in C^1(U)$ and $\nabla \times F = 0$. However, we claim there does not exist any $V : U \to \mathbb{R}$ such that $F = -\nabla V$. To see this let Γ be the unit circle with center 0 contained in the x_1-x_2 plane. A calculation we've done before shows

$$\oint_\Gamma F \cdot d\ell = 2\pi \neq 0.$$

But by the fundamental theorem we know $\oint \nabla V \cdot d\ell = 0$ for any closed curve, and thus F can not equal $-\nabla V$ for any $V \in C^1(U)$.

5. Divergence Theorem

DEFINITION 5.1. If $v : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1, vector field we define the *divergence* of v by

$$\nabla \cdot v = \sum_{i=1}^{3} \partial_i v_i.$$

REMARK 5.2. The divergence is often denoted by $\operatorname{div}(v)$, and measures the infinitesimal *outward flux* of a vector field at a given point. Indeed, suppose v represents the velocity field of a fluid and consider a small imaginary cube placed in the fluid. The difference in the horizontal components of the velocity on the right and left will contribute towards the horizontal outward flux, and is captured by the $\partial_1 v_1$ term. Similarly the $\partial_2 v_2$ and $\partial_3 v_3$ terms capture the outward fluxes parallel to the x_2 and x_3 axes respectively.

Regions of high divergence are associated with *sources* (e.g. where a fluid is being pumped in), and regions of low divergence are associated with *sinks* (e.g. where a fluid drains out).

THEOREM 5.3 (Divergence Theorem). *Let $U \subseteq \mathbb{R}^3$ be a bounded domain whose boundary is a (piecewise) C^1 surface denoted by ∂U. If $v : U \to \mathbb{R}^3$ is a vector field, then*

$$\int_U (\nabla \cdot v)\, dV = \oint_{\partial U} v \cdot \hat{n}\, dS,$$

where $\hat{n}$ is the outward pointing unit normal vector.

REMARK 5.4. Similar to our convention with line integrals, we denote surface integrals over *closed surfaces* with the symbol $\oint$.

REMARK 5.5. Let $B_R = B(x_0, R)$ and observe

$$\lim_{R\to 0} \frac{1}{\operatorname{vol}(B_R)} \int_{\partial B_R} v \cdot \hat{n}\, dS = \lim_{R\to 0} \frac{1}{\operatorname{vol}(B_R)} \int_{B_R} \nabla \cdot v\, dV = \nabla \cdot v(x_0),$$

which justifies our intuition that $\nabla \cdot v$ measures the outward flux of a vector field.

REMARK 5.6. If $V \subseteq \mathbb{R}^2$, $U = V \times [a, b]$ is a cylinder, and $v : \mathbb{R}^3 \to R^3$ is a vector field that doesn't depend on x_3, then the divergence theorem reduces to Greens theorem.

PROOF OF THE DIVERGENCE THEOREM. Suppose first that the domain C is the unit cube $(0, 1)^3 \subseteq \mathbb{R}^3$. In this case

$$\int_C \nabla \cdot v\, dV = \int_C (\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3)\, dV.$$

Taking the first term on the right, the fundamental theorem of calculus gives

$$\begin{aligned}\int_C \partial_1 v_1\, dV &= \int_{x_3=0}^1 \int_{x_2=0}^1 (v_1(1, x_2, x_3) - v_1(0, x_2, x_3))\, dx_2\, dx_3 \\ &= \int_L v \cdot \hat{n}\, dS + \int_R v \cdot \hat{n}\, dS,\end{aligned}$$

where L and B are the left and right faces of the cube respectively. The $\partial_2 v_2$ and $\partial_3 v_3$ terms give the surface integrals over the other four faces. This proves the divergence theorem in the case that the domain is the unit cube.

Now given an arbitrary domain U, suppose there exists a C^2 coordinate change function $\varphi : C \to U$. By interchanging x_1 and x_2 if necessary, we can guarantee $\det(D\varphi) > 0$ in all of C. Using the coordinate change formula for surface integrals (from your homework) observe

$$\int_{\partial U} v \cdot \hat{n}\, dS = \int_{\partial C} \big(\operatorname{adj}(D\varphi) v \circ \varphi\big) \cdot \hat{n}\, dS = \int_C \nabla \cdot w\, dV,$$

where

$$w = \operatorname{adj}(D\varphi)\, v \circ \varphi.$$

Now, with the product and chain rule, we compute

$$\nabla \cdot w = \det(D\varphi)\, (\nabla \cdot v) \circ \varphi,$$

and by the coordinate change for volume integrals

$$\int_{\partial U} v \cdot \hat{n}\, dS = \int_C \nabla \cdot w\, dV = \int_U \nabla \cdot v\, dV,$$

concluding the proof. □

PROPOSITION 5.7 (Gauss's gravitational law). *Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the gravitational field of a mass distribution (i.e. $g(x)$ is the force experienced by a point mass located at x). If Σ is any closed (C^1) surface, then*

$$\oint_\Sigma g \cdot \hat{n}\, dS = -4\pi G M,$$

where M is the mass enclosed by the region M. Here G is the gravitational constant, and $\hat{n}$ is the outward pointing unit normal vector.

PROOF. The crux of the matter is the following calculation. Given a fixed $y \in \mathbb{R}^3$, define the vector field F by

$$F(x) = \frac{x - y}{|x - y|^3}.$$

Then

$$\oint_\Sigma F \cdot \hat{n}\, dS = \begin{cases} 4\pi & \text{if } y \text{ is in the region enclosed by } \Sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

For simplicity, we subsequently assume $y = 0$.

To prove (5.1), observe

$$\nabla \cdot F = 0,$$

when $x \neq 0$. Let U be the region enclosed by Σ. If $0 \notin U$, then the divergence theorem will apply to in the region U and we have

$$\oint_\Sigma g \cdot \hat{n}\, dS = \int_U \nabla \cdot g\, dV = 0.$$

On the other hand, if $0 \in U$, the divergence theorem will not directly apply, since $F \notin C^1(U)$. To circumvent this, let $\varepsilon > 0$ and $U' = U - B(0, \varepsilon)$, and Σ' be the boundary of U'. Since $0 \notin U'$, F is C^1 on all of U' and the divergence theorem gives

$$0 = \int_{U'} \nabla \cdot F\, dV = \int_{\partial U'} F \cdot \hat{n}\, dS,$$

and hence

$$\oint_\Sigma F \cdot \hat{n}\, dS = -\oint_{\partial B(0,\varepsilon)} F \cdot \hat{n}\, dS = \oint_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon^2}\, dS = -4\pi,$$

as claimed. (Above the normal vector on $\partial B(0, \varepsilon)$ points outward with respect to the domain U', and *inward* with respect to the ball $B(0, \varepsilon)$.)

Now, in the general case, suppose the mass distribution has density ρ. Then the gravitational field $g(x)$ will be the super-position of the gravitational fields at x due to a point mass of size $\rho(y)\, dV$ placed at y. Namely, this means

$$g(x) = -G \int_{\mathbb{R}^3} \frac{\rho(y)(x - y)}{|x - y|^3}\, dV(y).$$

Now using Fubini's theorem,

$$\int_\Sigma g(x)\cdot\hat n(x)\,dS(x) = -G\int_{y\in\mathbb{R}^3}\rho(y)\int_{x\in\Sigma}\frac{x-y}{|x-y|^3}\cdot\hat n(x)\,dS(x)\,dV(y)$$
$$= -4\pi G\int_{y\in U}\rho(y)\,dV(y) = -4\pi GM,$$

where the second last equality followed from (5.1). □

We saw earlier that $\nabla\times(\nabla V)=0$, and conversely, in simply connected domains, any function for which $\nabla\times F=0$ must satisfy $F=-\nabla V$ for some V. A similar result is true for the divergence and curl.

PROPOSITION 5.8. *For any C^2 vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ we must have*

$$\nabla\cdot(\nabla\times F)=0.$$

Conversely, if $v:\mathbb{R}^3\to\mathbb{R}^3$ is a C^1 vector field for which $\nabla\cdot v=0$, there must exist a C^1 vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ such that $v=\nabla\times F$.

REMARK 5.9. If the vector field v is only defined on a domain $U\subseteq\mathbb{R}^3$, then the above proposition is still true, *provided* the domain U has "no holes". More precisely, for any closed surface $\Sigma\subseteq U$, the entire region enclosed by Σ must also be contained in U.

PROOF. Using Clairaut's theorem, we can directly check $\nabla\cdot(\nabla\times F)=0$. For the converse, suppose $\nabla\cdot v=0$. We need to find a C^1 vector field F such that $\nabla\times F=v$. Note that $\nabla\times(F+\nabla\phi)=\nabla\times F$ for any C^2 function ϕ. Set $G=F+\nabla\phi$, and define

$$\phi(x) = -\int_0^{x_3} F_3(x_1,x_2,t)\,dt\,,$$

and observe now that $G_3=0$. Thus if $v=\nabla\times F$ for some F, we can always find a vector field G such that $G_3=0$ and $v=\nabla\times G$. Since this is simpler, we will directly construct a vector field G, with $G_3=0$, such that $\nabla\times G=v$.

Note that if $G_3=0$, then

$$\nabla\times G = \begin{pmatrix} -\partial_3 G_2\\ \partial_3 G_1\\ \partial_1 G_2-\partial_2 G_1\end{pmatrix}.$$

Since we want $\nabla\times G=v$, we must have

$$G_2(x) = -\int_0^{x_3} v_1(x_1,x_2,t)\,dt + C_2(x_1,x_2)\,,$$
$$G_1(x) = \int_0^{x_3} v_2(x_1,x_2,t)\,dt + C_1(x_1,x_2)\,,$$

where C_1, C_2 are two functions that only depend on x_1 and x_2. This shows $(\nabla\times G)\cdot e_i=v_i$ for $i=1,2$, and to finish the proof we only need to verify the same identity for $i=3$.

For this, recall $\nabla\cdot v=0$, and hence $\partial_3 v_3=-\partial_1 v_1-\partial_2 v_2$. Consequently,

$$\nabla\times G\cdot e_3 = \partial_1 G_2-\partial_2 G_1 = \int_0^{x_3}(-\partial_1 v_2-\partial_2 v_2)\,dt+\partial_1 C_2-\partial_2 C_1$$
$$= \int_0^{x_3}\partial_3 v_3(x_1,x_2,t)\,dt+\partial_1 C_2-\partial_2 C_1$$

$$= v_3(x_1, x_2, x_3) - v_3(x_1, x_2, 0) + \partial_1 C_2 - \partial_2 C_1 \,.$$

To finish the proof we only need to choose C_1 and C_2 so that

$$v_3(x_1, x_2, 0) = \partial_1 C_2 - \partial_2 C_1 \,.$$

This is easily arranged. Indeed, choose $C_1 = 0$ and

$$C_2(x_1, x_2) = \int_0^t v_3(t, x_2, 0)\, dt \,.$$

With this choice of G, C_1 and C_2 we have $v = \nabla \times G$ as desired. □