Assignment 15 (assigned 2018-05-02, due Never).

1. Let $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ be C^2 vector fields, $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^2 scalar functions. Prove the following identities.
   (a) $\nabla(fg) = f\nabla g + g\nabla f$.
   (b) $\nabla \cdot (fu) = (\nabla f) \cdot u + f\nabla \cdot u$.
   (c) $\nabla \times (fu) = f\nabla \times u + (\nabla f) \times u$
   (d) $\nabla \times (u \times v) = u(\nabla \cdot v) - v(\nabla \cdot u) + (v \cdot \nabla)u - (u \cdot \nabla)v$
   (e) $\nabla \cdot (u \times v) = (\nabla \times u) \cdot v - u \cdot (\nabla \times v)$
   (f) $\nabla \times (\nabla u) = 0$
   (g) $\nabla \cdot (\nabla \times u) = 0$
   (h) $\nabla \times (\nabla \times u) = -\Delta u + \nabla(\nabla \cdot u)$
2. Let $F = (2x, y^2, z^2)$.
   (a) Compute $\int_\Sigma F \cdot \hat{n}\, dS$, where $\Sigma \subseteq \mathbb{R}^3$ is the sphere of radius 1.
   (b) Compute $\oint_\Gamma F \cdot d\ell$, where Γ is the intersection of Σ above and the plane $x + 2y + 3z = 0$.
3. In each of the following cases show that

$$\int_U (\nabla \cdot v)\, dV \neq \int_{\partial U} v \cdot \hat{n}\, dS.$$

   Also explain why this does not contradict the divergence theorem.
   (a) $U = B(0, 1) \subseteq \mathbb{R}^3$, and $v = x/|x|^4$.
   (b) $U = \{x \in \mathbb{R}^3 \mid z > x^2 + y^2\}$, and $v(x, y, z) = (0, 0, e^{-z})$.
4. *(Greens identity)* Let $U \subseteq \mathbb{R}^3$ be a bounded domain whose boundary is a C^1 surface. If $f, g : U \to \mathbb{R}$ be C^2, show

$$\int_U (f\Delta g)\, dV = \int_{\partial U} f\nabla g \cdot \hat{n}\, dS - \int_U (\nabla f \cdot \nabla g)\, dV.$$

   Recall $\Delta g = \sum_i \partial_i^2 g = \nabla \cdot (\nabla g)$, and by convention $\hat{n}$ is the *outward pointing* unit normal on ∂U.
5. Suppose $U, V \subseteq \mathbb{R}^3$ are domains, $\varphi : U \to V$ is C^2 and $v : V \to \mathbb{R}^3$ is C^1. Define $w = \operatorname{adj}(D\varphi)(v \circ \varphi) : U \to \mathbb{R}^3$. Show that

$$\nabla \cdot w = \det(D\varphi)(\nabla \cdot v) \circ \varphi.$$

   [This was a detail used in the proof of the divergence theorem.]
6. *(Optional challenge)* Suppose $u : \mathbb{R}^3 \to \mathbb{R}^3$ is C^1. Show that $\nabla \cdot u = 0$ if and only if there exists a C^1 vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that $u = \nabla \times v$.