21-268 Multidimensional Calculus: Final.

2018-05-14

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 9 questions and* 90 *points.*
- *You may use any result proved in class or any regular homework problem* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are all comparable in terms of difficulty, and are arranged in the order the material was covered. Depending on your intuition, you may find some questions easier than other so look over the whole exam before spending too much time on any one question. Good luck* ⌣̈.

$\boxed{10}$ 1. Does $\lim_{(x,y)\to 0} \dfrac{2xy}{|x| + y^2}$ exist? Prove your answer.

$\boxed{10}$ 2. Let $U \subseteq \mathbb{R}^d$ be open and $f\colon \mathbb{R}^d \to \mathbb{R}^n$ be continuous. True or false: For every open set $V \subseteq \mathbb{R}^n$, the set $f^{-1}(V)$ is also open. Prove it, or find a counter example.

[This was a question on your homework. Please provide a complete proof (or counterexample here) without relying on this particular result from the homework.]

3. Consider the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Let f be a C^1 function.

$\boxed{5}$ (a) Express $\partial_\theta f$ in terms of x, y, $\partial_x f$ and $\partial_y f$.

$\boxed{5}$ (b) Express $\partial_x f$ in terms of $\partial_r f$, $\partial_\theta f$, r and θ.

$\boxed{10}$ 4. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, and $a \in \mathbb{R}^2$. If f attains a local minimum at a, show that the Hessian Hf_a is positive semi-definite.

[This was a result we proved in class. Please provide a complete proof here without quoting, or relying on this particular result from class.]

$\boxed{10}$ 5. Find the constrained maximum of xyz under the constraint $2x^2 + y^2 + z^2 = 1$.

$\boxed{10}$ 6. Let $S \subseteq \mathbb{R}^3$ be the set

$$S \stackrel{\text{def}}{=} \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; xy + 2yz^2 - x^2 z = 2\,,\ \text{and } 2x + 6y - 8z = 0\right\}.$$

Determine whether S is a curve or a surface. Find a basis of the tangent space at the point $(1,1,1)$.

$\boxed{10}$ 7. Let Γ be the curve parametrized by $\gamma(t) = (2\sin t, \cos t)$, for $t \in [0, 2\pi]$. Compute the area of the region enclosed by Γ.

$\boxed{10}$ 8. Let $U \subseteq \mathbb{R}^3$ be defined by $U = \{x \in \mathbb{R}^3 \mid x_3 > 0,\ |x| < 2,\ \text{and } x_1^2 + x_2^2 < 1\}$. Find the volume of U.

$\boxed{10}$ 9. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x) = (0, 0, 2x_3)$, and $\Sigma \subseteq \mathbb{R}^3$ be the surface defined by

$$\Sigma = \{x \in \mathbb{R}^3 \mid x_3 > 0,\ |x| = 2,\ \text{and } x_1^2 + x_2^2 < 1\}.$$

Let $\hat{n}$ be the upward pointing unit normal on Σ (i.e. $\hat{n}$ is the unit normal for which $\hat{n} \cdot e_3 > 0$). Compute the surface integral $\int_\Sigma F \cdot \hat{n}\, dS$.