CHAPTER 5

Line Integrals

1. Line integrals

DEFINITION 1.1. If a force F acting on a body produces an instantaneous displacement v, then the work done by the force is $F \cdot v$.

Let $\Gamma \subseteq \mathbb{R}^3$ be a curve, with a given direction of traversal, and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a (vector) function. Here F represents the force that acts on a body and pushes it along the curve Γ. The work done by the force can be approximated by

$$W = \sum_{i=0}^{N-1} F(x_i) \cdot (x_{i+1} - x_i)$$

where $x_0, x_1, \ldots, x_{N-1}$ are N points on Γ, *chosen along the direction of traversal.* The limit as the largest distance between neighbours approaches 0 is defined to be the line integral.

DEFINITION 1.2. Let $\Gamma \subseteq \mathbb{R}^d$ be a curve (with a given direction of traversal), and $F : \Gamma \to \mathbb{R}^d$ be a (vector) function. The *line integral* of F over Γ is defined to be

$$\int_\Gamma F \cdot d\ell = \lim_{\|P\| \to 0} \sum_{i=0}^{N-1} F(x_i) \cdot (x_{i+1} - x_i).$$

Here $P = \{x_0, x_1, \ldots, x_{N-1}\}$, the points x_i are chosen along the direction of traversal, and $\|P\| = \max|x_{i+1} - x_i|$.

REMARK 1.3. If $F = (F_1, \ldots, F_d)^T$, where $F_i : \Gamma \to \mathbb{R}$ are functions, then one often writes the line integral in the *differential form* notation as

$$\int_\Gamma F \cdot d\ell = \int_\Gamma F_1 \, dx_1 + \cdots + F_d \, dx_d = \int_\Gamma \sum_{i=1}^{d} F_i \, dx_i.$$

PROPOSITION 1.4. *If $\gamma : [a, b] \to \mathbb{R}^d$ is a parametrization of Γ (in the direction of traversal), then*

$$\int_\Gamma F \cdot d\ell = \int_a^b F \circ \gamma(t) \cdot \gamma'(t) \, dt \tag{1.1}$$

In the differential form notation (when $d = 2$) say

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $f, g : \Gamma \to \mathbb{R}$ are functions. Then Proposition 1.4 says

$$\int_\Gamma F \cdot d\ell = \int_\Gamma f \, dx + g \, dy = \int_\Gamma \big(f(x(t), y(t)) \, x'(t) + g(x(t), y(t)) \, y'(t) \big) \, dt$$

REMARK 1.5. Sometimes (1.1) is used as the definition of the line integral. In this case, one needs to verify that this definition is *independent* of the parametrization. Since this is a good exercise, we'll do it anyway a little later.

EXAMPLE 1.6. Suppose a body of mass M is placed at the origin. The force experienced by a body of mass m at the point $x \in \mathbb{R}^3$ is given by $F(x) = \frac{-GMx}{|x|^3}$, where G is the *gravitational constant.* Compute the work done when the body is moved from a to b along a straight line.

SOLUTION. Let Γ be the straight line joining a and b. Clearly $\gamma : [0, 1] \to \Gamma$ defined by $\gamma(t) = a + t(b - a)$ is a parametrization of Γ. Now

$$W = \int_\Gamma F \cdot d\ell = -GMm \int_0^1 \frac{\gamma(t)}{|\gamma(t)|^3} \cdot \gamma'(t) \, dt = \frac{GMm}{|b|} - \frac{GMm}{|a|}. \qquad \square$$

REMARK 1.7. If the line joining through a and b passes through the origin, then some care has to be taken when doing the above computation. We will see later that gravity is a *conservative force*, and that the above line integral only depends on the endpoints and not the actual path taken.

2. Parametrization invariance and arc length

So far we have always insisted all curves and parametrizations are differentiable or C^1. We now relax this requirement and subsequently only assume that all curves (and parametrizations) are *piecewise differentiable*, or *piecewise* C^1.

DEFINITION 2.1. A function $f : [a, b] \to \mathbb{R}^d$ is called *piecewise* C^1 if there exists a finite set $F \subseteq [a, b]$ such that f is C^1 on $[a, b] - F$, and further both left and right limits of f and f' exist at all points in F.

DEFINITION 2.2. A (connected) curve Γ is *piecewise* C^1 if it has a parametrization which is continuous *and* piecewise C^1.

REMARK 2.3. A piecewise C^1 function need not be continuous. But curves are always assumed to be at least continuous; so for notational convenience, we define a piecewise C^1 curve to be one which has a parametrization which is both continuous and piecewise C^1.

EXAMPLE 2.4. The boundary of a square is a piecewise C^1 curve, but not a differentiable curve.

PROPOSITION 2.5 (Parametrization invariance). *If $\gamma_1 : [a_1, b_1] \to \Gamma$ and $\gamma_2 : [a_2, b_2] \to \Gamma$ are two parametrizations of Γ that traverse it in the same direction, then*

$$\int_{a_1}^{b_1} F \circ \gamma_1(t) \cdot \gamma_1'(t) \, dt = \int_{a_2}^{b_2} F \circ \gamma_2(t) \cdot \gamma_2'(t) \, dt.$$

PROOF. Let $\varphi : [a_1, b_1] \to [a_2, b_2]$ be defined by $\varphi = \gamma_2^{-1} \circ \gamma_1$. Since γ_1 and γ_2 traverse the curve in the same direction, φ must be increasing. One can also show (using the inverse function theorem) that φ is continuous and piecewise C^1. Now

$$\int_{a_2}^{b_2} F \circ \gamma_2(t) \cdot \gamma_2'(t)\, dt = \int_{a_2}^{b_2} F(\gamma_1(\varphi(t))) \cdot \gamma_1'(\varphi(t))\varphi'(t)\, dt.$$

Making the substitution $s = \varphi(t)$ finishes the proof. $\square$

DEFINITION 2.6. If $\Gamma \subseteq \mathbb{R}^d$ is a piecewise C^1 curve, then

$$\operatorname{arc\,len}(\Gamma) = \lim_{\|P\| \to 0} \sum_{i=0}^{N} |x_{i+1} - x_i|,$$

where as before $P = \{x_0, \dots, x_{N-1}\}$. More generally, if $f : \Gamma \to \mathbb{R}$ is any scalar function, we define[1]

$$\int_\Gamma f\, |d\ell| \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{i=0}^{N} f(x_i)\, |x_{i+1} - x_i|,$$

The *arc length* of a curve can be computed by taking the line integral of the unit tangent vector.

PROPOSITION 2.7. *Let $\Gamma \subseteq \mathbb{R}^d$ be a piecewise C^1 curve, $\gamma : [a, b] \to \mathbb{R}$ be any parametrization (in the given direction of traversal), $f : \Gamma \to \mathbb{R}$ be a (scalar) function, and $\tau : \Gamma \to \mathbb{R}^d$ is the unit tangent vector (i.e. $|\tau| \equiv 1$ and τ is always tangent to Γ) along the direction of traversal. Then*

$$\int_\Gamma f\, |d\ell| = \int_\Gamma f\tau \cdot d\ell = \int_a^b f(\gamma(t))\, |\gamma'(t)|\, dt,$$

and consequently

$$\operatorname{arc\,len}(\Gamma) = \int_\Gamma 1\, |d\ell| = \int_a^b |\gamma'(t)|\, dt.$$

EXAMPLE 2.8. Compute the circumference of a circle of radius r.

REMARK 2.9. A very useful way to describe curves is to parametrize them by arc length. Namely, let $\gamma(s) \in \Gamma$ be the unique point so that the portion of Γ traversed up to the point $\gamma(s)$ has arc length exactly s.

3. The fundamental theorem

THEOREM 3.1 (Fundamental theorem for line integrals). *Suppose $U \subseteq \mathbb{R}^d$ is a domain, $\varphi : U \to \mathbb{R}$ is C^1 and $\Gamma \subseteq \mathbb{R}^d$ is any differentiable curve that starts at a, ends at b and is completely contained in U. Then*

$$\int_\Gamma \nabla\varphi \cdot d\ell = \varphi(b) - \varphi(a).$$

[1] Unfortunately $\int_\Gamma f\, |d\ell|$ is also called the line integral. To avoid confusion, we will call this the *line integral with respect to arc-length* instead.

PROOF. Let $\gamma : [0, 1] \to \Gamma$ be a parametrization of Γ. Note

$$\int_\Gamma \nabla\varphi \cdot d\ell = \int_0^1 \nabla\varphi(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 \frac{d}{dt}\varphi(\gamma(t))\, dt = \varphi(b) - \varphi(a). \qquad \square$$

DEFINITION 3.2. A *closed curve* is a curve that starts and ends at the same point. A *simple closed curve* is a closed curve that never crosses itself. (More precisely, a simple closed curve is a compact 1-dimensional manifold with no boundary.)

If Γ is a closed curve, then line integrals over Γ are denoted by

$$\oint_\Gamma F \cdot d\ell.$$

COROLLARY 3.3. *If $\Gamma \subseteq \mathbb{R}^d$ is a closed curve, and $\varphi : \Gamma \to \mathbb{R}$ is C^1, then*

$$\oint_\Gamma \nabla\varphi \cdot d\ell = 0.$$

DEFINITION 3.4. Let $U \subseteq \mathbb{R}^d$, and $F : U \to \mathbb{R}^d$ be a vector function. We say F is a *conservative force* (or *conservative vector field*) if

$$\oint F \cdot d\ell = 0,$$

for all closed curves Γ which are completely contained inside U.

Clearly if $F = -\nabla V$ for some C^1 function $V : U \to \mathbb{R}$, then F is conservative. The converse is also true provided U is *simply connected*, which we'll return to later.

EXAMPLE 3.5. If φ fails to be C^1 even at one point, the above can fail quite badly. Let $\varphi(x, y) = \tan^{-1}(y/x)$, extended to $\mathbb{R}^2 - \{(x, y) \mid x \leqslant 0\}$ in the usual way. Then

$$\nabla\varphi = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

which is defined on $\mathbb{R}^2 - (0, 0)$. In particular, if $\Gamma = \{(x, y) \mid x^2 + y^2 = 1\}$, then $\nabla\varphi$ is defined on all of Γ. However, you can easily compute

$$\oint_\Gamma \nabla\varphi \cdot d\ell = 2\pi \neq 0.$$

The reason this doesn't contradict the previous corollary is that Corollary 3.3 requires φ itself to be defined on all of Γ, and not just $\nabla\varphi$! This example leads into something called the *winding number* which we will return to later.

4. Greens theorem

THEOREM 4.1 (Greens Theorem). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain whose exterior boundary is a piecewise C^1 curve Γ. If Ω has holes, let $\Gamma_1, \dots, \Gamma_N$ be the interior boundaries. If $F : \bar{\Omega} \to \mathbb{R}^2$ is C^1, then*

$$\int_\Omega \left(\partial_1 F_2 - \partial_2 F_1\right) dA = \oint_\Gamma F \cdot d\ell + \sum_{i=1}^{N} \oint_{\Gamma_i} F \cdot d\ell,$$

where all line integrals above are computed by traversing the exterior boundary counter clockwise, *and every interior boundary* clockwise.

REMARK 4.2. A common convention is to denote the *boundary* of Ω by $\partial\Omega$ and write

$$\partial\Omega = \Gamma \cup \Big(\bigcup_{i=1}^{N} \Gamma_i\Big).$$

Then Theorem 4.1 becomes

$$\int_{\Omega} \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA = \oint_{\partial\Omega} F \cdot d\ell,$$

where again the exterior boundary is oriented *counter clockwise* and the interior boundaries are all oriented *clockwise*.

REMARK 4.3. In the differential form notation, Greens theorem is stated as

$$\int_{\Omega} \Big(\partial_x Q - \partial_y P\Big)\, dA = \int_{\partial\Omega} P\,dx + Q\,dy,$$

$P, Q : \bar{\Omega} \to \mathbb{R}$ are C^1 functions. (We use the same assumptions as before on the domain Ω, and orientations of the line integrals on the boundary.)

REMARK 4.4. Note, Greens theorem requires that Ω is bounded and F (or P and Q) is C^1 on *all* of Ω. If this fails at even one point, Greens theorem need not apply anymore!

PROOF. The full proof is a little cumbersome. But the main idea can be seen by first proving it when Ω is a square, and then applying a coordinate transformation. Indeed, suppose first $\Omega = (0,1)^2$. Then the fundamental theorem of calculus gives

$$\int_{\Omega} \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA = \int_{y=0}^{1} \big(F_2(1,y) - F_2(0,y)\big)\, dy - \int_{x=0}^{1} \big(F_1(x,1) - F_1(x,0)\big)\, dx$$

The first integral is the line integral of F on the two vertical sides of the square, and the second one is line integral of F on the two horizontal sides of the square. This proves Theorem 4.1 in the case when Ω is a square.

Now let U be an arbitrary region for which there exists a C^2 coordinate transformation $\varphi : \Omega \to U$, where Ω is the unit square. We assume that φ also maps $\partial\Omega$ to ∂U and *preserves the orientation of the boundaries*. (One can show that this will imply $\det D\varphi > 0$ in U.) Now, using Greens theorem on the square,

$$\oint_{\partial U} F \cdot d\ell = \oint_{\partial\Omega} (D\varphi)^T F \circ \varphi \cdot d\ell = \int_{\Omega} (\partial_1 G_2 - \partial_2 G_1)\, dA,$$

where

$$G = (D\varphi)^T F \circ \varphi = \sum_{i,j} \partial_i \varphi_j F_j \circ \varphi\, e_i$$

Now we compute using the chain rule

$$\partial_1 G_2 - \partial_2 G_1 = \sum_{i,j} \partial_2 \varphi_j\, \partial_i F_j\big|_{\varphi}\, \partial_1 \varphi_i - \partial_1 \varphi_j\, \partial_i F_j\big|_{\varphi}\, \partial_2 \varphi_i = \big(\partial_1 F_2 - \partial_2 F_1\big) \circ \varphi\, \det(D\varphi).$$

Thus, by the change of variable theorem,

$$\int_{\Omega} \big(\partial_1 G_2 - \partial_2 G_1\big)\, dA = \int_{\Omega} \big(\partial_1 F_2 - \partial_2 F_1\big) \circ \varphi\, \det(D\varphi)\, dA = \int_{U} \big(\partial_1 F_2 - \partial_2 F_1\big)\, dA,$$

finishing the proof. □

REMARK 4.5. The above strategy will only work if the domain has no holes. In the presence of holes, you can make one or more cuts and then find a coordinate transformation $\varphi : \Omega \to U$ as above. The only difference is now part of the boundary of Ω will be mapped to the cut you just made. The boundary integral over this piece, however, will cancel since it will now be traversed twice in opposite directions.

COROLLARY 4.6. *If $\Omega \subseteq \mathbb{R}^2$ is bounded with a C^1 boundary, then*

$$\operatorname{area}(\Omega) = \frac{1}{2} \int_{\partial\Omega} \big(-y\,dx + x\,dy\big) = \int_{\partial\Omega} -y\,dx = \int_{\partial\Omega} x\,dy$$

REMARK 4.7. A *planimeter* is a measuring instrument used to determine the area of an arbitrary two-dimensional shape. The operational principle of the planimeter can be proved using the previous corollary.

COROLLARY 4.8 (Surveyor's Formula). *Let $P \subseteq \mathbb{R}^2$ be a (not necessarily convex) polygon whose vertices, ordered counter clockwise, are (x_1, y_1), ..., (x_N, y_N). Then*

$$\operatorname{area}(P) = \frac{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_N y_1 - x_1 y_N)}{2}.$$