$$X(t) = \int_0^{W(t)} e^{-s^2}\, ds.$$

$$f\cancel{A}(x) = \int_0^{x} e^{-s^2}\, ds. \qquad \partial_t f = 0 \quad \partial_x f = e^{-x^2}.$$

$$\partial_x^2 f = -2x\, e^{-x^2}.$$

$$\text{Itô: } dX = \partial_x f \; d\overset{W}{\cancel{t}} + \frac{1}{2}\partial_x^2 f \; d[W,W].$$

$$= e^{-W(t)^2}\, dW(t) + \frac{1}{2}\left(-2W(t)\right)\left(e^{-W(t)^2}\right) dt$$

$$\boxed{\begin{array}{l} Y(t) = \sin(X_t). \\ \text{Find Itô Decomp \& Q.V.} \end{array}}$$

$Y(t) = W\left(\frac{1}{t}\right) \cdot t$ $\qquad$ $W \to$ BM.

Let $B(t) = t\, W\left(\frac{1}{t}\right)$.

<u>Claim:</u> B is a B.M.

Note: $\mathcal{F}^W_t$ = filtration of W.

B is <u>NOT ADAPTED</u> to $\mathcal{F}^W$

Let $\mathcal{F}_t^B$ = filtration generated by B.

B is adapted w.r.t. $\mathcal{F}_t^B$.

To see B is a BM.

① $B(t) \sim t \cdot N(0, \frac{1}{t}) \Rightarrow B$ is Gaussian.

$$EB^2(t) = t^2 \, E W(\tfrac{1}{t})^2 = t^2 \cdot \frac{1}{t} = t.$$

② Also check: $B(t) - B(s) = tW(\frac{1}{t}) - sW(\frac{1}{s})$.

Gaussian.

Compute $E(B(t)-B(s))^2 \longrightarrow$ Get $t-s$ $(s<t)$.

Check: $\left(tW\left(\frac{1}{t}\right) - sW\left(\frac{1}{s}\right)\right)^2 = t^2W\left(\frac{1}{t}\right)^2 + s^2W\left(\frac{1}{s}\right)^2$

$$- 2stW\left(\frac{1}{t}\right)W\left(\frac{1}{s}\right).$$

$$\Rightarrow E(\quad)^2 = t + s - 2st\left[\left(\frac{1}{t}\right)\wedge\left(\frac{1}{s}\right)\right].$$

$$= t + s - \frac{2st}{t} = t - s.$$

③ Check $B(t) - B(s)$ ind of $\mathcal{F}_s^B \Rightarrow B$ is a B.M.

$B(0)$: Define $B(0) = \lim_{t\to 0} tW\left(\frac{1}{t}\right). = 0$ a.s.

$$\hat{c}(T, S(T)) = \left(S_T^2 - K\right)^+.$$

find $\hat{c}(T, x)$.

<u>Option 1:</u> RNP formula: $\hat{c}(t)$

$$\hat{c}(t, S(t)) = \tilde{E}\left(e^{-r(T-t)}\left(S_T^2 - K\right)^+ \,\middle|\, \mathcal{F}_t\right).$$

Option 2: Use hint.

$$\tilde{S} = S^2. \quad d\tilde{S} = 2S\,dS + d[S,S].$$

$$= 2S(\mu S\,dt + \sigma S\,dW) + \sigma^2 S^2\,dt.$$

$$= 2\mu S^2 dt + 2\sigma S^2 dW + \sigma^2 S^2 dt$$

$$= \tilde{S}(2\mu + \sigma^2) dt + 2\sigma \tilde{S} dW$$

$\Rightarrow \tilde{S}$ is a GBM mean return $2\mu + \sigma^2$.

& vol 2σ.

$\hat{c}$ = call option on $\tilde{S}$ with strike K & mat T.

$$\Rightarrow \hat{c} = x N(d_+) - K e^{-r(T-t)} N(d_-).$$

$$d_\pm = \frac{1}{2\sigma\sqrt{\tau}} \left(\ln \frac{x}{K} + \left(r \pm \frac{1}{2} (2\sigma)^2 \tau \right) \right).$$

Q: $W \to$ BM.

$$B(t) = \int_0^t \operatorname{sign}(W(s))\, dW(s). \qquad \operatorname{sign}(x) = \begin{cases} +1 & x > 0 \\ -1 & x \leq 0 \\ \boxed{0 \quad x = 0} \end{cases}$$

① Compute $d[W,B] = \operatorname{sign}(W(t))\, d[W,W](t)$.

$$= \operatorname{sign}(W(t))\, dt.$$

② Is B a BM?

Yes: Levy. ① Cts ✓, mg ✓ (true bc Ito integrals).

② $d[B,B] = dt$

$$d[B,B] = [\operatorname{sign}(W(t))]^2 \, dt = dt.$$

By Levy $\Rightarrow$ B is a B.M.

$$[B,B](t) = \int_0^t \operatorname{sign}(W(s))^2 \, ds$$

$$= \int_0^t \mathbb{1}_{\{W(s) \neq 0\}} \, ds \overset{a.s.}{=} \int_0^t 1 \, ds.$$

$B \to$ BM.
$W \to$ B.M.
Q: ~~Is BW a mg?~~
$\mathrm{Cov}(B W)$?

Compute $E\,BW$

$$d(BW) = B\,dW + W\,dB + d[B, W].$$

$$\Rightarrow d(BW) = B\,dW + W\,dB + \mathrm{sign}(W(s))\,ds.$$

$$E\,B(t)W(t) = E\left(\int_0^t B(s)\,dW(s) + W(s)\,dB(s)\right) + \underbrace{\int_0^t E\,\mathrm{sign}(W(s))\,ds}_{0}.$$

$$\Rightarrow E(BW) = 0$$

Q: Are B & W independent?

① $B(t) \sim N(0,t)$

② $W(t) \sim N(0,t)$

③ $E\, B(t) W(t) = 0$

Guess B & W are ind! FALSE!

B & W are NOT Jointly Gaussian.

Check B & W are not Independent:

Option 1: Use Hint & suffer ($E\, B(t) W(t)^2$)

Option 2: M & N are cts ind mg $\Rightarrow [M,N] = 0$

$d[B,W] = \text{sign}(W(t))\, dt \neq 0 \Rightarrow$ B, W not ind!!

$$\cancel{\int_0^t} \quad \cancel{[X+Y] =}$$

$$[X+Y, X+Y] = [X,X] + [Y,Y] + 2[X,Y].$$

Claim: $[X, Y+Z] = [X,Y] + [X,Z].$

Intution: $[X,Y] = \lim\limits_{\|P\| \to 0} \sum_i \Delta_i X \; \Delta_i Y.$

$$\Delta_i X = X(t_{i+1}) - X(t_i).$$

$$\sum (\Delta_i X)(\Delta_i (Y+Z)) = \sum (\Delta_i X)(\Delta_i Y + \Delta_i Z).$$

Proof check: $X = M + B_1$ $\quad$ M, N, L mg's.

$Y = N + B_2$. $\quad$ B_1, B_2, B_3 finite finest var.

$Z = L + B_3$

$$[X, Y+Z] = [M, L+N].$$

$$M(L+N) - [M,L] - [M,N]$$

$$= (ML - [M,L]) + (MN - [M,N]).$$

is a mg.

$$\Rightarrow [M,L] + [M,N] = [M, L+N]$$

QED!

$$X(t) = \int_0^t s \, dW(s) \qquad Y(t) = \int_0^t W(s) \, ds .$$

$$E X(t)^n \quad \& \quad E Y(t)^n$$

$$X(t) = \int_0^t \sigma(s) \, dW(s) \quad \& \ \sigma \text{ is } \boxed{\text{not random.}}$$

$\Rightarrow$ $X(t)$ is gaussian &

$$X(t) \sim N\left(0, \int_0^t \sigma(s)^2 ds\right).$$

Eg $Y(t) = \int_0^t (t-s)\, dW(s)$.

Is Y a mg?

Recall: $I(t) = \int_0^t \sigma(s)\, dW(s)$, σ adapted.

$\Rightarrow I(t)$ is a mg.

But $\int_0^t \sigma(s,t)\, dW(s)$ NEED not be a mg!

Check Y is not a mg:

① $Y(t) = \int_0^t t\,dW(s) - \int_0^t s\,dW(s).$

$$= tW(t) - \underbrace{\int_0^t s\,dW(s)}_{\text{Is a mg}}.$$

Pick $r < t$: Compute $E(Y(t) \mid \mathcal{F}_r) = tW(r) - \int_0^r s\,dW(s).$

$$\neq Y(r)$$

$\Rightarrow Y$ is not a mg.

$$M(t) = \int_0^t \mathrm{Sign}(W(s))\, dW(s)$$

$$E\, M(t)^2 = E \int_0^t \mathrm{Sign}(W(s))^2\, ds. = \int_0^t ds. = t.$$

$$\boxed{\lim_{t\to\infty} M(t)^2 = \infty. \quad \text{OK??}}$$

Need for any $\underline{\underline{t < \infty}}$ $E\, M(t)^2 < \infty$.

OK for $\lim_{t\to\infty} E M(t)^2 = +\infty$.