① Interview Favorite:

f — $\boxed{\text{NOT random}}$ (Deterministic).

$$X(t) = \int_0^t f(s)\, W(s)\, ds.$$

~~find $E\,X(t)^n$~~

find dist of X.

$\boxed{\text{OPTION 1:}}$ Set $F(t) = \int_0^t f(s)\, ds$: $F' = f$.

Compute $d(FW) = F(t)\, dW(t) + W(t)\, dF(t) + \underbrace{d[F, W]}_{0}$.

$$= F(t)\, dW(t) + W(t) f(t)\, dt$$

$$\Rightarrow F(t)W(t) = \int_0^t F(s)\,dW(s) + \underbrace{\int_0^t W(s)\,f(s)\,ds}_{X(t).}$$

$$\Rightarrow X(t) = \underbrace{F(t)W(t)}_{\text{Gaussian.}} - \underbrace{\int_0^t F(s)\,dW(s).}_{\text{Gaussian. (Midterm Review).}}$$

Compute $\underbrace{E\,X(t)}_{0}$ & $\underbrace{E\,X(t)^2}_{\text{little work} \to \text{You check.}}$ & get the distribution.

$\boxed{\text{OPTION 2}}$ $X(t) = \int_0^t f(s) W(s)\, ds.$

$$= \lim_{\|P\| \to 0} \sum f(s_i) W(s_i) (s_{i+1} - s_i).$$

$$\big(W(s_0), W(s_1) \dots W(s_n)\big) \leftarrow \text{Gaussian.}$$

$$\Rightarrow \sum f(s_i) W(s_i) (s_{i+1} - s_i) \text{ is Gaussian!}$$

$$\int_0^t f(s) W(s)\, ds = \lim \big(\text{Gaussians}\big) \Rightarrow X(t) \text{ is Gaussian.}$$

Compute $EX(t)$ & $EX(t)^2$.

① $E X(t) = E\int_0^t f(s)\, W(s)\, ds = \int_0^t E f(s) W(s)\, ds = 0.$

② $E X(t)^2 = E\left(\int_0^t f(s) W(s)\, ds\right)^2$

$= E\left(\int_0^t f(s) W(s)\, ds\right)\left(\int_0^t f(r) W(r)\, dr\right).$

$= E \int_0^t\int_0^t f(s) f(r) W(s) W(r)\, dr\, ds.$

$= \int_0^t\int_0^t f(s) f(r)\, E(W(s) W(r))\, dr\, ds$

Recall $E\big(W(r)\,W(s)\big) = r \wedge s$.

$$\Rightarrow E\,X(t)^2 = \int_0^t \int_0^t f(r) f(s)\,(r \wedge s)\;dr\,ds.$$

& compute.

Q2: X, Y two $\boxed{\text{martingales}}$

σ, ρ, τ $\boxed{\text{NON RANDOM}}$ $(\sigma, \tau \geqslant 0)$.

$$d[X,X](t) = \sigma(t)\,dt \qquad d[Y,Y](t) = \tau(t)\,dt$$

$$d[X,Y](t) = \rho(t)\,dt.$$

① MGF: ~~$M_{\lambda X + \mu Y}$~~ Compute. $M_{X(t),Y(t)}(\lambda, \mu) = E\, e^{\lambda X(t) + \mu Y(t)}$.

Let $\varphi(t) = M_{X(t),Y(t)}(\lambda,\mu) = E\, \underline{e^{\lambda X(t) + \mu Y(t)}}$

$$Z(t) = e^{\lambda X(t) + \mu Y(t)} = f(X, Y)$$

where $f(x, y) = e^{\lambda x + \mu y}$

Itô: $dZ = \partial_t f \, dt + \partial_x f \, dX + \partial_y f \, dY$

$$+ \frac{1}{2}\left(\partial_x^2 f \, d[X, X] + \partial_y^2 f \, d[Y, Y] + 2 \partial_x \partial_y f \, d[X, Y]\right).$$

$\partial_t f = 0$

$\partial_x f = \lambda f$, $\partial_y f = \mu f$, $\partial_x^2 f = \lambda^2 f$, $\partial_y^2 f = \mu^2 f$.

$\partial_x \partial_y f = \lambda \mu f$.

$$\Rightarrow dZ = 0\,dt + \lambda Z\,dX + \mu Z\,dY + \frac{1}{2}\Big(\lambda^2 Z\,\sigma(t)\,dt + \mu^2 Z\,\tau(t)\,dt + 2\lambda\mu\,\rho(t) Z\,dt\Big).$$

$$\cancel{E Z}\,\varphi(t) - \varphi(0) = E Z(t) - E Z(0) = E\int_0^t \lambda Z\,dX + \mu Z\,dY + \frac{1}{2} E\int_0^t \big(\lambda^2 Z\sigma + \mu^2 Z\tau + 2\lambda\mu Z\rho\big)\,ds.$$

X, Y mg $\Rightarrow \int_0^t \lambda Z\, dX$ & $\int_0^t \lambda Z\, dY$ are mg

$$\Rightarrow E(\ \ ) = E\int_0^t \lambda Z\, dY = 0.$$

$$\Rightarrow \varphi(t) = \varphi(0) + \frac{1}{2}\int_0^t \Big(\lambda^2 \sigma(s)\varphi(s) + \mu^2 \tau(s)\varphi(s) + 2\lambda\mu\rho(s)\varphi(s)\Big)\, ds.$$

$$\Rightarrow \varphi(t) = \frac{1}{2}\int_0^t (\lambda^2\sigma + \mu^2\tau + 2\lambda\mu\rho)\,\varphi(s)\, ds.$$

$$\Rightarrow \partial_t \varphi = \frac{\left(\lambda^2 \sigma + \mu^2 \tau + 2\lambda\mu\rho\right)}{2} \varphi$$

τ_u^u

$$\boxed{\begin{aligned} \Rightarrow \varphi(t) &= \underbrace{\varphi(0)}_{1} \cdot \exp\left(\frac{1}{2}\int_0^t \left(\lambda^2 \sigma(s) + \mu^2 \tau(s) + 2\lambda\mu\, \rho(s)\right) ds.\right) \\ \| \\ M_{X(t),Y(t)}&(\lambda,\mu). \end{aligned}}$$

$$X(0) = Y(0) = 0 \Rightarrow \varphi(0) = 1$$

Part (b). Suppose $\sigma = 1$, $\tau = 1$, $\rho = 0$

$\sigma \to d[X,X]$, $\tau \to d[Y,Y]$, $\rho \to d[X,Y]$.

Levy: X, Y mg, cts. & have same JQV as 2D BM.

$\Rightarrow (X,Y)$ is a 2D BM.

Pf: $M_{X(t), Y(t)}(\lambda, \mu) = \exp\left(\frac{\lambda^2 t}{2} + \frac{\mu^2 t}{2} + 0\right)$.

$=$ MGF of 2D BM.

$\Rightarrow X$ & Y are ind. $X(t) \sim N(0, t)$, $Y(t) \sim N(0,t)$.

Can also check $X(t) - X(s)$ ind of $\mathcal{F}_s$. $\Rightarrow (X,Y)$ 2D BM!!

$\boxed{Q3:}$ Market

① Stock : $GBM(\alpha, \sigma)$.

$S(t)$ = stock price.

$$dS = \alpha S\, dt + \sigma S\, dW.$$

② Money Market : Return rate r.

Security : Pays $V(T) = \frac{1}{T}\int_0^T S(s)\, ds$.

① Compute AFP at time $t \leq T$.

② Compute Trading strategy.

Sol: Use RNP formula. ($\tau = T - t$).

$$V(t) = \tilde{E}\Big(\underbrace{e^{-r\tau}}_{e^{-\int_t^T R(s)\,ds}} V(T) \,\Big|\, \mathcal{F}_t \Big).$$

$$\cancel{d\tilde{P} = Z(T)\,dP \qquad Z = \exp(\quad)} \longleftarrow \text{DONT CARE}$$

Under $\tilde{P}$: $\tilde{W}$ is a B.M.

& $$dS = rS\,dt + \sigma S\,d\tilde{W}.$$

Recall $S(t) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}(t)\right)$.

$\frac{dS}{S} = r\,dt + \sigma\, d\tilde{W}$.

Compute $d(\ln S)$.

$$\Rightarrow V(t) = \tilde{E}\left(\frac{e^{-r\tau}}{T}\int_0^T S(0)e^{\left(r-\sigma^2/2\right)s + \sigma\tilde{W}(s)}\,ds \,\middle|\, \mathcal{F}_t\right).$$

$$= \frac{e^{-r\tau}}{T} \widetilde{E}\left(S(t) \int_t^T e^{\left(r-\frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\widetilde{W}(s)-\widetilde{W}(t)\right)} ds \,\middle|\, \mathcal{F}_t \right).$$

$$+ \frac{e^{-r\tau}}{T} \widetilde{E}\left(\int_0^t S(0) e^{\left(r-\frac{\sigma^2}{2}\right)s + \sigma \widetilde{W}(s)} ds \,\middle|\, \mathcal{F}_t \right).$$

$$= \frac{e^{-r\tau}}{T} S(t)\, \widetilde{E} \int_t^T e^{\left(r-\frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\widetilde{W}(s)-\widetilde{W}(t)\right)} ds.$$

$$+ \frac{e^{-r\tau}}{T} \cancel{\widetilde{E}} \int_0^t S(s)\, ds.$$

$$\therefore V(t) = \frac{e^{-r\tau}}{T}\int_0^t S(s)\,ds$$

$$+ \frac{e^{-r\tau}}{T} S(t) \int_t^T e^{(r-\frac{\sigma^2}{2})(s-t)}\, e^{\frac{\sigma^2}{2}(s-t)}\,ds.$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S(s)\,ds + S(t)\int_t^T e^{r(s-t)}\,ds\right).$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S(s)\,ds\right) + \left(\frac{e^{-r\tau}}{rT}\left(e^{r\tau}-1\right)\right)S(t).$$

$$\boxed{\underset{\overset{\parallel}{X}}{V(t)} = \frac{e^{-r\tau}}{T}\int_0^t S(s)\,ds + \left(\frac{1-e^{-r\tau}}{rT}\right)S(t) \ldots}$$

AFP.

Trading strategy: $X(t)$ → $\Delta(t)$ shares of Stock.
→ Rest cash.

$X(t) = V(t)$ (AFP & X is R-Portfolio).

Know $dX = \Delta(t)\,dS + r(X - \Delta S)\,dt$.

Claim:

$$\boxed{\Delta(t) = \frac{1 - e^{-r\tau}}{rT}}$$

(You check: Use $X(t) = V(t) =$ formula.

Compute dX. Look at the coeff of dS.

Get Δ from there!).

Q6: $M \to$ Mg
$\varphi \to$ convex fn.

Q: Is $\varphi(M)$ a mg?
Is $\varphi(M)$ a sub mg?
Is $\varphi(M)$ a super mg?

submg: $E(M(t) \mid \mathcal{F}_s) \geq M(s)$.

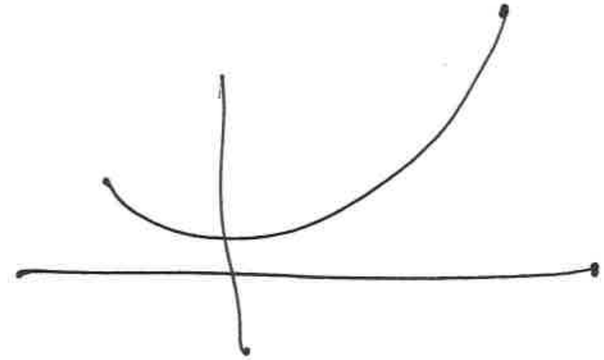
sup mg: $E(M(t) \mid \mathcal{F}_s) \leq M(s) \leftarrow$ Life.

~~Option 1~~ Claim: $\varphi(M)$ is a sub mg.

Option 1: Jensen

Option 2: Itô.

φ convex $\quad$ φ' is inc



$\varphi'' \geq 0.$

$$d(\varphi(M)) = \varphi'(M)\,dM + \frac{1}{2}\varphi''(M)\,d[M,M].$$

$$\varphi(M(t)) - \varphi(M(0)) = \int_0^t \varphi'(M(s))\,dM(s) + \frac{1}{2}\int_0^t \varphi''(M(s))\,d[M,M](s).$$

$r < t$

$$E(\varphi(M(t)) \mid \mathcal{F}_r) = \varphi(M(0)) + \int_0^r \varphi'(M(s))\,dM(s) + \frac{1}{2}\int_0^r \varphi''(\;)\;\cdots$$

$$+ \frac{1}{2} E\left(\int_r^t \varphi''(M)\,d[M,M]\right),$$

$$= M(r) + \frac{1}{2} E \int_r^t \underbrace{\varphi''(M(s))}_{\geq 0} \underbrace{d[M,M]}_{\geq 0}$$

(φ convex) (QV is inc $\Rightarrow d[M,M] \geq 0$).

$\Rightarrow M$ is a sub mg.

Q5 $x_0, \mu, \theta, \sigma \in \mathbb{R}$. (Ornstein Uhlenbeck)

X an Itô process +

$$dX = \theta(\mu - X(t))\,dt + \sigma\, dW$$

(1) Find a formula for $X(t)$.

Set $Y(t) = e^{\theta t} X(t)$.

Compute $dY = e^{\theta t} dX + X \theta e^{\theta t} dt + d[e^{\theta t}, X]$

$=$

$$= e^{\theta t}\left(\theta(\mu - X(t))\,dt + \sigma\,dW\right)$$

$$+ X\theta e^{\theta t}\,dt + 0.$$

$$dY = \theta\mu e^{\theta t}\,dt + e^{\theta t}\sigma\,dW.$$

$$\Rightarrow Y(t) = Y(0) + \mu(e^{\theta t} - 1) + \int_0^t \sigma e^{\theta s}\,dW(s).$$

$$\Rightarrow X(t) = e^{-\theta t}Y(t) = e^{-\theta t}x_0 + \mu(1 - e^{-\theta t})$$

$$+ e^{-\theta t}\int_0^t \sigma e^{\theta s}\,dW(s).$$

(b) Compute $E X(t)$ & $\operatorname{cov}(X(s), X(t))$.

(1) $$E X(t) = e^{-\theta t} x_0 + \mu(1-e^{-\theta t}) + 0.$$

(2) $$\operatorname{Cov}(X(s), X(t)) = E\big(X(s) - EX(s)\big)\big(X(t) - EX(t)\big).$$

$$= E\left(e^{-\theta s}\int_0^s \sigma e^{\theta r}\, dW(r)\right)\left(e^{-\theta t}\int_0^t \sigma e^{\theta r}\, dW(r)\right).$$

$$= \sigma^2 e^{-\theta(s+t)} E\left(\int_0^s e^{\theta r}\, dW(r) \int_0^t e^{\theta r}\, dW(r)\right).$$

Set $M(t) = \int_0^t e^{\theta r}\, dW(r)$.

$$\Rightarrow \operatorname{Cov}\big(X(s), X(t)\big) = \sigma^2 e^{-\theta(s+t)}\, E\, M(s) M(t)$$

$$= \sigma^2 e^{-\theta(s+t)}\, E\, E\big(M(s) M(t) \,\big|\, \mathcal{F}_s\big)$$

$$= \sigma^2 e^{-\theta(s+t)}\, E\, M(s)^2$$

$$= \sigma^2 e^{-\theta(s+t)} \int_0^{\cancel{t}\, s} e^{2\theta r}\, dr$$ & compute ...