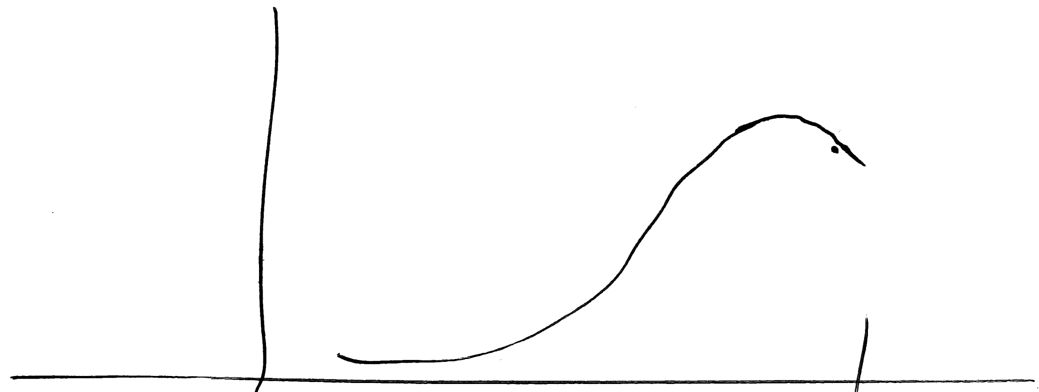
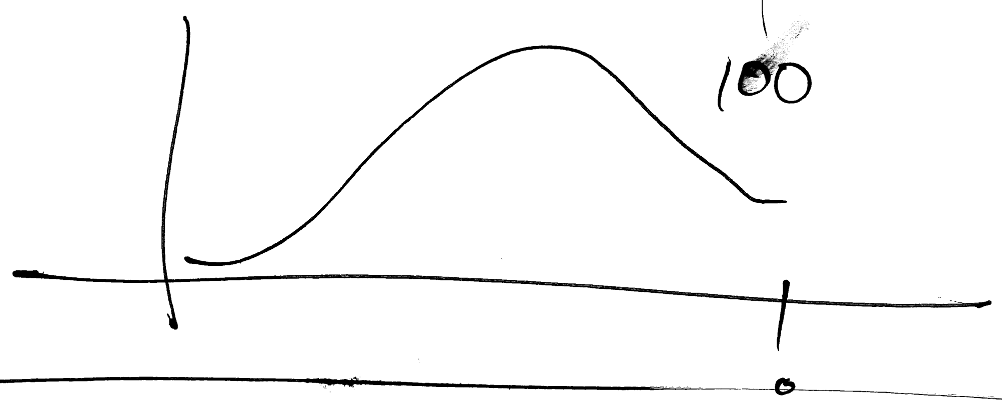
Reminder: 1 Fill out FCE's.

2 Final → $\boxed{\text{Sat}}$.

Midterm



Final



Last time: Girsanov Theorem (C-M-G thm).

① $d\tilde{W} = b(t)\,dt + dW$ (W d-dim BM, $b = (b_1, b_2, \dots b_d)$).

(b adapted).

② $$Z(t) = \exp\Big(-\underbrace{\int_0^t b(s)\cdot dW(s)}_{\text{mg}} - \frac{1}{2}\underbrace{\int_0^t |b(s)|^2\,ds}_{\text{qv of}}\Big)$$

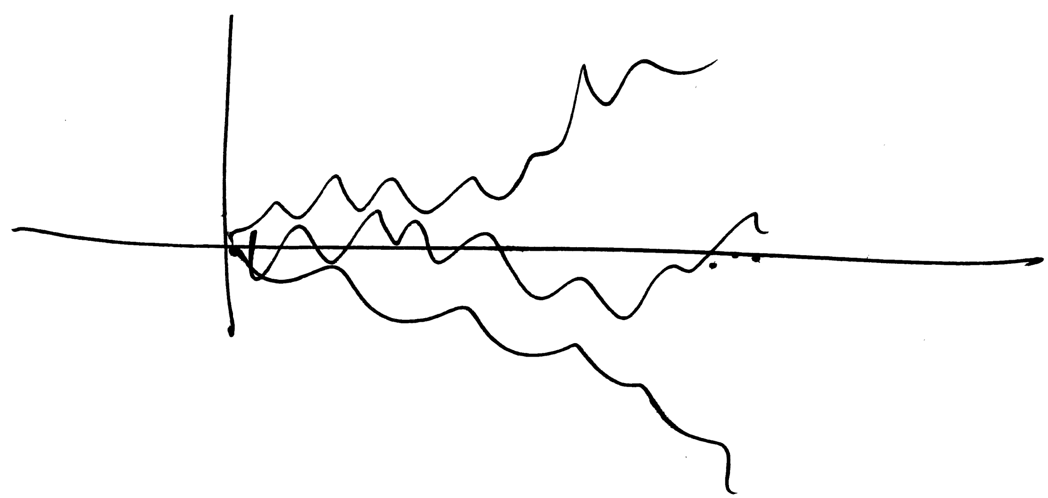
$$\Big(dZ = -Z(t)\, b(t)\cdot dW(t)\Big).$$

$$= -Z(t)\sum_j b_j(t)\, dW_j(t).$$

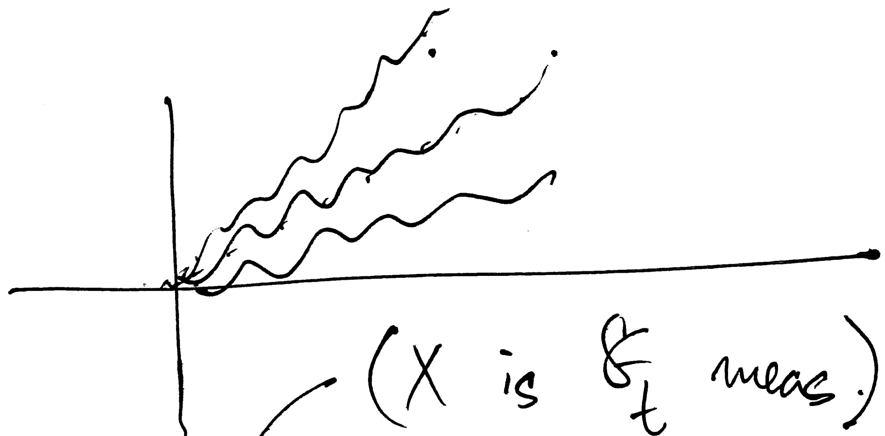
Define a new measure $\tilde{P}$ by $d\tilde{P} = Z(T)\, dP$.
(Here T = fixed = maturity time).
Then $\tilde{W}$ is a d-dim BM under $\tilde{P}$.

<u>Recall:</u> $\tilde{P}(A) = \int_A Z(T)\, dP = E(\mathbb{1}_A Z(T))$.

W:



Say $b(t) = t$. $\quad \tilde{W}(t) = t + W(t)$



Proof: Lemma ① $\tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t) X \mid \mathcal{F}_s)$.

(X is $\mathcal{F}_t$ meas.).

old meas.

② Lemma: M is a mg wrt $\tilde{P}$

$\Longleftrightarrow$ ZM is a mg wrt P.

Pf of $\Rightarrow$ of Lema ②:

~~Assume ZM is a mg wrt P.~~

Assume M is a mg wrt $\tilde{P}$.

NTS ZM is a mg wrt P.

Compute $E(Z(t)M(t) \mid \mathcal{F}_s) = Z(s)\tilde{E}(M(t) \mid \mathcal{F}_s)$.

$= Z(s)\,M(s).$ QED.

Proof of Girsanov:

1 Lévy: $\tilde{W}$ is a BM (under $\tilde{P}$).

$\iff$ 1 $[\tilde{W}_i, \tilde{W}_j] = 1_{i=j}\, t$.

& 2 Each $\tilde{W}_i$ is a mg under $\tilde{P}$.

Know $[\tilde{W}_i, \tilde{W}_j] = [W_i, W_j] = 1_{i=j}\, t. \Rightarrow$ 1.

For 2, only NTS. $\tilde{W}_i Z$ is a mg under P.

Product rule: $d(\tilde{W}_i Z) = \tilde{W}_i \, dZ + Z \, d\tilde{W}_i + d[\tilde{W}_i, Z]$.

$$= \tilde{W}_i \, dZ + Z(b_i \, dt + dW_i) + (-Z b_i \, dt)$$

$$= \tilde{W}_i \, dZ + Z \, dW_i$$ which is a mg.

(Since Z is a mg). QED.

Risk Neutral Pricing:

1. Risky asset. Price given by a ~~GBM~~. generalized GBM.
$\underbrace{\text{Risky asset}}_{\text{Stock}}$.

$$dS = \alpha(t) S \, dt + \sigma(t) S \, dW(t).$$

$\alpha, \sigma \to 2$ adapted processes.

$\boxed{\sigma > 0}$ (assumption).

2. Money Market acount: return rate is $R(t)$.
R is an adapted process.

Discount Process: $D(t) = e^{-\int_0^t R(s)\,ds}$.

$$\boxed{\text{Note } dD(t) = -R(t)D(t)\,dt}$$

<u>Def</u>: A <u>risk neutral measure</u> is a measure that is equivalent to $\mathbb{P}$ under which the process

$$D(t)S(t) \text{ is a mg.}$$

① Existence of RNM $\Longleftrightarrow$ no arbitrage.

② Uniqueness of RNM $\Longleftrightarrow$ Every derivative sec can be hedged.

FTAP

Compute RMM in our market:

$$d(DS) = D\,dS + S\,dD + d\cancel{[S,D]}^{\,\to 0}$$

$$= DS(\alpha\,dt + \sigma\,dW) - SRD\,dt$$

$$= \sigma SD\left(\left(\frac{\alpha - R}{\sigma}\right)dt + dW\right).$$

Let $\theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$. [Market price of risk].

$$d(DS) = \sigma SD\underbrace{(\theta\,dt + dW)}_{d\tilde{W}}$$

Let $d\tilde{W} = \theta\, dt + dW$. $(\tilde{W}(0) = 0)$.

<u>Girsanov:</u> $Z(t) = \exp\left(- \int_0^t \theta(s)\, dW(s) - \frac{1}{2} \int_0^t \theta(s)^2\, ds \right)$.

Fix $T > 0$. Let $d\tilde{P} = Z(T)\, dP$.

(Assume Z is a mg). Girsanov $\Rightarrow \tilde{W}$ is a BM under $\tilde{P}$.

$\boxed{\Rightarrow d(DS) = \sigma S D\, d\tilde{W}}$ <u>This is a mg under $\tilde{P}$!</u>

$\Rightarrow d\tilde{P} = Z(T)\, dP$ is a RNM

Theorem (Useful): Risk Neutral Pricing formula.

Fix. $T > 0$. Let $V(T)$ be any $\mathcal{F}_T$ meas R.V.

$$V(T) = \text{Payoff at maturity of a derivative security.}$$

Market $= \{$ Money market R + Stock S.

$\tilde{P}$ = Risk Neutral measure above. Then the arbitrage free price of this security is

$$V(t) = \tilde{E}\left(e^{-\int_t^T R(s)\,ds} V(T) \,\middle|\, \mathcal{F}_t \right).$$

Main Intuition: Under a RNM. any discounted hedg'g Pf. is a mg !!

Reason:

$$dS = \alpha(t) S\, dt + \sigma S\, dW.$$

$$= \cancel{\alpha(t)}\, \alpha S\, dt + \sigma S \left(d\tilde{W} - \theta\, dt \right).$$

$$= \alpha S\, dt + \sigma S \left(d\tilde{W} - \frac{\alpha - R}{\sigma}\, dt \right).$$

$$= \underline{+ R S\, dt} + \sigma S\, d\tilde{W}$$

$M(t)$ = value of cash in money market, $dM = \underline{R M\, dt}$

Lemma: Let Δ be any adapted process.

$X(t)$ = wealth of a Pf
- (1) $\Delta(t)$ shares of stock.
- (2) Rest in M.M.

Self financing.

Then $\underbrace{D(t)\,X(t)}_{\text{Discounted wealth.}}$ is a mg under $\underbrace{\widetilde{P}}_{\text{RNM.}}$.

Proof: Know $dX(t) = \Delta(t)\, dS(t) + R(t)\big(X(t) - \Delta(t) S(t)\big)\, dt$.

$$= \underbrace{\Delta(RS\,dt} + \sigma S\, d\tilde{W}) + RX\,dt - \underbrace{R\Delta S\,dt}$$

$$\boxed{dX = RX\,dt + \Delta\sigma S\, d\tilde{W}}$$

$$d(DX) = D\,dX + X\,dD + 0$$

$$= \underbrace{D(RX\,dt} + \Delta\sigma S\, d\tilde{W}) + \underbrace{X(-RD\,dt)}$$

$$= D\Delta\sigma S\, d\tilde{W} \leftarrow \text{mg under } \tilde{P}!$$

Pf of the RN Pricing Formula:

Let $X(t)$ = wealth of the replicating Portfolio.

Lemma $\Rightarrow$ DX is a mg under $\widetilde{\mathbb{P}}$!

By def ① Price at time t $V(t) = X(t)$.

② At maturity $\cancel{X(t) =}\ X(T) = V(T)$.

$$\Rightarrow V(t) = X(t) = \frac{1}{D(t)} D(t)X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}\left(D(T)X(T) \mid \mathcal{F}_t\right).$$

$$= \widetilde{\mathbb{E}}\left(\frac{D(T)}{D(t)} V(T) \,\Big|\, \mathcal{F}_t\right) = \widetilde{\mathbb{E}}\left(e^{-\int_t^T R(s)\,ds}\, V(T) \,\Big|\, \mathcal{F}_t\right) \quad \text{QED.}$$

Remark: Say $V(T) = f(S_T)$ (Eg. European Call).

Markov Prop $\implies V(t) = c(t, S_t)$

Ito: $d(c(t, S(t))) = \partial_t c \, dt + \boxed{\partial_x c \, dS} + \frac{1}{2} \partial_x^2 c \, d[S, S]$.

Also $dV = dX = \boxed{\Delta(t) \, dS} + R(\quad) dt$

Delta Hedging rule: $\Delta(t) = \partial_x c(t, S(t))$.

Eg: Black Scholes formula.

$$\alpha(t) \longrightarrow \text{anything}.$$

$$\sigma(t) \longrightarrow \text{constant}.$$

$$R(t) \longrightarrow \text{constant} \longleftarrow R(t) = r.$$

European call, strike K: $V(T) = (S(T) - K)^+$

$$RNP \Rightarrow c(t, S(t)) = \tilde{E}\left(e^{-r(T-t)}(S(T)-K)^+ \,\middle|\, \mathcal{F}_t\right).$$

Know $S(t) = \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}(t)\right) S(0).$

$$c(t, S(t)) = e^{-r\tau}\,\tilde{E}\left(\left(\exp\left(\left(r - \frac{\sigma^2}{2}\right)\cancel{t}\,T + \sigma \tilde{W}(T)\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right)$$

$$= e^{-r\tau}\,\tilde{E}\left(\left(\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}(T) - \tilde{W}(t)\right) + \sigma \tilde{W}(t)\right) \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)t\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right)$$

$$= e^{-r\tau}\,\tilde{E}\left(\left(S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\underbrace{\tilde{W}(T) - \tilde{W}(t)}_{N(0,\,T-t)}\right)\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right)$$

Independence lemma, simplify & get Black Scholes!