1. Change of Measures & Girsanov Theorem

2. Lévy Characterization of B.M.

3. Theory of Risk Neutral Pricing

   ① Replication $\rightarrow$ X replicates V_T

   ② Switch to Q $\leftarrow$ risk neutral measure.

   Under Q, $\{e^{-rt} S_t\}$ is m'gale

   $\Rightarrow$ $\{e^{-rt} X_t\}$ is also a m'gale

$$V_t = X_t = e^{rt} e^{-rt} X_t = e^{rt} \mathbb{E}^{Q}(e^{-rT} X_T | \mathcal{F}_t)$$

$$\boxed{V_t = \mathbb{E}^{Q}\left(e^{-r(T-t)} V_T | \mathcal{F}_t\right)}$$

Ex 1 $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, W

$$Z_1 = W_1^2 \geq 0, \quad \mathbb{E}(Z_1) = 1$$

Define Q via $\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_1} = Z_1$

$$Q(A) = \mathbb{E}(Z_1 \mathbb{1}_A) \quad A \in \mathcal{F}_1$$

<u>Goal</u>: Identity the dynamics of W, under Q

Girsanov: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, W, Θ

$$\tilde{Z}_t = e^{-\int_0^t \Theta_s dW_s - \frac{1}{2}\int_0^t \Theta_s^2 ds}$$

≥ 0 ← stochastic exponential

$$\mathbb{E}(Z_T) = \mathbb{E}(Z_0) = 1$$

$$\left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T$$. Then, under Q,

$$W_t^Q = W_t - \int_0^t \Theta_s ds$$ is a Q-B.M.

$$dW_t = dW_t^Q + \boxed{\Theta_t} dt$$

$T = 1, \quad Z_1 = W_1^2.$

$$Z_t = \mathbb{E}(Z_1 | \mathcal{F}_t) = \mathbb{E}(W_1^2 | \mathcal{F}_t) = \mathbb{E}(W_1^2 - 1 | \mathcal{F}_t) + 1$$

$$= \underline{W_t^2 - t + 1}$$

$$\left.\begin{aligned} dZ_t &= 2W_t\, dW_t \\ \underline{d\tilde{Z}_t} &\underline{= -\tilde{Z}_t \Theta_t\, dW_t} \end{aligned}\right\} \not\Rightarrow \quad \underline{2W_t = -\tilde{Z}_t \Theta_t}$$

$$\Theta_t = -\frac{2W_t}{\tilde{Z}_t}$$

$$d \log Z_t = \frac{1}{Z_t} dZ_t - \frac{1}{2} \frac{1}{Z_t^2} d[Z,Z]_t$$

$$= \frac{2W_t}{\underline{W_t^2 - t + 1}} dW_t - \frac{1}{\underline{2(W_t^2 - t + 1)^2}} \underline{4W_t^2}\, dt$$

$$d(\log \tilde{Z}_t) = \frac{1}{\tilde{Z}_t} d\tilde{Z}_t - \frac{1}{2\tilde{Z}_t} d[\tilde{Z},\tilde{Z}]_t$$

$$= \underline{-\Theta_t}\, dW_t - \frac{1}{2}\underline{\Theta_t^2}\, dt$$

$$\boxed{\Theta_t = -\frac{2W_t}{W_t^2 - t + 1}}$$

Under Q,

$$W_t^Q = W_t - \int_0^t \Theta_s\, ds \quad \text{is a } Q\text{-B.M.}$$

$$\boxed{dW_t = dW_t^Q + \frac{2W_t}{W_t^2 - t + 1} dt}$$

Ex2: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, (W, B).

$$[W,B]_t = \int_0^t \rho_s \, ds, \quad \rho \text{ adapted}, \ -1<\rho<1.$$

$$\left[X_t = \int_0^t \Delta_s \, dW_s + \int_0^t \Gamma_s \, dB_s, \ W \right].$$

Find Δ, Γ, s.t. (X, W) is a 2-dim B.M.

Apply Lévy:

$$[X,X]_t = t$$

$$[X,W]_t = 0$$

$$[X,X]_t = \int_0^t \Delta_s^2 \, ds + \int_0^t \Gamma_s^2 \, ds + 2\int_0^t \Delta_s \Gamma_s \rho_s \, ds = t$$

$$\Delta_t^2 + \Gamma_t^2 + 2\Delta_t \Gamma_t \rho_t = 1 \quad - \quad (1)$$

$$[X,W]_t = \int_0^t \Delta_s \, ds + \int_0^t \Gamma_s \rho_s \, ds = 0$$

$$\Delta_t + \Gamma_t \rho_t = 0 \quad - \quad (2)$$

$$\Rightarrow \begin{cases} \Delta_t = -\dfrac{\rho_t}{\sqrt{1-\rho_t^2}} \\[2ex] \Gamma_t = \dfrac{1}{\sqrt{1-\rho_t^2}} \end{cases} \checkmark$$

$$\left[\int_0^t a_s\,dW_s, \int_0^t b_s\,dB_s\right] = \int_0^t a_s b_s\, d[W,B]_s$$

$$Z_1 = X_1^2 \geq 0, \quad \mathbb{E}(Z_1) = 1 \quad \left.\frac{dQ}{d\mathbb{P}}\right|_{\mathcal{F}_1} = Z_1$$

Show, under Q, W_1, X_1 are independent.

$$Q(W_1 \in A, X_1 \in B) = \underline{Q(W_1 \in A)\, Q(X_1 \in B)} \quad \checkmark$$

$$LHS = \mathbb{E}^Q\left(\mathbb{1}_{\{W_1 \in A\}} \mathbb{1}_{\{X_1 \in B\}}\right) = \mathbb{E}^{\mathbb{P}}\left(X_1^2\, \mathbb{1}_{\{W_1 \in A\}} \mathbb{1}_{\{X_1 \in B\}}\right)$$

$$= \mathbb{E}^{\mathbb{P}}\left(X_1^2\, \mathbb{1}_{\{X_1 \in B\}}\right) \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\{W_1 \in A\}}\right)$$

$$= \mathbb{E}^Q\left(\mathbb{1}_{\{X_1 \in B\}}\right) \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\{W_1 \in A\}}\right)$$

$$= Q(X_1 \in B)\, \underline{\mathbb{P}(W_1 \in A)}$$

$$Q(W_1 \in A) = \mathbb{E}^{\mathbb{P}}\left(X_1^2 \, \mathbb{1}_{\{W_1 \in A\}}\right) = \underline{\mathbb{E}^{\mathbb{P}}(X_1^2)}\, \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\{W_1 \in A\}}\right)$$

$$= \mathbb{P}(W_1 \in A)$$

3° Risk-Neutral pricing

$$V_t = \mathbb{E}^Q\left(e^{-r(T-t)} V_T \mid \mathcal{F}_t\right)$$

<u>Ex1</u>: (call on call)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \qquad \frac{dB_t}{B_t} = r dt$$

Q r.n.m. Under Q: $\dfrac{dS_t}{S_t} = r dt + \sigma dW_t^Q$

$C(t, x, K, T)$ = price of European call option at time t, given $S_t = x$

$$= \mathbb{E}^Q\left(e^{-r(T-t)} (S_T - K)^+ \mid \mathcal{F}_t, S_t = x\right)$$

Fix $T_0 \leq T$

strike

$$\hat{C}(T_0, S_{T_0}) = \underline{\left(c(T_0, S_{T_0}, K, T) - \overset{\downarrow}{K'}\right)^+}$$

$\hat{C}(t, x)$ = price of call on call at t, given $S_t = x$.

<u>Q</u>:

$$\hat{C}(t,x) \ \begin{matrix} \leq ? \\ = \\ \geq \end{matrix} \ c(t, x, K, T)$$

$$\underline{\hat{C}(t, S_t)} = \mathbb{E}^Q\left(e^{-r(T_0-t)}\left(c(T_0, S_{T_0}, K, T) - K'\right)^+ \middle| \mathcal{F}_t\right)$$

$$\leq \mathbb{E}^Q\Big(\underbrace{e^{-rT_0}\, c(T_0, S_{T_0}, K, T)}\ \Big|\ \mathcal{F}_t\Big)\, e^{rt}$$

$$= \cancel{e^{-rt}}\, c(t, S_t, K, T) \cdot \cancel{e^{rt}}$$

$t \approx T_0$, what's the optimal hedging strategy?

$$\Delta_t = \hat{C}_x(t, S_t)$$

Look at $\underline{\hat{C}(t,x)} \cong \big(\underbrace{c(t,x,K,T)} - K'\big)^+$

$$\left.\begin{array}{l} c(t,0,K,T) = 0 \\ \lim\limits_{x\to\infty} c(t,x,K,T) = \infty \end{array}\right\} \Rightarrow \hat{x} \text{ s.t.}$$

$$c(t,\hat{x},K,T) = K'$$

$$\hat{C}(t,x) \approx \begin{cases} c(t,x,K,T) - K' & x > \hat{x} \\ 0 & x \le \hat{x} \end{cases}$$

$$\hat{C}_x(t,x) = \begin{cases} c_x(t,x,K,T) & x > \boxed{\hat{x}} \\ 0 & x \le \hat{x} \end{cases}$$

Ex2: (forward start option)

Fix $t_1 \in [0, T]$.

payoff at $T = (S_T - S_{t_1})^+$

Goal: Find a.f.p of this option at $t=0$.

Let $\mathbb{Q}$ r.n.m.

$$\underline{V_0 = \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(S_T - S_{t_1})^+\right)}$$

$=$ in term of C, some functions of t.

$C(\tau, x, K)$ = Price of European call on S, with time to maturity τ, strike K, given $S_{T-\tau} = x$.

$$C(\tau, x, K) = \mathbb{E}^{Q}\left(e^{-r\tau}(S_T - K)^+ \,\middle|\, \mathcal{F}_{T-\tau}, S_{T-\tau} = x \right)$$

(a) Show $\underline{C(\tau, \lambda x, \lambda K) = \lambda C(\tau, x, K)}$

Under Q, $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t^Q}$

$$\Rightarrow \underline{S_T = S_{T-\tau} e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(W_T^Q - W_{T-\tau}^Q)}}$$

$$C(\tau, x, K) = \mathbb{E}^{Q}\left(e^{-r\tau}\left(x e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(\underbrace{W_T^Q - W_{T-\tau}^Q}) } - K\right)^+ \,\middle|\, \underline{\mathcal{F}_{T-\tau}, S_{T-\tau} = x} \right)$$

$$= \mathbb{E}^{Q}\left(e^{-r\tau}\left(x e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(\underbrace{W_T^Q - W_{T-\tau}^Q})} - K\right)^+ \right)$$

$$\underline{W_T^Q - W_{T-\tau}^Q} \overset{d}{\sim} \underline{W_\tau^Q}$$

$$V_0 = \mathbb{E}^Q\left(e^{-rT}(S_T - S_{t_1})^+\right)$$

$$= \mathbb{E}^Q\left(\underbrace{e^{-rt_1} S_{t_1}}\ \underbrace{e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}}-1\right)^+}\right)$$

$$= \mathbb{E}^Q\left(e^{-rt_1} S_{t_1}\ \underbrace{\mathbb{E}^Q\left(e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}}-1\right)^+ \Big| \mathcal{F}_{t_1}\right)}_{(*)}\right)$$

$$\boxed{\frac{S_T}{S_{t_1}}} = e^{(r-\frac{1}{2}\sigma^2)(T-t_1) + \sigma(\underbrace{W_T^Q - W_{T-t_1}^Q})} \perp\!\!\!\perp \mathcal{F}_{t_1}$$

$$\boxed{S_{T-t_1}} = S_0 e^{(r-\frac{1}{2}\sigma^2)(T-t_1) + \sigma W_{T-t_1}^Q}$$

$$(*) = \mathbb{E}^Q\left(e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}}-1\right)^+\right) = \underbrace{\mathbb{E}^Q\left(e^{-r(T-t_1)}\left(S_{T-t_1}-S_0\right)^+\right)}\frac{1}{S_0}$$

$$= C(T-t_1, S_0, S_0)\frac{1}{S_0} = C(T-t_1, 1, 1)$$

$$V_0 = \mathbb{E}^Q\left(e^{-rt_1} S_{t_1}\right) C(T-t_1, 1, 1)$$

$$= S_0\, C(T-t_1, 1, 1) = \underline{C(T-t_1, S_0, S_0)}$$

<u>Ex3</u>: In BS Model

$C(t, x, r, \sigma^2)$ = price of European call.

$$S \sim GBM(r, \sigma^2)$$

Price the call option on S_T^2:

$$(S_T^2 - K)^+$$