Mistake from last time:

$\cancel{M}$, M, N ind, mg, cts. $\Longrightarrow [M,N] = 0$

(Correct).

$\rightarrow$ Last time $\longrightarrow$ Wrong proof.

not true in general.

Wrong $E\left(M(t)N(t) \mid \mathcal{F}_s\right) = E\left(M(t) \mid \mathcal{F}_s\right) E\left(N(t) \mid \mathcal{F}_s\right)$.

$$[M,N](T) = \lim_{\|P\| \to 0} \sum \underbrace{\left(M(t_{i+1}) - M(t_i)\right)}_{\Delta_i M} \underbrace{\left(N(t_{i+1}) - N(t_i)\right)}_{\Delta_i N}.$$

$$E\left(\sum \Delta_i M \, \Delta_i N\right)^2 = \sum E(\Delta_i M)^2(\Delta_i N)^2 + \boxed{2\sum_{i<j} E(\Delta_i M \, \Delta_i N)(\Delta_j M)(\Delta_j N)}.$$

Tower prop $\Rightarrow 0$ (Uses M & N are Ind).

$$E\sum (\Delta_i M)^2(\Delta_i N)^2 \le E \underbrace{\max_i (\Delta_i M)^2}_{\to\, 0 \text{ (M is cts).}} \underbrace{\boxed{\sum_i (\Delta_i N)^2}}_{\to\, [N,N]}$$

$\Rightarrow [M,N] = 0$

Last time: Multi D Itô $X_1, \dots X_n$ Ito proc.

$f \to C^1$ in t & C^2 in each x_i

$$d\left(f(t, X(t))\right) = \partial_t f \, dt + \sum \partial_i f \, dX_i + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f \, d[X_i, X_j].$$

d-dim BM: $W = (W_1, W_2, \dots W_d)$.

(1) Each W_i is a 1d BM.

& (2) W_i ind of W_j for $i \neq j$

$\Rightarrow [W_i, W_j] = \mathbb{1}_{\{i=j\}} t$

$\Rightarrow$ that $d(f(t, W(t))) = \partial_t f \, dt + \sum \partial_i f \, dW_i(t)$

$+ \frac{1}{2} \Delta f \, dt.$

$\Delta f = \text{Laplacian } f = \sum_{i=1}^{d} \partial_i^2 f.$

Waming Eg: $d = 2$. $f(x) = \ln |x|^2 =$

$x = (x_1, x_2)$. $|x|^2 = x_1^2 + x_2^2$.

You compute: $\partial_i f = \frac{2}{|x|} \frac{x_i}{|x|} = \frac{2x_i}{|x|^2}$.

$\Delta f = \cdots = 0$ (You check).

Set $Y = \ln |W|^2$.

then $dY = 0\,dt + \sum \frac{2W_i}{|W|^2} dW_i + 0$

$$\Rightarrow \boxed{dY = \sum \frac{2W_i}{|W|^2} dW_i.} \leftarrow \text{Mg?}$$

Claim Y is not a Mg:

Compute $EY(t) = E \ln |W|^2 = \iint_{\mathbb{R}^2} \frac{1}{2\pi t} e^{-|x|^2/2t} \ln |x|^2 \, dx_1 \, dx_2$.

You check: Not constant $\Rightarrow$ Y can not be a Mg.

Warning: $\int_0^t \sigma \, dW$ is only guaranteed to be a Mg

if $E \int_0^t \sigma^2(s) \, ds < \infty$.

(Not true for $\dfrac{2W_i}{|W|^2} dW_i$.)

Thm: (Levy). Say $M_1, M_2, \dots M_d$ are d cts Martingales such that $M_i(0) = 0$

& $[M_i, M_j](t) = \mathbb{1}_{\{i=j\}} t$.

Then $M = (M_1, M_2, \dots M_d)$ is a BM !.

Idea: 1D: Compute the MGF M_i.

Review problems: $\sigma(s)$ is not random. $X(t) = \int_0^t \sigma(s)\, dW(s)$.

Compute $E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = e^{\lambda^2 \int_s^t \sigma(r)^2 dr}$.

$\Rightarrow X(t) - X(s)$ is ind of $\mathcal{F}_s$

& $X(t) - X(s) \sim N\left(0, \int_s^t \sigma^2(r)\,dr\right)$.

If $\sigma^2 = 1 \Rightarrow X(t) - X(s) \sim N(0, t-s)$.

BM.

Multi D $\rightarrow$ Repeat the same trick.

Risk Neutral ~~Pricing~~ Measures.

① Girsanov: →

<u>Def:</u> Two Probability measures P & $\tilde{P}$ are $\boxed{\text{equivalent}}$.

if $\forall$ events A, $P(A)=0 \iff \tilde{P}(A)=0$

Eg: Say Z is R.V. $Z>0$ & $EZ=1$.

Define $\tilde{P}(A) = E\,\mathbb{1}_A Z = \int_A Z\,dP$.

Notation: Say $d\tilde{P} = Z\,dP$. $\left(\int f\,d\tilde{P} = \int f\,Z\,dP \quad \forall \text{ RV } f\right)$.

Note P & $\tilde{P}$ are equiv.

① If $P(A) = 0 \Rightarrow \int_A Z\,dP = 0$.

② If $\tilde{P}(A) = 0 \Rightarrow \int_A Z\,dP = 0 \Rightarrow P(A) = 0$

$(\because Z > 0)$.

Note Need $EZ = 1$ to ensure

$$\tilde{P}(\Omega) = EZ = 1.$$

<u>Theorem</u>: (Radon - Nicodym). If P & $\tilde{P}$ are equiv.

then $\exists Z \ni Z > 0$, $EZ = 1$.

Z is a RV $\ni$ $d\tilde{P} = Z\,dP$

$$\left(\text{i.e. } \tilde{P}(A) = \int_A Z\,dP\right).$$

Z is called the density of $\tilde{P}$ wrt P.

(RN derivative).

Notation $Z = \dfrac{d\tilde{P}}{dP}$.

Notation: $\tilde{E}$ = Expectation / cond Exp wrt to the measure $\tilde{P}$.

$$\tilde{E}(X) = \int X \, d\tilde{P} = \int X Z \, dP$$

(if $d\tilde{P} = Z \, dP$).

Also: $\tilde{E}(X \mid \mathcal{F}) \longrightarrow$ is a $\mathcal{F}$ meas r.v. such that

$$\forall A \in \mathcal{F}, \quad \int_A \tilde{E}(X \mid \mathcal{F}) \, d\tilde{P} = \int_A X \, d\tilde{P}$$

Thm: Cameron - Martin - Girsanov theorem.

Let $b(t) = (b_1(t), b_2(t), \dots, b_d(t))$.

be a d-dimensional adapted process.

$W \longrightarrow$ a d dimensional BM.

Set $\tilde{W}(t) = W(t) + \int_0^t b(s)\,ds$. ← vector obtained by integrating each coordinate.

Let $Z(t) = \exp\left(-\int_0^t b(s) \cdot dW(s) - \frac{1}{2}\int_0^t |b(s)|^2 ds\right)$.

Fix $T > 0$. Define $\tilde{P}_T = \tilde{P}$ by

$$d\tilde{P} = Z(T)\, dP.$$

Then: If Z is a mg then $\tilde{W}$ is a B.M. under the measure $\tilde{P}$.

Remark 1: $\int_0^t b(s) \cdot dW(s) \stackrel{\text{def}}{=} \sum_i \int_0^t b_i(s)\, dW_i(s)$.

& $|b(s)|^2 = \sum_i b_i(s)^2$.

Rmk: b_i & b_j mea not be ind.

But under $\tilde{P}$, $\tilde{W}_i$ & $\tilde{W}_j$ are ind for $i \neq j$.

Remark 2: Note $Z(0) = 1$

Set $M(t) = \int_0^t \sigma(s) \cdot dW(s)$.

$$f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t |\sigma(s)|^2 \, ds\right).$$

$Z(t) = f(t, M(t))$. $\partial_x f = -f$, $\partial_x^2 f = +f$.

& $\partial_t f = \cancel{f} \cdot \left(-\frac{1}{2}|\sigma(t)|^2\right) f$.

$$\Rightarrow dZ = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2}\partial_x^2 f \, d[M,M].$$

$$= -\frac{1}{2}|\sigma(t)|^2 Z(t)\, dt - Z\, \sigma(t) \cdot dW(t) + \frac{1}{2} Z(t) |\sigma(t)|^2 dt$$

$$\Rightarrow dZ = -Z\, b(t) \cdot dW(t)$$

$$\Rightarrow dZ = -\sum_{i=1}^{d} Z\, b_i(t)\, dW_i(t)$$

Note Only guaranteed Z is a mg if $\boxed{E\int_0^t Z(s)^2 b_i(s)^2\, ds < \infty}$

Note Z a mg $\Rightarrow E\,Z(t) = E\,Z(0) = 1.$

Idea behind the Proof:

Note $[\tilde{W}_i, \tilde{W}_j] = [W_i, W_j] = 1_{\{i=j\}} t$, & $\tilde{W}$ is cts.

To prove Girsanov only NTS $\tilde{W}$ is a mg under $\tilde{P}$.

Lemma: Let $0 \leq s \leq t \leq T$.

X is $\mathcal{F}_t$ - meas. $d\tilde{P} = Z(T)\, dP$.

Z is a mg (under P).

Then $\tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t) X \mid \mathcal{F}_s)$.

Pf: Let $A \in \mathcal{F}_s$.

$$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} = \int_A X \, d\tilde{P} = \int_A X Z(T)\, dP$$

$$= \int_A E(X Z(T) \mid \mathcal{F}_t)\, dP$$

$$= \int_A X\cancel{(t)}\, Z(t)\, dP \qquad (\because Z \text{ is a mg } \& \ X \text{ is } \mathcal{F}_t \text{ meas}).$$

$$= \int_A E(X\cancel{(t)}\, Z(t) \mid \mathcal{F}_s)\, dP \quad \ldots\ldots (1).$$

Also $\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} = \int_A \tilde{E}(X \mid \mathcal{F}_s)\, Z(T)\, dP$

$$= \int_A E\left(\tilde{E}(X \mid \mathcal{F}_s)\, Z(T) \,\Big|\, \mathcal{F}_s \right) dP.$$

$$= \int_A Z(s)\, \tilde{E}(X \mid \mathcal{F}_s)\, dP. \quad \cdots \quad (2).$$

$\Rightarrow \forall A \in \mathcal{F}_s, \ \int_A Z(s)\, \tilde{E}(X \mid \mathcal{F}_s)\, dP = \int_A E\left(X Z(t) \,\big|\, \mathcal{F}_s\right) dP.$

$\Rightarrow Z(s)\, \tilde{E}(X \mid \mathcal{F}_s) = E\left(X Z(t) \,\big|\, \mathcal{F}_s\right).$

QED.

Lemma 2: An adapted process M is a mg under $\tilde{P}$

$\iff Z(t)M(t)$ is a mg under P

Pf: ① Say $Z(t)M(t)$ is a mg under P.

NTS M is a mg under $\tilde{P}$

$\iff$ NTS $\tilde{E}(M(t)|\mathcal{F}_s) = M(s)$.

Note $\tilde{E}(M(t)|\mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t)M(t)|\mathcal{F}_s)$.

$= \frac{1}{\cancel{Z(s)}} \cancel{Z(s)} M(s) = M(s)$ QED.