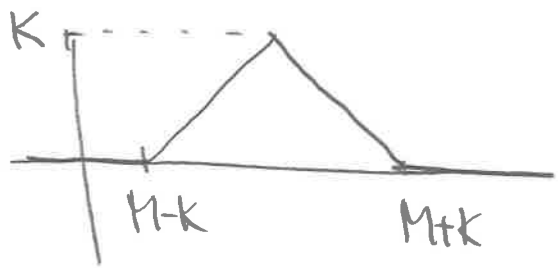
1° Black-Scholes Model

- risky asset $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
- risk-free asset $\frac{dB_t}{B_t} = r dt$

Goal: price financial derivatives.

e.g.
- European call $(S_T - K)^+$

- Butterfly option $(M - |S_T - K|)^+$.



- asian option $\frac{1}{T}\int_0^T S_t dt$
- forward start option $(S_T - S_{t_1})^+$

- Arbitrage free price.

Start $\boxed{x}$ ← initial captial

invest into the market (S, B)

If at T, $\underline{(S_T - K)^+}$ $\Longrightarrow$ fair price $= X$

Need to find $\left\langle \begin{array}{l} X \text{ - initial captial} \\ \Delta \text{ -trading strategy.} \quad \Delta_t = \# \text{ of shares} \end{array} \right.$

<u>Method</u>: Replication

Wealth Process

$$- X_t^{x,\Delta} = \Delta_t S_t + (X_t^{x,\Delta} - \Delta_t S_t)$$

$$dX_t = \Delta_t dS_t + \frac{X_t^{x,\Delta} - \Delta_t S_t}{B_t} dB_t$$

$$= \Delta_t(\mu S_t dt + \sigma S_t dW_t) + (X_t - \Delta_t S_t) r\, dt$$

$$= [r X_t + (\mu - r)\Delta_t S_t]\, dt + \sigma \Delta_t S_t dW_t$$

Value of call at t, $c(t, S_t)$

$$dc(t,S_t) = \left\{ c_t(t,S_t) + c_x(t,S_t)\,\mu S_t + \frac{1}{2} c_{xx}(t,S_t)\,\sigma^2 S_t^2 \right\} dt$$

$$+ \, c_x(t,S_t)\,\sigma S_t \, dW_t$$

Matching dt term & dW_t term.

$$\begin{cases} \underline{c_t(t,x) - r\,c(t,x) + c_x(t,x)\,r x + \frac{1}{2} c_{xx}(t,x)\,\sigma^2 x^2 = 0} \\ c(T,x) = (x-K)^+ \\ \Delta_t = c_x(t,S_t) \end{cases}$$

<u>Ex1</u>: $X = S_0$,

Invest into call & Bank account

Goal replicate S_t

$$X_t = \Gamma_t\, c(t,S_t) + \left(X_t - \Gamma_t\, c(t,S_t)\right) \overset{\text{Goal}}{\equiv} S_t$$

$$dX_t = \Gamma_t\, dc(t,S_t) + \frac{X_t - \Gamma_t\, c(t,S_t)}{B_t}\, dB_t$$

<u>$d(e^{-rt}X_t)$</u>

$$= \Gamma_t \Big(\left[c_t(t,S_t) + c_x(t,S_t)\,\mu S_t + \tfrac{1}{2} c_{xx}(t,S_t)\,\sigma^2 S_t^2 \right] dt + \underline{c_x(t,S_t)\,\sigma S_t\, dW_t} \Big) + r\left(\underline{X_t} - \Gamma_t\, c(t,S_t)\right) dt$$

$$= \Big\{ r S_t + \Gamma_t \Big(\underbrace{c_t(t,S_t) + \overset{(r)}{\mu} S_t\, c_x(t,S_t)} + \tfrac{1}{2}\sigma^2 S_t^2\, c_{xx}(t,S_t) - \underline{r\, c(t,S_t)} \Big) \Big\}\, dt$$

$$+ \Gamma_t\, c_x(t,S_t)\, \sigma S_t\, dW_t$$

(BS PDE)

$$= \left\{ \underline{rS_t + \Gamma_t (\mu - r) S_t C_x(t, S_t)} \right\} dt + \underline{\Gamma_t C_x(t, S_t)\sigma S_t} dW_t$$

$$dS_t = \underline{S_t \mu} dt + \underline{S_t \sigma} dW_t.$$

$$\begin{cases} \Gamma_t C_x(t, S_t) \sigma S_t = \sigma S_t \\ rS_t + \Gamma_t (\mu - r) S_t C_x(t, S_t) = S_t \mu \end{cases}$$

$$\boxed{\Gamma_t = \frac{1}{C_x(t, S_t)}} = \frac{1}{\Delta_t}$$

Assume $S \sim GBM(r, \sigma^2)$ $\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t$.

$$d\left[e^{-rt} c(t, S_t)\right] = e^{-rt}\Big[\underbrace{-rc + c_t + \mu S_t c_x + \tfrac{1}{2}\sigma^2 S_t^2 c_{xx}}_{=0}\Big]dt$$

$$+ \; e^{-rt} \sigma S_t c_x \, dW_t$$

$\Rightarrow$ $\underline{e^{-rt} c(t, S_t)}$ is a m'gale!

$$\mathbb{E}\left(e^{-rT} c(T, S_T) \mid \mathcal{F}_t\right) = \underline{e^{-rt} c(t, S_t)} \quad (*)$$

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} \implies \frac{S_T}{S_t} = e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

$$\begin{aligned}
LHS &= \mathbb{E}\left(e^{-rT}(S_T - K)^+ \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{E}\left(e^{-rT}\left(\underline{S_t} e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\underline{(W_T - W_t)}} - K\right)^+ \,\middle|\, \mathcal{F}_t\right) \\
&= \underline{g(S_t)}
\end{aligned}$$

$$g(x) = \mathbb{E}\left(e^{-rT}\left(x e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)} - K\right)^+\right)$$

Given $S_t = x$,

$$c(t,x) = \mathbb{E}\left(e^{-r(T-t)}\left(x e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)} - K\right)^+\right)$$

$$= \ldots\ldots$$

$X = C(0, S_0) \leftarrow$ price of European call

Ex2: $S \sim GBM(r, \sigma^2)$

$$\underline{S^g_T} = e^{\underline{\frac{1}{T}\int_0^T \log(S_t)\,dt}}$$

Geometric average

Goal: $(S^g_T - K)^+$

Find the distribution of S^g_T.

$$e^{\frac{1}{T}\int_0^T \log S_t\,dt} \qquad \frac{1}{T}\int_0^T \log S_t\,dt = \frac{1}{T}\int_0^T \log\left(S_0\, e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}\right)dt$$

$$= \frac{1}{T}\int_0^T \log S_0 + \left(r-\tfrac{1}{2}\sigma^2\right)t + \sigma W_t\,dt$$

$$= \log S_0 + \frac{1}{2}\left(r-\tfrac{1}{2}\sigma^2\right)T + \underbrace{\frac{\sigma}{T}\int_0^T W_t\,dt}_{\sim N\left(0,\frac{\sigma^2 T}{3}\right)}$$

$$d(tW_t) = t\,dW_t + W_t\,dt \Rightarrow \int_0^T W_t\,dt = TW_T - \int_0^T t\,dW_t$$

$$= \int_0^T \underline{(T-t)}\,dW_t$$

$$\sim N\left(0, \int_0^T (T-t)^2\,dt\right) = N\left(0, \frac{T^3}{3}\right)$$

$$\overset{d}{=} S_0\, e^{\frac{1}{2}(r-\frac{1}{2}\sigma^2)T + \underbrace{N\left(0,\frac{\sigma^2 T}{3}\right)}_{=\frac{\sigma}{\sqrt{3}}\tilde{W}_T}}$$

$$\boxed{S_T^g} \overset{d}{=} S_0 e^{\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)T + \frac{\sigma}{\sqrt{3}}\tilde{W}_T}$$

$$= \underbrace{S_0 e^{\left(\tilde{r}-\frac{1}{2}\tilde{\sigma}^2\right)T + \tilde{\sigma}\tilde{W}_T}}_{GBM(\tilde{r}, \tilde{\sigma}^2)}$$

$$\Rightarrow \begin{cases} \tilde{\sigma} = \frac{\sigma}{\sqrt{3}} \\ \tilde{r} = \frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right) + \frac{\sigma^2}{6} = \frac{1}{2}r - \frac{1}{12}\sigma^2 \end{cases}$$

Ex 3: W, B indep B.M.

$$X_t = \frac{1}{2}(W_t - B_t)^2.$$

(a) dX_t

(b) $M_t = \mathbb{E}(X_1 | \mathcal{F}_t) = \underline{\underline{f}}(t, X_t)$, $0 \le t \le 1$
?

(c) $g(t, X_t)$ is a m'gale

(d) $C_t = h(t) X_t$

find h s.t. C_t has constant expectation

But C_t is not a m'gale.

(a) $dX_t = (W_t - B_t)\,dW_t - (W_t - B_t)\,dB_t \underbrace{+ d[W,B]_t}_{=0}$ $+\,dt$

$\mathbb{E}(W_t B_t \mid \mathcal{F}_s) = W_s B_s \Rightarrow [W,B] = 0$

$\parallel$

$\mathbb{E}\big((W_t - W_s + W_s)(B_t - B_s + B_s) \mid \mathcal{F}_s\big)$

$= \mathbb{E}\big((W_t - W_s)(B_t - B_s) + W_s(B_t - B_s) + B_s(W_t - W_s) + W_s B_s \mid \mathcal{F}_s\big)$

$= W_s B_s$

$WB - [W,B]$

(b) $\mathbb{E}\left(\frac{1}{2}(W_1 - B_1)^2 \mid \mathcal{F}_t\right) = \mathbb{E}\left(\frac{1}{2}W_1^2 + \frac{1}{2}B_1^2 - W_1 B_1 \mid \mathcal{F}_t\right)$

$= \cancel{\frac{1}{2} + W} \qquad - W_t B_t$

$W_t^2 - t$

$B_t^2 - t$

$$\mathbb{E}\left(\tfrac{1}{2}(W_1^2 - 1 + 1) + \tfrac{1}{2}(B_1^2 - 1 + 1) \,\middle|\, \mathcal{F}_t \right) - W_t B_t$$

$$= \tfrac{1}{2}(W_t^2 - t) + \tfrac{1}{2}(B_t^2 - t) + 1 - W_t B_t$$

$$= \underline{\tfrac{1}{2}(W_t - B_t)^2} - t + 1 = X_t - t + 1$$

$$\boxed{f(t,x) = x - t + 1}$$

$X^2 - [X,X]$ - m'gale

Ex: $XY - \underline{[X,Y]}$ m'gale $\underline{\int X\,dY} + \underline{\int Y\,dX}$

$$d(XY) = X\,dY + Y\,dX + d[X,Y].$$

$$
v_t(t,y) := \boxed{\gamma_1 e^{\gamma_1 t} c(T-t, y+\gamma_2(T-t)) + e^{\gamma_1 t}\left\{ -c_t(T-t, y+\gamma_2(T-t)) - \gamma_2 c_x(T-t, y+\gamma_2(T-t)) \right\}}
$$

$$
v_y(t,y) = e^{\gamma_1 t} c_x(T-t, y+\gamma_2(T-t))
$$

$$
v_{yy}(t,y) = e^{\gamma_1 t} c_{xx}(T-t, y+\gamma_2(T-t)) \cdot K
$$

$$
\boxed{\gamma_1 c - c_t + \gamma_2 c_x = K c_{xx}}
$$

$$
c(t, e^y)
$$

$$
c(t,x) \longrightarrow c(t, e^y) = f(t,y)
$$

$$
V(t,y) := e^{\gamma_1 t} f(T-t, y+\gamma_2(T-t))
$$