Last time: Black Scholes - Merton.

Set up: ① $dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$.

(Geom. B.M.). $\alpha \to$ mean return rate.

$\sigma \to$ volatility.

$S \to$ model for stock price.

European call: Strike K, Maturity T.

$c(x,t)$ = A.F.P. of the call at time t, given $S(t) = x$.

Common sense: $c(x,T) = (x-K)^+$ (b)

& $c(0,t) = 0$ (c)

BSM: (1) If ~~c(x,t)~~ $c(t,x)$ is AFP then.

(a) $$\left.\partial_t c + r x \partial_x c + \frac{\sigma^2 x^2}{2} \partial_x^2 c = r c \right\} \text{Last time.}$$

[r = interest rate in M.M.]

(2) If c satisfies (a) (b) & (c) then

$c(t, S(t))$ is the AFP.

Last time: $\cancel{X(t)}$ for 1. Let $X(t)$ = value of R. Pf.

→ hold $\Delta(t)$ shares of stock.

& $X(t) - \Delta(t) S(t)$ in cash.

If c = AFP. Equate $\boxed{c(t, S(t)) = X(t)}$

Itô & equate dt & dW terms. $\Rightarrow$ (a)

<u>CONVERSE</u> (Part 2).

Will $\boxed{\text{construct}}$ a R. Pf.

Let $X(t)$ = value of a pf.

$\Delta(t)$ shares of stock.

$X(t) - \Delta(t)S(t)$ in M.M.

Choose $\Delta(t) = \partial_x c(t, S(t))$.

~~Claim~~: Choose $X(0) = \cancel{c(t,} \underline{c(0, S(0))}$

Know $dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)dt$

$\Rightarrow dX(t) = \partial_x c(t, S(t))\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)dt$

Claim: $X(t) = c(t, S(t)) \quad \forall\, t < T$

($\Rightarrow$ By continuity $X(T) = c(T, S(T))$

$\Rightarrow X$ is a R. Pf.).

Pf: Compute $d(e^{-rt} X(t)) =$.

$f(t,x) = e^{-rt} x$.

$\partial_t f = -r e^{-rt} x$

$\partial_x f = e^{-rt}$

$\partial_x^2 f = 0$

$$d(e^{-rt} X(t)) = -r e^{-rt} X \, dt + e^{-rt} dX.$$

$$\Rightarrow d\left(e^{-rt}X(t)\right) = -re^{-rt}X\,dt + \cancel{e^{-rt}}$$

$$+\, e^{-rt}\left(\partial_x c\, dS(t) + r\left(X(t) - \partial_x c\, S(t)\right)dt\right).$$

$$= \cancel{\partial_x c\, dS}\; e^{-rt}\partial_x c\, dS - re^{-rt}\partial_x c\, S\, dt$$

$$= e^{-rt}\partial_x c\, S\sigma\, dW + e^{-rt}\partial_x c\,(\alpha - r)S\,dt.$$

$$= e^{-rt}\partial_x c\, S\sigma\, dW + e^{-rt}\partial_x c(\alpha - r)S\,dt.$$

Compute $d\left(e^{-rt} c(t, S(t))\right) = -r e^{-rt} c \, dt + e^{-rt} dc$

$$= -r e^{-rt} c \, dt + e^{-rt} \left(\partial_t c \, dt + \partial_x c \, dS + \frac{1}{2} \partial_x^2 c \, d[S,S] \right).$$

$$= e^{-rt} \left(-rc + \partial_t c + \partial_x c \, \alpha S + \frac{\sigma^2}{2} \partial_x^2 c \, S^2 \right) dt$$

$$+ e^{-rt} \partial_x c \, \sigma S \, dW$$

$$= e^{-rt} \left(-r S \partial_x c + \partial_x c \, \alpha S \right) dt + e^{-rt} \partial_x c \, \sigma S \, dW$$

(by (a))

$$= d\left(e^{-rt} X(t)\right).$$

$$\Rightarrow d\left(e^{-rt}c(t,S(t))\right) = d\left(e^{-rt}X(t)\right) \quad (t<T).$$

Since $X(0) = c(0,S(0))$. $\Rightarrow X(t) = c(t,S(t))$

~~$\forall t \geq 0$~~

$\forall t \in [0,T)$.

$\Rightarrow$ QED.

Solve ⓐ ⓑ & ⓒ get.

$$c(x,t) = x N(d_+) - K e^{-r(T-t)} N(d_-)$$

$$d_\pm = d_\pm(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right)$$

Put Call Parity:

$p(t,x) =$ AFP. Put option.

maturity T, strike K.

$$c(t,x) - p(t,x) = S(t) - Ke^{-r(T-t)}.$$

$$c(T,x) - p(T,x) = (S-K)^+ - (S(T)-K)^-$$

$$= \underbrace{S(T) - K}_{\text{forward contract}}.$$

"Greeks" $\longrightarrow$ partials of c.

① Delta: $\partial_x c$ $\boxed{\Delta(t) = \partial_x c(t, S(t))}$

Compute $\partial_x c = N(d_+) + x N'(d_+) d_+'$

$$- K e^{-r(T-t)} N'(d_-) d_-'$$

Turns out $x N'(d_+) d_+' = K e^{-r(T-t)} N'(d_-) d_-'$

(You check).

$$\boxed{\Rightarrow \partial_x c = N(d_+) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-y^2/2} \, dy}$$

② Gamma: $\partial_x^2 c = N'(d_+)\, d_+'$

$$= \frac{1}{x\sigma\sqrt{2\pi\tau}} e^{-d_+^2/2}. \qquad (\tau = T - t).$$

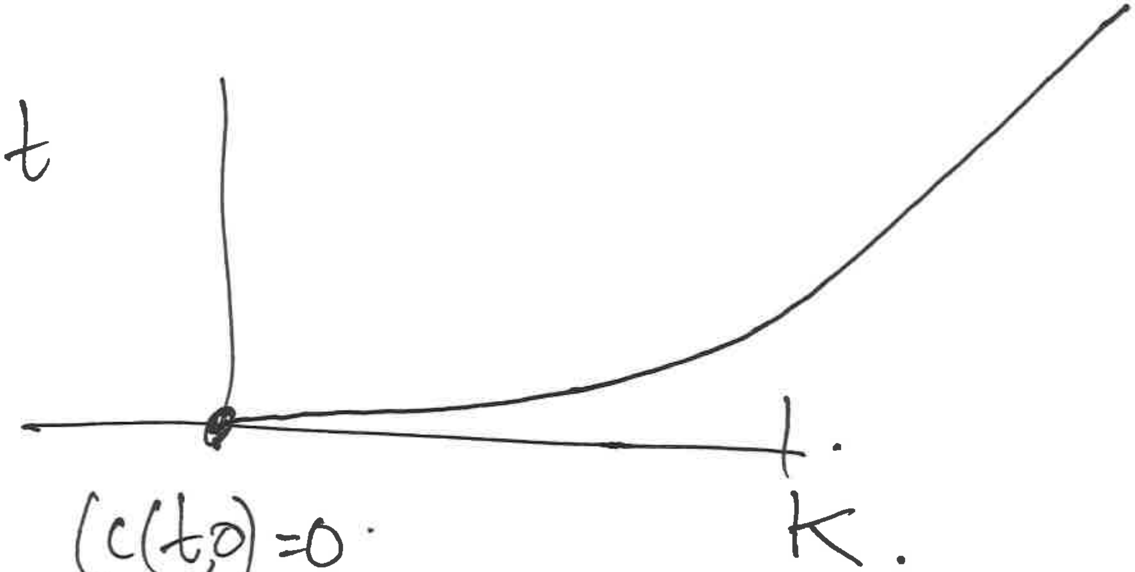
③ Theta: $\partial_t c = -rKe^{-r\tau} N(d_-) - \frac{\sigma x}{2\sqrt{\tau}} N'(d_+)$

Prop: ① c is inc as a fn of x ($\because \partial_x c > 0$).

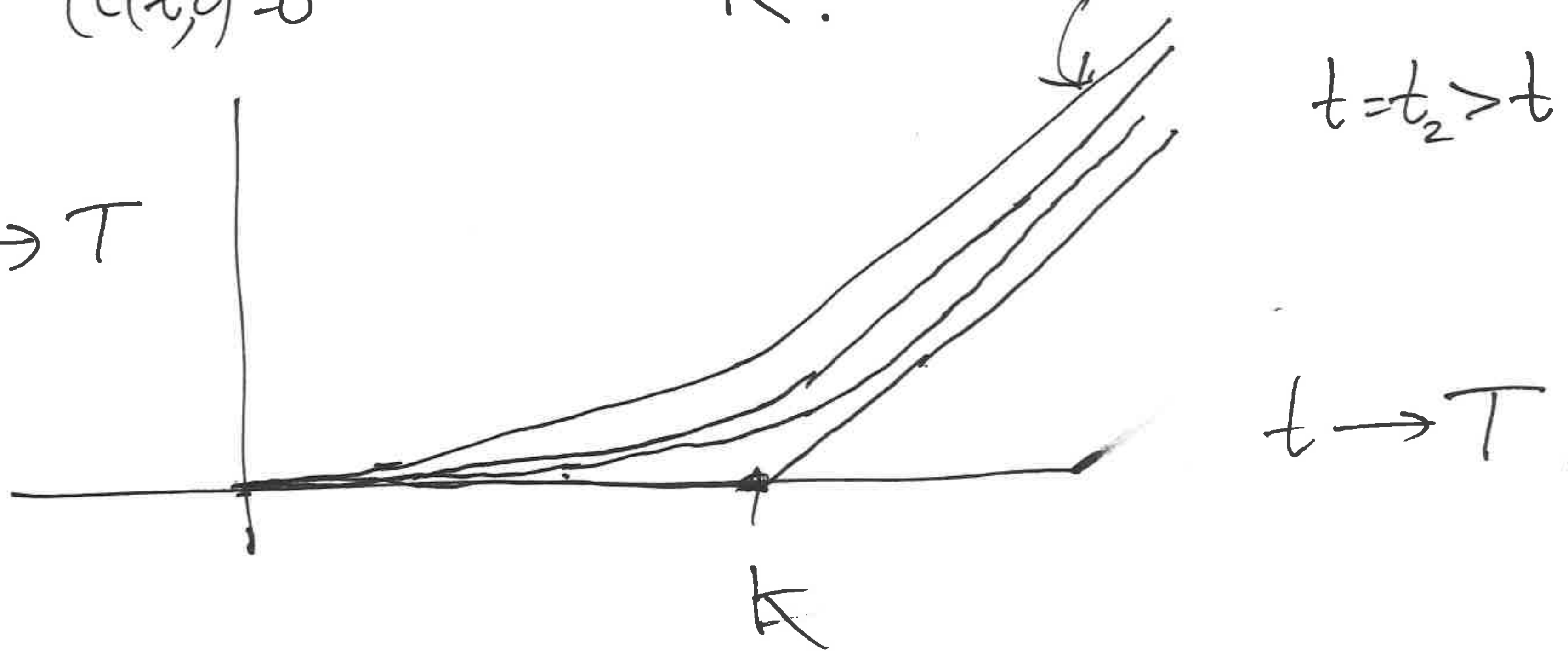
② c is convex as a fn of x ($\because \partial_x^2 c > 0$).

③ c is decreasing as a of t ($\because \partial_t c < 0$).

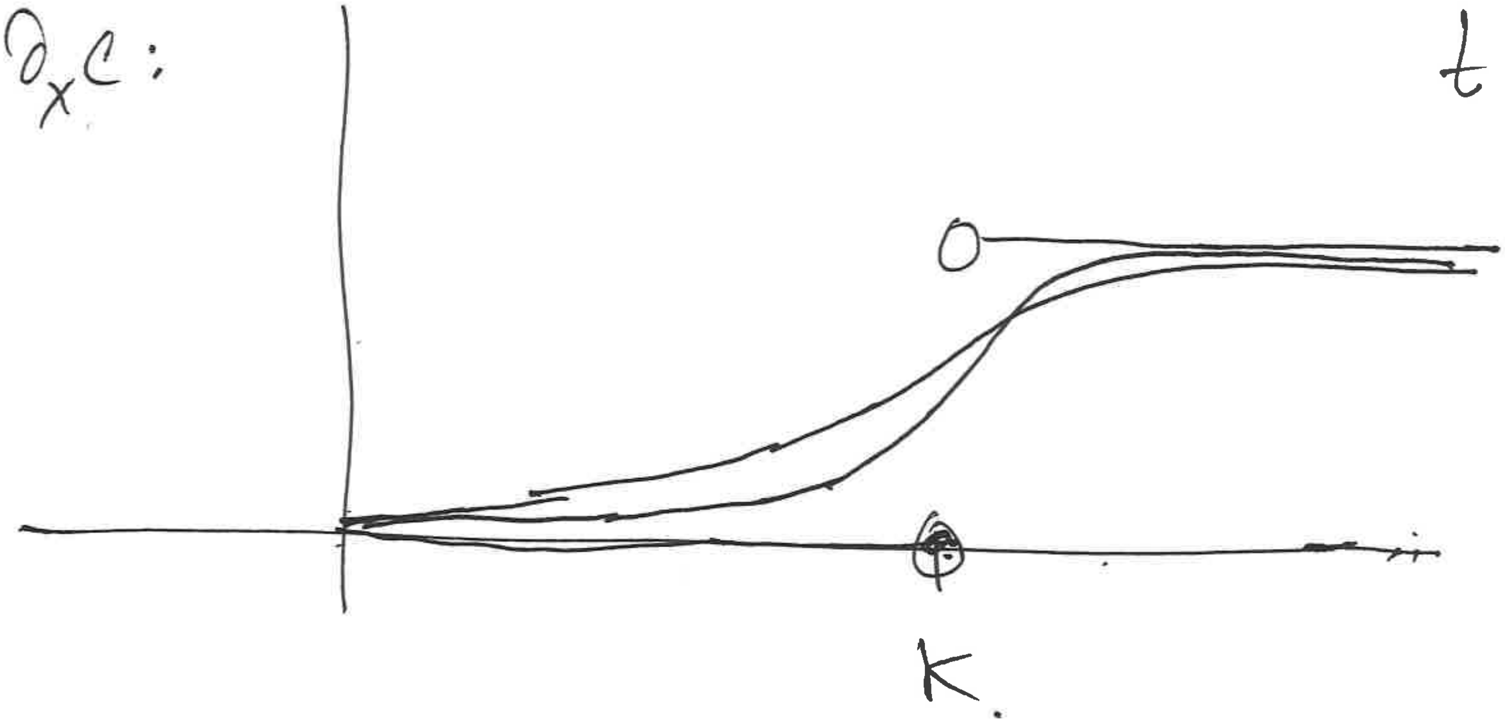
Fix t



$t \to T$



$\partial_x c$: $t = T$.



Hedging a short call: Sell a call option.
value $c(t,x)$.

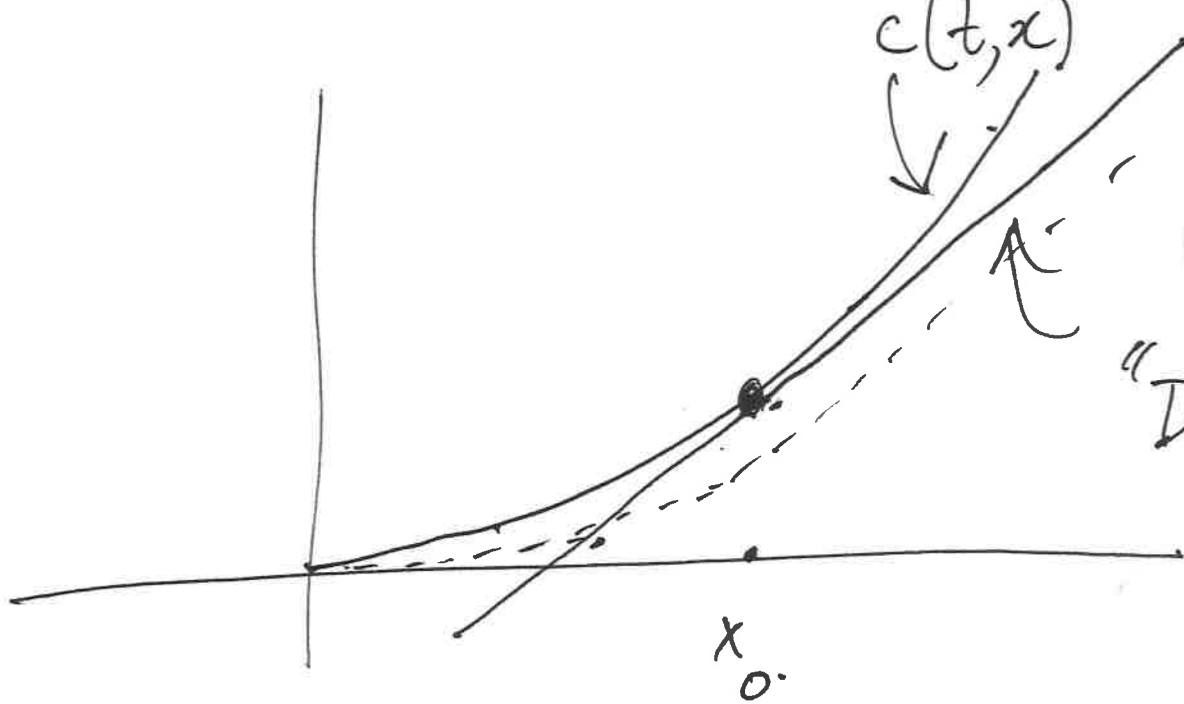
Invest \$ $c(t,x)$ in stock & M.M.

Should buy $\partial_x c(t,x)$. shares of stock.

Cash value : $c(t,x) - x\partial_x c(t,x)$.

$$= \cancel{x N(d_+)} - k e^{-r\tau} N(d_-) - \cancel{x N(d_+)}.$$

$$= -k e^{-r\tau} N(d_-) < 0$$



(Fixed t).

(Stock price $= x_0$).

"Delta neutral".

Set up a portfolio that: short $\partial_x c(t, x_0)$ shares of stock.

Buy ① call option (valued at $c(t, x_0)$.)

② Balance in cash.

$$M = x_0 \partial_x c(t, x_0) - c(t, x_0).$$

HOLD POSITION.

Say <u>instantaneously</u> stock price becomes x.

$$\text{Pf. value} = \cancel{c(t,x) - x\partial(t,}$$

$$= c(t,x) - x\,\partial_x c(t,x_0) + M.$$

$$= c(t,x) - x\,\partial_x c(t,x_0) + x_0\,\partial_x c(t,x_0) - c(t,x_0).$$

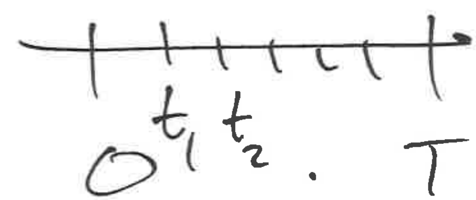
$$= c(t,x) - \underbrace{\Big[c(t,x_0) + (x - x_0)\,\partial_x c(t,x_0)\Big]}_{\text{tangent line.}} > 0$$

Multi D Ito Formula

X, Y two Ito processes.

Expect $\begin{vmatrix} X(t+\delta t) - X(t) \\ Y(t+\delta t) - Y(t) \end{vmatrix} \begin{matrix} \approx \sqrt{\delta t}. \\ \approx \sqrt{\delta t}. \end{matrix}$

Q.V. $\displaystyle\sum_{i=0}^{n-1} \left(X(t_{i+1}) - X(t_i)\right)^2.$



Joint QV: Def $[X,Y] = \displaystyle\lim_{\|P\| \to 0} \cancel{\sum \left(X(t_{i+1}) - X(t_i)\right)\left(Y(t_{i+1}) - Y(t_i)\right)}$

$$[X,Y](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \big(X(t_{i+1}) - X(t_i)\big)\big(Y(t_{i+1}) - Y(t_i)\big)$$

$P =$ partition of $[0,T]$.

$$4ab = (a+b)^2 - (a-b)^2$$

You check: $$[X,Y] = \frac{1}{4}\Big([X+Y, X+Y] - [X-Y, X-Y]\Big).$$

Joint QV   QV   QV.

Product Rule: If X & Y are two Itô processes.

then $d(XY) = \underbrace{X\,dY + Y\,dX}_{\text{Usual product rule.}} + \underset{\text{Extra}}{d[X,Y]}$.

(Recall $(fg)' = f'g + g'f$.)

Pf: $d(X+Y)^2 = 2(X+Y)\,d(X+Y) + d[X+Y, X+Y]$.

$= 2X\,dX + 2Y\,dY + 2X\,dY + 2Y\,dX + d[X+Y, X+Y]$.

$$d(X-Y)^2 = 2X\,dX + 2Y\,dY - 2X\,dY - 2Y\,dX + d[X-Y, X-Y].$$

$$d(4XY) = d\left((X+Y)^2 - (X-Y)^2\right).$$

$$\Rightarrow \cancel{4}\,d(XY) = \cancel{4}(X\,dY + Y\,dX) + \cancel{4}\,d[X,Y].$$

QED.