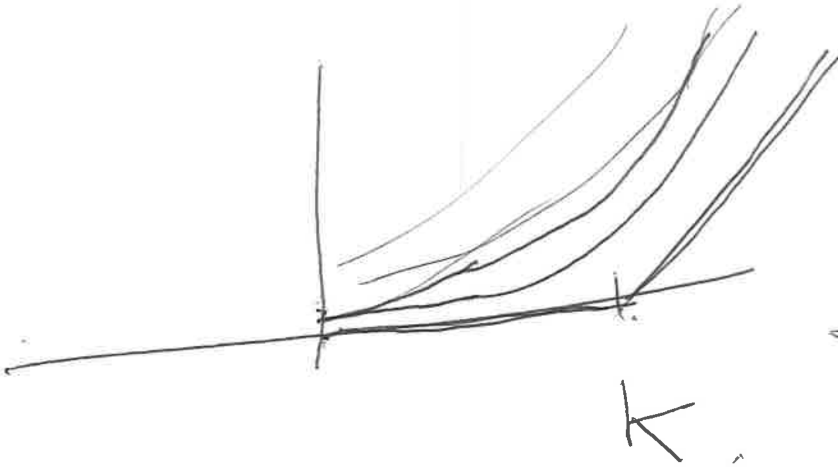
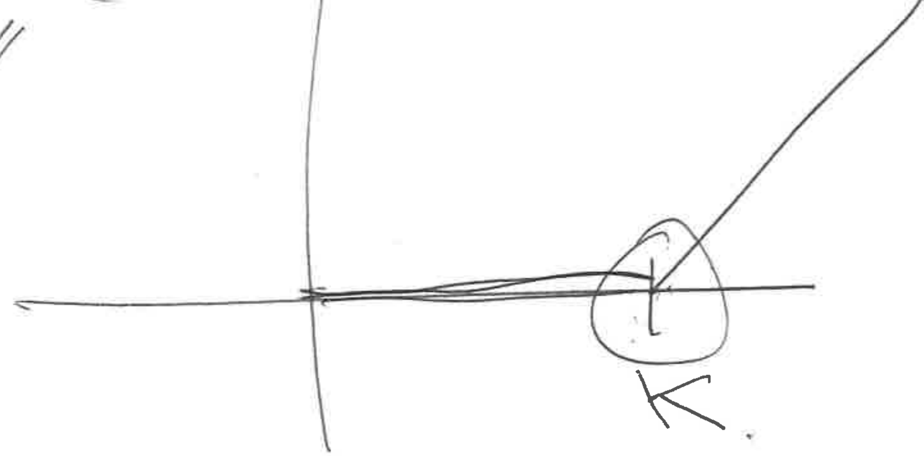
$(x-K)^+$ time T





Midterm: $\longrightarrow$ $E\left(\int_0^t b(s)\,ds \,\middle|\, \mathcal{F}_r\right) = \int_0^t E\left(b(s) \,\middle|\, \mathcal{F}_r\right) ds.$

$$E\left(\lim \sum b(s_i)(s_{i+1}-s_i) \,\middle|\, \mathcal{F}_r\right).$$

$$\|$$

$$\lim \sum E\left(b(s_i) \,\middle|\, \mathcal{F}_r\right)(s_{i+1}-s_i)$$

$$E\left(\int_0^t b(s)\,dW(s) \,\middle|\, \mathcal{F}_r\right) \neq \int_0^t E\left(b(s) \,\middle|\, \mathcal{F}_r\right) dW_s.$$

$$E\left(\sum b(s_i)\left(W(s_{i+1})-W(s_i)\right) \,\middle|\, \mathcal{F}_r\right) = \sum E\left(b(s_i)\underbrace{\left(W(s_{i+1})-W(s_i)\right)}_{\text{random}} \,\middle|\, \mathcal{F}_r\right).$$

Black Shotes Merton:

'73.

Uniqueness of the Itô Decomposition (Semi Mg decomposition).

$X \longrightarrow$ Itô process. $X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s)$

$= X(0) + B(t) + M(t)$

$B \longrightarrow$ finite 1^{st} var.

$M \longrightarrow$ Mg.

Prop: If $X = X_0 + B_1(t) + M_1(t)$

$$= X_0 + B_2(t) + M_2(t).$$

$$\left.\begin{array}{l} B_1, B_2 \longrightarrow \text{finite } 1^{st} \text{ var.} \\ M_1, M_2 \longrightarrow \text{Mg} \end{array}\right\} \Longrightarrow B_1 = B_2 \ \& \ M_1 = M_2.$$

Practically: If

$$\left.\begin{array}{l} dX(t) = b_1(t)\,dt + \sigma_1(t)\,dW(t) \\ \& \ dX(t) = b_2(t)\,dt + \sigma_2(t)\,dW(t). \\ \text{Then: } b_1(t) = b_2(t) \ \& \ \sigma_1(t) = \sigma_2(t). \end{array}\right]$$

Proof: Say $X = X_0 + B_1 + M_1 = X_0 + B_2 + M_2$.

$$\Rightarrow \underbrace{B_1 - B_2}_{B} = \underbrace{M_2 - M_1}_{M}.$$

$B \longrightarrow$ Finite 1^{st} var. $\Longrightarrow M$ has finite 1^{st} var.

$$\Longrightarrow [M,M](t) = 0$$

Know: $E M(t)^2 = E \underbrace{[M,M](t)}_{0} + \underbrace{\cancel{E M(0)^2}}$.

$\Rightarrow E M(t)^2 = 0 \Longrightarrow M = 0 \Rightarrow M_1 = M_2$ & $B_1 = B_2$. QED.

① Model Stock Prices: Geometric BM.

Def: A process S that satisfies

$$dS(t) = \alpha S(t)\,dt + \underbrace{\sigma S(t)\,dW(t)}$$

is called a Geometric B.M. $(\alpha, \sigma \in \mathbb{R})$.

Set $Y(t) = \ln(S(t))$ ⟵ $\boxed{f(t,x) = \ln x,\ \partial_t f = 0,\ \partial_x f = \frac{1}{x},\ \partial_x^2 f = \frac{-1}{x^2}}$

Itô: $dY(t) = \frac{1}{S(t)}\,dS(t) + \frac{1}{2}\left(\frac{-1}{S(t)^2}\right)\sigma^2 S(t)^2\,dt$

$$\Rightarrow dY(t) = \alpha\, dt + \sigma\, dW - \frac{1}{2}\sigma^2 dt$$

$$= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW.$$

If $\alpha = \frac{\sigma^2}{2}$: $\ln(S(t)) = \ln S(0) + \sigma W(t)$.

$$\Leftrightarrow \ln\left(\frac{S(t)}{S(0)}\right) = \sigma W(t).$$

In general: $\boxed{S(t) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)}$

$S(t) \longrightarrow$ Price of risky asset.

European call option: Maturity T, strike price $\boxed{K}$.

(Right to buy asset at price K at time T)

<u>Theorem:</u> Say AF market.
① risky asset (Price given by S)
② Money Market (return rate r)

Consider European call, strike K, maturity T:

① Say $c = \cancel{c(x,t)}\ c(t,x)$ is such that

$\forall t \leq T$, $\cancel{c(x,t)}\ c(t,x) =$ A.F.P. of call option, given that $S(t) = x$.

($\Leftarrow$) $c(t, S(t)) = $ AFP of the call.)

PDE.

Then: (a) $\boxed{\partial_t c + r x \partial_x c + \frac{\sigma^2 x^2}{2} \partial_x^2 c - r c = 0}$ $\leftarrow$ $x > 0$, $t \leq T$

Boundary condition $\to$ (b) $c(t, 0) = 0$

Terminal cond. $\to$ (c) $c(T, x) = (x - K)^+ = \max\{0, x - K\}$.

(2) Conversely: Say c satisfies (a), (b) & (c).

Then $\boxed{c(t, S(t))} = $ A.F.P of the call.

<u>Assumptions:</u> (1') <u>Frictionless</u> (no transaction costs)

(2) Liquidity (can trade fractions of S).

(3) Borrowing & lending rates are equal.

Remark: ⓐ ~~&~~ ⓑ - ⓒ $\rightarrow$ Partial Dif Eq.

Explicit Solution to ⓐⓑ&ⓒ:

$$c(t,x) = x\,N\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big)$$

$\tau = T-t$

$$d_\pm = d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right)$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

Strategy of Proof: $\boxed{\text{Replicating Portfolio}}$

$X(t) \rightarrow$ value of portfolio: $\Delta(t)$ shares of ~~in~~ the asset.

Rest in cash.

Goal: at maturity $X(T) = (S(T) - K)^+$

(same cash flow as the call).

$X(t) = \text{AFP}$.

Proof of BSM Part I:

Know $c(t, S(t)) = AFP$.

Let R portfolio hold $\Delta(t)$ shares of S & rest cash.

$$dX(t) = \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt.$$

If X is a R. Pf. then $\boxed{c(t, S(t)) = X(t)}$.

Itô: $$dc(t, S(t)) = \partial_t c\,dt + \partial_x c\,dS(t) + \tfrac{1}{2}\partial_x^2 c\,d[S,S]$$
$$= \left(\partial_t c + \partial_x c\,(\alpha S) + \frac{\sigma^2}{2}\partial_x^2 c \cdot S^2\right)dt + \partial_x c\,\sigma S\,dW(t),$$

Shall have $d(c(t,S(t))) = dX(t)$

$$= \Delta(t)\,\sigma\, S(t)\, dW(t)$$
$$+ \left(\Delta(t)\,\alpha\, S(t) + r\left(X(t) - \Delta(t) S(t)\right)\right) dt.$$

$$= \left(\Delta(t)(\alpha - r) S + r X(t)\right) dt + \Delta(t)\,\sigma\, S(t)\, dW(t).$$

Equate dt & dW terms:

1. $\Delta(t) \cancel{\sigma} \cancel{S(t)} = \partial_x c \cancel{\sigma} \cancel{S(t)}$.

$$\boxed{\Rightarrow \Delta(t) = \partial_x c(t, S(t))}$$ Delta Hedging rule.

2. $\partial_x c(\alpha - r) S + r c = \partial_t c + \partial_x c(\alpha S) + \frac{\sigma^2}{2} \partial_x^2 c\, S^2$.

$$\iff \partial_t c^{(t,S(t))} + r S \partial_x c^{(t,S(t))} + \frac{S^2 \sigma^2}{2} \partial_x c^{(t,S(t))} = r\, c(t, S(t)).$$

(Note α cancelled).

Put $x = S(t)$.

$$\boxed{\partial_t c(t,x) + r x \partial_x c(t,x) + \frac{x^2 \sigma^2}{2} \partial_x c(t,x) = r\, c(t,x).}$$

Converse: Start with a sol of (a) (b) & (c).

NTS $c = AFP$.

Let X be a portfolio. $X(0) = c(0, S(0))$.

$\boxed{\text{Let } \Delta(t) = \partial_x c(t, S(t))}$

Claim: $X(t) = c(t, S(t)) \quad \forall\, t < T$

Trick: Set $Y(t) = e^{-rt} X(t)$.

Compute dY & compute $d\left(e^{-rt} c(t, S(t))\right)$.

Will check $dY = d(\qquad)$.

$\Rightarrow e^{-rt} X(t) = e^{-rt} c(t, S(t)) \Rightarrow X(t) = c(t, S(t))$.

$\Rightarrow X(T) = c(T, S(T)) = (S(T) - K)^+$