~~Recall: Itô~~ MIDTERM.

Recall: Itô Processes: $X(t) = \underbrace{X(0)}_{\text{not random}} + \overbrace{\int_0^t b(s)\,ds}^{\text{Rieman}} + \overbrace{\int_0^t \sigma(s)\,dW(s)}^{\text{Itô}}$.

$\sigma, b \longrightarrow$ adapted processes. (Random).

Short hand Notation:

$$\int_0^t dX(s) = X(t) - X(0) = \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s).$$

$$dX(\cancel{t}\,s) = b(s)\,ds + \sigma(s)\,dW(s)$$

Last time: $B(t) = \int_0^t b(s)\,ds$ & $M(t) = \int_0^t \sigma(s)\,dW(s)$,

$B \longrightarrow$ Bounded Var (Finite First variation).

$M \longrightarrow$ Martingale (Finite Quadratic Var).

$$\boxed{\begin{array}{l} \text{Finite QV} \Longrightarrow \text{First var} = \infty \\ \text{Finite FV} \Longrightarrow QV = 0 \end{array}}$$

$X(t) = X(0) + B(t) + M(t)$; $[X, X](t) = [M, M](t)$.

$$= \int_0^t \sigma(s)^2 \, ds.$$

Itô Formula: Chain rule:

Let $f = f(t, x)$. $\quad$ ($x \in \mathbb{R}$, $t \geq 0$).

Say $X(t)$ is a DIFFRENTIABLE fn of t

(Will NEVER happen for Ito processes).

Compute
$$\frac{d}{dt}\Big(f\big(t, X(t)\big)\Big) = \partial_t f\big(t, X(t)\big) + \underline{\partial_x f\big(t, X(t)\big)} \frac{dX(t)}{dt}.$$

Let $Y(t) = f\big(t, X(t)\big)$.

$$Y(t) - Y(0) = \int_0^t \frac{dY}{ds}\, ds. = \int_0^t \Big\{ \partial_t f\big(s, X(s)\big) ds + \int_0^t \partial_x f\big(s, X(s)\big). \frac{dX}{ds}\, ds$$

$$\boxed{\text{CHAIN RULE: } dY(t) = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t).}$$

~~Itô~~ ~~f~~

Eg: $f(t,x) = t e^{tx}$

$$\partial_t f = t\left(x e^{tx}\right) + e^{tx} \cdot 1.$$

$$\partial_x f = t^2 e^{tx}.$$

Itô formula: $X \longrightarrow$ Itô Process. $f = f(t,x)$ a fn.

<u>Theorem</u>: Set $Y(t) = f(t, X(t))$.

$$dY = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t)$$

$$+ \frac{1}{2} \partial_x^2 f(t, X(t))\, d[X, X](t).$$

Integral form:

$$f(t,X(t)) - f(0,X(0)) = \int_0^t \partial_t f(s,X(s))\,ds$$

$$+ \int_0^t \partial_x f(s,X(s))\,dX(s)$$

$$+ \frac{1}{2}\int_0^t \partial_x^2 f(s,X(s))\,d[X,X](s).$$

CRUCIAL ASSUMPTION: ① ~~f~~ f is diff wrt t

& ② f is TWICE diff wrt x

③ & $\partial_t f$, $\partial_x f$ & $\partial_x^2 f$ are all cts fns.

Substitute: $dX(t) = b(t)\,dt + \sigma(t)\,dW(t)$.

$$[X,X](t) = \int_0^t \sigma(s)^2\,ds \Rightarrow d[X,X](t) = \sigma^2(t)\,dt.$$

Itô formula: $f(t,X(t)) - f(0,X(0)) = \int_0^t \partial_t f(s,X(s))\,ds$

$$+ \int_0^t \partial_x f(s,X(s))\,b(s)\,ds + \int_0^t \partial_x f(s,X(s))\,\sigma(s)\,dW(s)$$

$$\underbrace{+ \frac{1}{2}\int_0^t \partial_x^2 f(s,X(s))\,\sigma(s)^2\,ds}_{\text{Itô correction.}}$$

Intution Behind Itô's formula:

Simplest case: $f = f(x)$. (doesn't depend on t).

$$X = W.$$

Itô: $Y(t) = f(W(t))$.

$$dY(t) = 0\,dt + f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t))\,dt.$$

$$Y(t+\delta t) - Y(t) = f(W(t+\delta t)) - f(W(t)).$$

$$= \quad + (W(t+\delta t) - W(t)) f'(W(t))$$

$$+ \frac{1}{2} (W(t+\delta t) - W(t))^2 f''(W(t)) + \cdots$$

$$Y(T) - Y(0) = \sum_{0}^{T/\delta t} Y((k+1)\delta t) - Y(k\delta t).$$

$$\xrightarrow{\delta t \to 0} \int_0^T f'(W(t))\, dW$$

$$= \sum_{0}^{T/\delta t} \left(W((k+1)\delta t) - W(k\delta t)\right) f'(W(k\delta t)).$$

$$+ \sum_{0}^{T/\delta t} \left[\frac{\left(W((k+1)\delta t) - W(k\delta t)\right)^2}{2}\, f''(\cancel{k\delta t}\; W(k\delta t)) \right]$$

$$\delta t + \delta t$$

$$+ \quad \text{~~smaller terms~~}$$

$$\longrightarrow \int f''(W(t))\, dt$$

$$\left(W((k+1)\delta t) - W(k\delta t)\right)^2 \sim N(0, \delta t)^2.$$

Mean $= 0$

Variance $= (\delta t)^2$.

$$\left(W((k+1)\delta t) - W(k\delta t)\right)^2 - \delta t \sim \boxed{N(0, \delta t)^2 - \delta t}$$

$$f(x+h) - f(x) = h f'(x) + \frac{1}{2} h^2 f''(x) + \underbrace{o(h^2)}_{\text{smaller than } h^2}.$$

$$\cancel{h = \delta t}$$

$$x = W(t)$$

$$x + h = W(t+\delta t).$$

$$h = W(t+\delta t) - W(t).$$

$$\sum \left[\frac{\left(W((k+1)\delta t) - W(k\delta t) \right)^2}{2} - \delta t + \delta t \right] f''(W(k\delta t))$$

$$= \underbrace{\sum_{0}^{T/\delta t} \left(\frac{\left(W((k+1)\delta t) - W(k\delta t) \right)^2}{2} - \frac{\delta t}{2} \right) f''(W(k\delta t))}_{\to\, 0}$$

① Each term voriance $(\delta t)^2$.

$$+ \underbrace{\sum \cancel{W} \left(\frac{\delta t}{2} \right) f''(W(k\delta t))}_{\to\, \frac{1}{2}\int_0^t f''(W(s))\, ds}$$

Examples: $f(t,x) = x^2$, $X(t) = W(t)$.

$\partial_t f = 0$

$\partial_x f = 2x$

$\partial_x^2 f = 2$

$\boxed{\text{Compute QV of } W(t)^2}$

Put $Y(t) = f(t, W(t)) = W(t)^2$.

Ito: $dY = 0\,dt + 2W(t)\,dW + \frac{1}{2}\cdot 2 \underbrace{dt}_{d[W,W]}$.

$$W(t)^2 - W(0)^2 = Y(t) - Y(0)$$

$$= \int_0^t 2W(s)\,dW(s) + t$$

$$\therefore [Y, Y](t) = \boxed{\int_0^t \underbrace{4W(s)^2}_{\sigma(s)^2} ds} \quad //$$

Note $W(t)^2 = \int_0^t 2W(s)\,dW(s) + t$

$$\Leftrightarrow W(t)^2 - t = \underbrace{2\int_0^t W(s)\,dW(s)}_{\text{Mg.}}.$$

$$Y(t) = W(t)^2 = \underbrace{\int_0^t 2W(s)\,dW(s)} + \int_0^t ds.$$

$$\int_0^t \sigma(s)^2 ds. \quad \xleftarrow{\text{QV.}} \quad \int_0^t \sigma(s)\,dW(s) \quad + \int_0^t b(s)\,ds.$$

Eg: $M(t) = W(t)$
$N(t) = W(t)^2 - t$

Know M, N are martingales.

$\boxed{\text{Q: Is } MN \text{ a mg?}}$

$$Y(t) = W(t)^3 - tW(t).$$

Compute dY: Let $f(t,x) = x^3 - tx$.

Then $Y(t) = f(t, W(t))$.

$\partial_t f = -x$
$\partial_x f = 3x^2 - t$
$\partial_x^2 f = 6x$

$$dY = \partial_t f(t,W(t))\,dt + \partial_x f(t,W(t))\,dW(t) + \frac{1}{2}\partial_x^2 f(t,W(t))\,d[W,W](t).$$

$$= -W(t)\,dt + (3W(t)^2 - t)\,dW + \frac{1}{2}\cdot 6W(t)\,dt$$

$$= 2W(t)\,dt + (3W(t)^2 - t)\,dW.$$

$$\Rightarrow Y(t) - Y(0) = \underbrace{\int_0^t 2W(s)\,ds}_{\text{Not a Mg.}} + \underbrace{\int_0^t (3W(s)^2 - s)\,dW(s)}_{\text{Martingale.}}$$

Note: $\int_0^t b(s)\,ds$ is only a Mg if $b = 0$ identically!

$$\Rightarrow Y \text{ is not a mg.}$$

Deside if Y is a Mg.

Compute $dY = \boxed{(\ )\,dt} + (\ )\,dW$.

Y is only a mg if the "dt" part is 0.

Proof: Let $f = f(t,x)$. (∞'ly diff).

Define $M(t) = f(t, W(t)) - \int_0^t \left[\partial_t f(s, W(s)) + \frac{1}{2}\partial_x^2 f(s, W(s))\right] ds$.

Claim: M is a Martingale.

Proof: Set $Y(t) = f(t, W(t))$.

$$dY = \partial_t f(t, W(t))\, dt + \partial_x f(t, W(t))\, dW(t)$$
$$+ \frac{1}{2}\partial_x^2 f(t, W(t)) \underbrace{dt}_{d[W,W]}.$$

$$\Rightarrow dY = \left(\partial_t f(t, W(t)) + \frac{1}{2}\partial_x^2 f(t, W(t))\right) dt$$
$$+ \partial_x f(t, W(t))\, dW(t).$$

$$\Rightarrow Y(t) - Y(0) = f(t, W(t)) - f(0, W(0))$$
$$= \int_0^t \partial_t f(s, W(s)) + \frac{1}{2}\partial_x^2 f(s, W(s))\, ds + \int_0^t \partial_x f(s, W(s))\, dW(s).$$

$$\Rightarrow M(t) = f(0, W(0)) + \underbrace{\int_0^t \partial_x f(s, W(s))\, dW(s)}_{\text{Mg}}. \Rightarrow \text{done!}$$