Last Time:

$$\begin{cases} \Delta(t) \longrightarrow \text{Position at time } t \text{ on an asset.} \\ W(t) \longrightarrow \text{Std BM.} \end{cases}$$

$$\cancel{I(t)}\ I(T) = \text{"Profit / Winnings"} = \int_0^T \underbrace{\Delta(t)}_{\text{Random \& } \boxed{\text{adapted}}} \, dW(t).$$

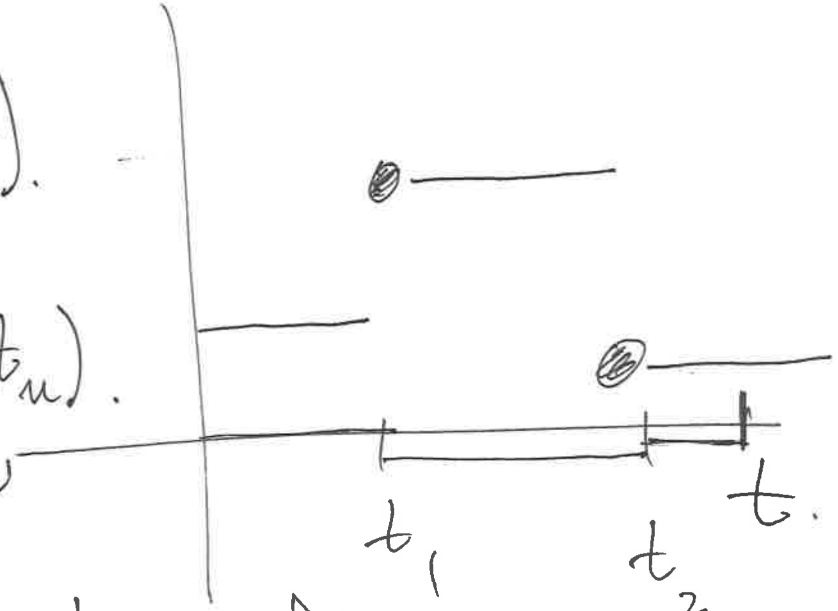
Case I: Say ~~of~~ the asset is only traded at times.

$P.\ \{0 = t_0 < t_1 < t_2 \dots\}$

$\Delta$ is constant on the interval $[t_i, t_{i+1})$.

$$I_p(t) = \sum_{i=0}^{n-1} \Delta(t_i)\left(W(t_{i+1}) - W(t_i)\right)$$

$$\underbrace{+\ \Delta(t_n)\left(W(t) - W(t_n)\right).}$$




when $t \in [t_n, t_{n+1})$.

Claim ① $I_p(t)$ is a mg

② $$E\, I_p(t)^2 = E\left[\sum_{i=0}^{n-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_n)^2 (t - t_n)\right].$$

(3) $[I_p, I_p](t) = \cancel{E} \underbrace{\sum_0^{n-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_n)^2 (t - t_n)}$.

when $t \in [t_n, t_{n+1})$.

KEY IDEA:

$$E\left(\Delta(t_i)\left(W(t_{i+1}) - W(t_i)\right)\right)^2$$

(tower) $$= E\left[E\left[\left(\Delta(t_i)\left(W(t_{i+1}) - W(t_i)\right)\right)^2 \middle| \mathcal{F}_{t_i}\right]\right].$$

$$= E\left(\Delta(t_i)^2 \, E\left(W(t_{i+1}) - W(t_i)\right)^2\right).$$

$$\boxed{= E\Delta(t_i)^2 (t_{i+1} - t_i).}$$

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General Itô integral: "$\lim\limits_{\|P\| \to 0}$"

Note $\lim\limits_{\|P\| \to 0} \sum\limits_{0}^{n-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_n)^2 (t - t_n)$.

$$= \int_0^t \Delta(s)^2 \, ds$$

Itô : ① As $\|P\| \to 0$ $\quad [I_P, I_P](t) \longrightarrow \int_0^t \Delta(s)^2\, ds$.

② Since $I_P$ is a Mg.

"M-conv thm" $\longrightarrow$ $I_P$ converges (as mg) to some process $I$.

$$\lim_{\|P\| \to 0} I_P(t) \overset{\text{def}}{=\!=} I(t) \overset{\text{Notation}}{=\!=} \underbrace{\int_0^t \Delta(s)\, dW(s)}_{\text{Itô integral}}.$$

Thm: If $\int_0^t \Delta(s)^2\,ds < \infty$ a.s.

then $\lim_{\|P\|\to 0} I_P(t)$ exists. Set $I(t) = \lim_{\|P\|\to 0} I_P(t)$.

1. $I(t)$ is a <u>cts</u> & adapted process.

2. If $E\int_0^t \Delta(s)^2\,ds < \infty$ then $I$ is a <u>Mg</u>.

In this case, $[I,I](t) = \int_0^t \Delta(s)^2\,ds$

$\int_0^t \Delta(s)\, dW(s)$ is a mg.

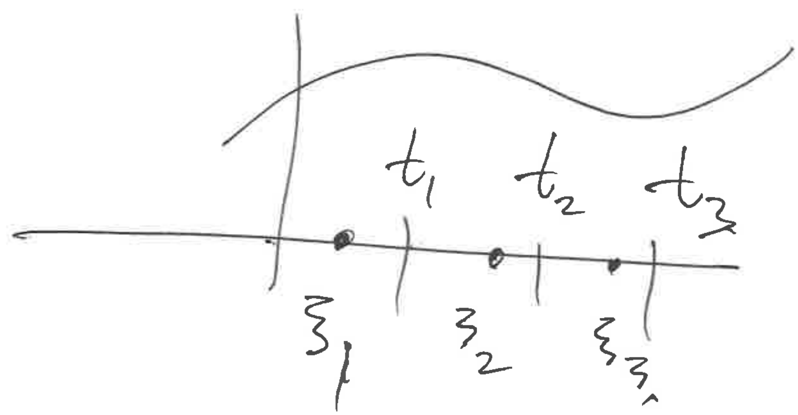
& $\left[\int_0^t \Delta(s)\, dW(s), \int_0^t \Delta(s)\, dW(s)\right] = \int_0^t \Delta(s)^2\, ds.$

Remark: ~~Crucial~~ Common fact used: $\sum \boxed{\Delta(t_i)}\left(W(t_{i+1}) - W(t_i)\right).$

Rieman Int: $\sum f(\xi_i)(t_{i+1} - t_i)$ where $\xi_i \in [t_i, t_{i+1})$

Rieman Sums.

$\int_0^t f(s)\, ds.$



Properties:

① (Itô Isometry): $$E\left(\int_0^t \Delta(s)\,dW(s)\right)^2 = E\int_0^t \Delta(s)^2\,ds.$$

(Any Mg: $EM(t)^2 - EM(0)^2 = E\,[M,M](t)$.)

② (Linearity): $\Delta_1$, & $\Delta_2$ two $\boxed{\text{adapted processes}}$.

$\alpha \in \mathbb{R}$. then

$$\int_0^t \left(\Delta_1(s) + \alpha\,\Delta_2(s)\right) dW(s) = \int_0^t \Delta_1(s)\,dW(s) + \alpha\int_0^t \Delta_2(s)\,dW(s).$$

3 $\boxed{\text{NO Positivity!}}$ $\Delta(s) \geq 0 \not\Rightarrow \int_0^t \Delta(s)\, dW(s) \geq 0.$

Eg: $\Delta(s) = 1 \ \forall s.$

Then $\int_0^t dW(s) = W(t) - \cancel{W(0)}$ $\boxed{\text{not} \geq 0}$

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$$\int_0^t W(s)\, dW(s)$$

$\boxed{\text{Itô Formula:}}$

1 $\boxed{\text{Itô Process}}$ : $b$ & $\sigma$ $\longrightarrow$ two adapted processes.

Assume $E\int_0^t \sigma(s)^2 \underbrace{ds} < \infty$ & $\int_0^t b(s) \underbrace{ds} < \infty$

Define a process $X$ by

$$X(T) = X(0) + \underbrace{\int_0^T b(t)\,dt}_{\text{Riemann Int.}} + \underbrace{\int_0^T \sigma(s)\,dW(s)}_{\text{Itô integral.}}$$

If $X(0)$ is constant (not random). $X$ is called an $\boxed{\text{Itô Process.}}$

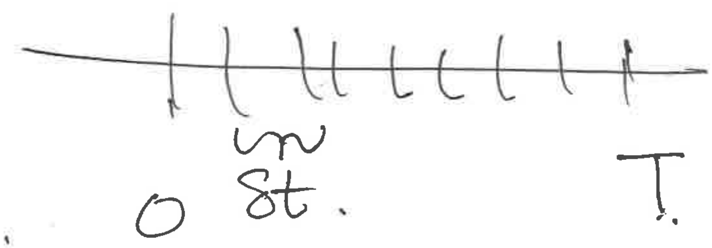
Q: What is $[X, X]$??

↳ Say $Y$ has finite Q.V.

Then $|Y(t+\delta t) - Y(t)|$ is of order $\sqrt{\delta t}$

$$\sum \left(Y(t_{i+1}) - Y(t_i)\right)^2$$

$$\sum_{0}^{\lfloor T/\delta t \rfloor} \underbrace{\left[Y((k+1)\delta t) - Y(k\delta t)\right]^2}_{\text{of size } \boxed{\delta t}}$$



$$E\,|W(t+\delta t) - W(t)| \sim E|N(0,\delta t)| =$$

$$= E\sqrt{\delta t}\,|N(0,1)| = \sqrt{\delta t}\,(\quad).$$

~~Let~~ $\cancel{B(t) = \int_0^T b}$ Let $B(t) = \int_0^t b(s)\,ds$

$$B(t+\delta t) - B(t) = \int_t^{t+\delta t} b(s)\,ds \longleftarrow$$ Finite First Variation.

of size $\approx \|b\|\,\boxed{\delta t}$

$$\sum_{0}^{T/\delta t} \Big( B((k+1)\delta t) - B(k(\delta t)) \Big)^2$$

$$\approx \sum_{0}^{T/\delta t} (\delta t)^2 = \frac{T}{\delta t} (\delta t)^2 \longrightarrow 0$$

Q: $[X+Y, X+Y] \neq [X,X] + [Y,Y]$ $\boxed{\text{NO}}$

$$X(T) = X(0) + \underbrace{\int_0^T b(s)\,ds}_{B(T)} + \underbrace{\int_0^T \sigma(s)\,dW(s)}_{M(T)}.$$

$$\big(X(t+\delta t) - X(t)\big)^2 = \overbrace{\big(B(t+\delta t) - B(t)\big)^2}^{\approx(\delta t)^2} + \overbrace{\big(M(t+\delta t) - M(t)\big)^2}^{\delta t}$$

$$+ \underbrace{2\big(B(t+\delta t) - B(t)\big)\big(M(t+\delta t) - M(t)\big)}_{(\delta t)^{3/2}}.$$

$$\sum_0^{T/\delta t} (\quad)^2 = O(\sqrt{\delta t}) + \sum \big(M(t+\delta t) - M(t)\big)^2.$$

Expect $[X,X](t) = [M,M](t) = \int_0^t \sigma(s)^2 ds$.

$B(t) = \int_0^t b(s)\, ds \longleftarrow$ Process of Bounded Variation.

(finite first Variation).

$M = \int_0^t \sigma(s)\, dW(s) \longleftarrow$ Mg.

$X = X(0) + B + M$.

Itô ~~Formula~~: integrals wrt $X$.

$\Delta(t) \longrightarrow$ adapted process (position on an asset).

$$X(t) = X(0) + B(t) + M(t)$$

$$B(t) = \int_0^t b(s)\, ds \quad \& \quad M(t) = \int_0^t \sigma(s)\, dW(s).$$

Def: $$I(t) \stackrel{\text{def}}{=} \int_0^t \Delta(s)\, dX(s) = \underbrace{\int_0^t \Delta(s)\, b(s)\, ds}_{\text{Riemann}} + \underbrace{\int_0^t \Delta(s)\sigma(s)\, dW(s)}_{\text{Itô}}.$$

Short hand Notation.

If $X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s)$.

We write $dX = b(s)\,ds + \sigma(s)\,dW(s)$.

Itô's formula: Let $f = f(t, x) \in C^{1,2}$

Set $Y(t) = f(t, X(t))$

$$dY = \overbrace{\partial_t f(t, X(t))\,dt + \partial_x f(t, X(t))\,dX(t)}^{\text{"Usual Chain rule"}}$$
$$+ \frac{1}{2}\,\partial_x^2 f(t, X(t))\,d[X,X](t).$$