Conditional Expectation: Mg: $\underbrace{E(M(t) \mid \mathcal{F}_s) = M(s)}$.

Last Compute $E(f(W(t)) \mid \mathcal{F}_s)$.

$$E(f(W(t)) \mid \mathcal{F}_s) = E(f(\underbrace{W(t) - W(s)}_{\text{Ind of } \mathcal{F}_s} + \underbrace{W(s)}_{\text{is } \mathcal{F}_s\text{-meas.}}) \mid \mathcal{F}_s).$$

General Principal:

1. Quantities that are already $\mathcal{F}_s$ meas $\to$ LEAVE ALONE!
2. Quantities that are IND of $\mathcal{F}_s$ $\to$ AVERAGE.

Eg: Let g be any fn.

Compute $E\left(\int_0^T g(W_t)\, dt \;\middle|\; \mathcal{F}_s \right)$.

$$= E\left(\int_0^s g(W(t))\, dt \;\middle|\; \mathcal{F}_s \right) + E\left(\int_s^T g(W(t))\, dt \;\middle|\; \mathcal{F}_s \right).$$

$$= \int_0^s g(W(t))\, dt + E\left(\int_s^T g\Big(\underbrace{W(t) - W(s)}_{\sim N(0,\, t-s)} + W(s)\Big)\, dt \;\middle|\; \mathcal{F}_s \right).$$

$$= \int_0^s g(W(t))\, dt + \int_{y=-\infty}^{\infty} \left(\int_s^T g\big(y + W(s)\big)\, G(t-s, y)\, dt\, dy \right).$$

$G(t-s, y) =$ Density of $N(0, t-s)$

$$= \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-y^2/2(t-s)}.$$

~~Claim~~: Know W is Mg.

$f \to$ some fn. Is $f(W(t))$ a mg?

Claim: $f(W(t)) - \int_0^t \frac{1}{2} f''(W(s))\, ds$. IS a MG!!

(If W is a STD BM).

$\boxed{\text{ITO INTEGRAL:}}$

Say $\Delta(t) \longrightarrow$ Your position on some Security.

$S(t) \longrightarrow$ Price of this security.

Say Only trade at times $0 = t_0 < t_1 < t_2 \cdots < t_n = T$

Δ constant on $[t_i, t_{i+1})$.

$$X(t_n) = \sum_{i=0}^{n-1} \Delta(t_i)\big(S(t_{i+1}) - S(t_i)\big). \longleftarrow \text{Profit/Winnings}$$

Want: $\displaystyle\lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \Delta(t_i)\big(S(t_{i+1}) - S(t_i)\big).$

$P =$ Partition $= \{0 = t_0 < t_1 < \cdots < t_n\}$.

$\|P\| \equiv \max_i \; t_{i+1} - t_i$

$$\lim_{\|P\| \to 0} \left(\quad \right) \equiv \underbrace{\int_0^T \Delta(t)\, dS(t)}_{\text{Rieman-Stieltjes Integral}}.$$

Ned for existece of R-S integral

Need $\boxed{\text{FINITE FIRST VARIATION}}$.

<u>Def:</u> $$V_{[0,T]}(S) \stackrel{\text{def}}{=\!=} \lim_{\|P\| \to 0} \sum_i |S(t_{i+1}) - S(t_i)|.$$

First Variation of S.

P = Partition of $[0,T] = \{0 = t_0 < \cdots < t_n = T\}$.

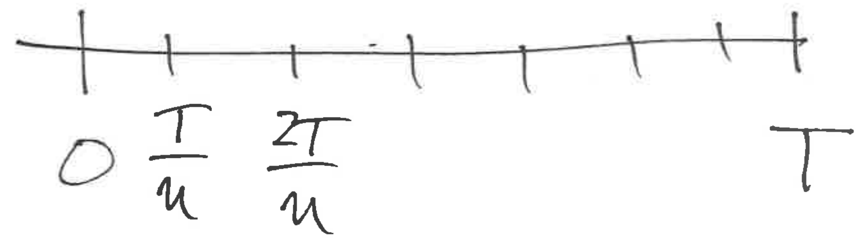
Prop: $W(t)$ DOES $\boxed{\text{NOT}}$ have finite first Variation.

Claim: $V_{[0,T]}(W) = +\infty$ almost surely!

(I.e. $\int_0^T \Delta(t)\, dW(t)$ DNE as a R-S int).

Intuition: Compute $E\, V_{[0,T]}(W) = +\infty$.

$$t_i = i\frac{T}{n}$$



$$E \sum_{i=0}^{n-1} \Big| \underbrace{W\Big(\frac{(i+1)T}{n}\Big) - W\Big(\frac{iT}{n}\Big)}_{\sim |N(0,\, T/n)|} \Big| =$$

$$= \cancel{E}$$

Note $E\left| W\Big(\frac{(i+1)T}{n}\Big) - W\Big(\frac{iT}{n}\Big) \right| = E\left|N\Big(0, \frac{T}{n}\Big)\right|$.

$$= \sqrt{\frac{T}{N}}\, E|N(0,1)|.$$

$$\Rightarrow E \sum_{i=0}^{N-1} \left| W\left(\frac{(i+1)T}{n}\right) - W\left(\frac{iT}{n}\right) \right|.$$

$$= \sum_{i=0}^{n-1} \sqrt{\frac{T}{n}} \; E\,|N(0,1)|.$$

$$= \sqrt{nT} \; E\,|N(0,1)| \xrightarrow{n \to \infty} \infty .$$

$\boxed{\text{Quadratic Variation:}}$

Def: Let M be any process.

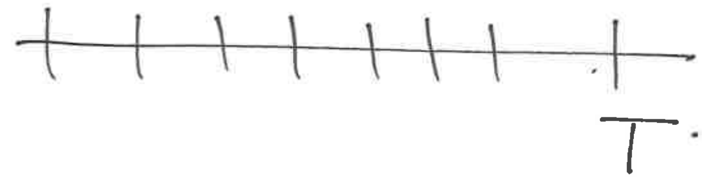
Define $[M, M](T) =$ Quadratic variation of M at time T.

$$= \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left(M(t_{i+1}) - M(t_i) \right)^2$$

<u>Prop</u>: If W is a std BM, then

$[W, W](T) = T$ a.s.

Intution: $t_i = \frac{iT}{n}$



$$\sum_{i=0}^{n} \left(W(t_{i+1}) - W(t_i) \right)^2 - T$$

$$= \underbrace{\sum_{i=0}^{n-1} \left[\left(W\left(\frac{(i+1)T}{n} \right) - W\left(\frac{iT}{n} \right) \right)^2 - \frac{T}{n} \right]}_{\xi_i}.$$

① ξ_i's are IID.

② $\xi_i \sim N\left(0, \frac{T}{n}\right)^2 - \frac{T}{n}$.

$$\Rightarrow E\xi_i = 0 \;\&\; E\xi_i^2 = E\left(N\left(0,\frac{T}{n}\right)^4 + \frac{T^2}{n^2} - 2N\left(0,\frac{T}{n}\right)^2\right)$$

$$= E\left(N(0,1)^4\right)\cdot\left(\frac{T}{n}\right)^2 + \frac{T^2}{n^2} - \frac{2T^2}{n^2}$$

$$= \underbrace{\left(EN(0,1)^4 - 1\right)}_{C}\cdot\frac{T^2}{n^2}$$

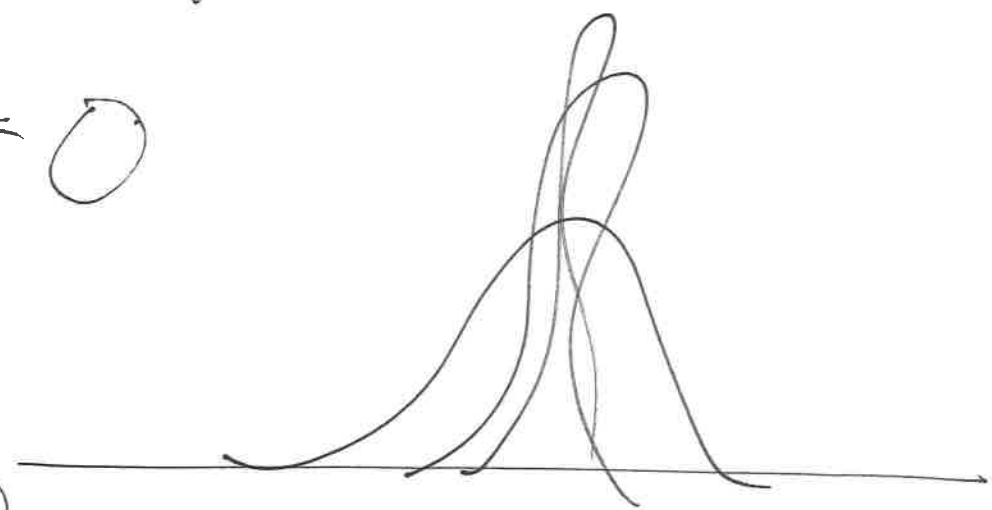
Now $\sum_0^{n-1}\left(W(t_{i+1}) - W(t_i)\right)^2 - T = \underbrace{\sum_{i=0}^{n-1}\xi_i}$

Mean : 0 & Var $= \frac{n \cdot C T^2}{n^2}$

$$\Rightarrow \mathrm{Var}\left(\sum_{0}^{n-1} \xi_i\right) = \frac{CT^2}{n} \xrightarrow[n \to \infty]{} 0$$

$$E\left(\quad \right) = 0$$

$$\Rightarrow \sum_{0}^{n-1} \xi_i \longrightarrow 0$$



Gen Claim: $W(t)^2 - t$ is a mg

Claim: $W(t)^2 - [W,W](t)$ is a mg.

In general: If M is a Mg. M^2 is not a mg.

But $M(t)^2 - [M,M](t)$ IS a mg.

Also: If $A(t)$ is any ~~cts~~ cts adapted process &

$M(t)^2 - A(t)$ is a mg $\Longrightarrow A(t) = [M,M](t)$.

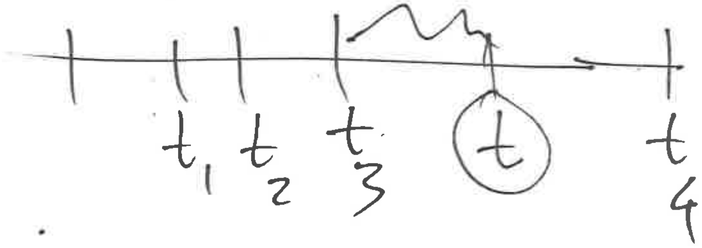
Check $W(t)^2 - t$ is a Mg.

Compute $E\left(W(t)^2 - t \mid \mathcal{F}_s\right) = E\left(\left(W(t) - W(s) + W(s)\right)^2 \mid \mathcal{F}_s\right) - t$.

$$= E\left((W(t)-W(s))^2 + W(s)^2 + 2(W(t)-W(s))W(s) \mid \mathcal{F}_s \right) - t$$

$$= t - s + W(s)^2 + 0 - t = W(s)^2 - s.$$

Construction of the Integral.



$\Delta(t)$. $\longrightarrow$ Position at time t.

$W(t) \longrightarrow$ Price of security $=$ B.M.

<u>Lemma:</u> Let $P = \{0 = t_0 < t_1 \ldots t_n = T\}$.

$\boxed{\Delta \text{ only traded at times } t_i}$. $\boxed{\begin{array}{l}\Delta \text{ is a process} \\ \Delta \text{ is ADAPTED}\end{array}}$

Define $I_p(t) = \sum_{i=0}^{n-1} \Delta(t_i)\left(W(t_{i+1}) - W(t_i)\right) \cancel{dt_i}$

$$+ \Delta(t_n)\left(W(t) - W(t_n)\right).$$

If $t \in [t_n, t_{n+1})$.

Thm: (i) $I_p(t)$ is a mg.

$EX^2 = E(X^2)$
$\cancel{(EX)^2}$

(2) $E\, I_p(t)^2 = E\left[\sum_{i=0}^{n-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_n)^2 (t - t_n)\right]$.

if $t \in [t_n, t_{n+1})$.

(3) $[I_p, I_p](t) = \sum_{i=0}^{n-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_n)^2 (t - t_n)$.

if $t \in [t_n, t_{n+1})$.

Check 2: Pretend first $t = t_M$.

Will show $E\left[\left(\sum_{0}^{M-1} \Delta(t_i)(W(t_{i+1}) - W(t_i))\right)^2\right] = E\sum_{0}^{M-1} \Delta(t_i)^2 (t_{i+1} - t_i).$

$$E\left[\sum_{0}^{M-1} \Delta(t_i)^2 . (W(t_{i+1}) - W(t_i))^2 . + 2\sum_{i<j} \Delta(t_i)\Delta(t_j) (W(t_{i+1}) - W(t_i)) . (W(t_{j+1}) - W(t_j))\right].$$

$$\mathbb{E}\, \Delta(t_i)^2 \big(W(t_{i+1}) - W(t_i)\big)^2$$

$$= \mathbb{E}\, \mathbb{E}\Big(\Delta(t_i)^2 \big(W(t_{i+1}) - W(t_i)\big)^2 \,\Big|\, \mathcal{F}_{t_i}\Big).$$

$$= \mathbb{E}\, \Delta(t_i)^2 \, \underbrace{\mathbb{E}\big(W(t_{i+1}) - W(t_i)\big)^2}_{(t_{i+1} - t_i)}.$$