Martingales:

1 - Martingales: $\longrightarrow$ "Fair Game" $\boxed{E(M(t) \mid \mathcal{F}_s) = M(s)}$

Info up to time s.

2

3 **Conditional Expectation:**

Say X is a R.V. $(\Omega, \mathcal{G})$.

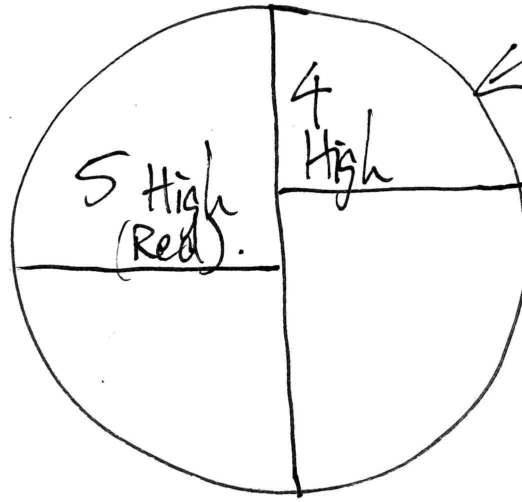
X is $\mathcal{G}$ - measurable.

$\mathcal{F} \subseteq \mathcal{G}$ $(\mathcal{F} \to \sigma$-alg $(\sigma$-sub alg of $\mathcal{G})$

"Best approximation" of X using only the σ-alg $\mathcal{F}$

is called the **conditional expectation**

Example:



$\Omega = 30$ cards.

Red $\rightarrow$ R (10 cards) B $\leftarrow$ Black (20 cards).

Game: $\left.\begin{array}{l} \text{High} \longrightarrow +1. \\ \text{Not High} \longrightarrow -1. \end{array}\right\}$ Play though a Dealor.

Dealer: (1) Pays winnigs.

(2) Tells you the COLOR.

$X \longrightarrow$ outcome of game $= 1_H - 1_{H^c}$

$\mathcal{G} = \mathcal{P}(\Omega)$. $\longleftarrow$ all info.

$\mathcal{F} =$ info you get by talking to dealer.

$= \{\phi, R, B, \Omega\}$.

X is NOT $\mathcal{F}$ measurable.

$E(X \mid \mathcal{F}) =$ "Best approx of X only given $\underline{\underline{\mathcal{F}}}$"

$= 0 \, 1_R + \left(\frac{1}{5} - \frac{4}{5}\right) 1_B$

Def: $\mathcal{G}$ – σ alg. $\mathcal{F} \subseteq \mathcal{G}$ is a σ-alg.

$X \to$ RV wrt $\mathcal{G}$ (X is $\mathcal{G}$-meas).

Define $E(X|\mathcal{F})$ to be a RV & such that

1. $E(X|\mathcal{F})$ ~~ta~~ is $\mathcal{F}$-measurable.

2. (Partial Averaging): $\forall F \in \mathcal{F}$, $\int_F X \, dP = \int_F E(X|\mathcal{F}) \, dP$.

Remark: Existence $\longrightarrow$ "RN theorem"

Remark: Uniqueness: If Y is any RV +

$$\left.\begin{array}{l} \text{① } Y \text{ is } \mathcal{G}\text{-meas.} \\ \text{② } \int_F Y \, dP = \int_F X \, dP \quad \forall F \in \mathcal{G} \end{array}\right\} \Rightarrow Y = E(X \mid \mathcal{G}).$$

$\boxed{\text{R.V.}}$

Eg 1: Say X is $\mathcal{G}$ meas.

$$E(X \mid \mathcal{G}) = X.$$

Eg 2: If X is independent of $\mathcal{G}$

then $E(X \mid \mathcal{G}) = EX$

Def: X is ind of $\mathcal{G}$ if $\forall A \in \sigma(X)$, $\forall B \in \mathcal{G}$

we have $P(A \cap B) = P(A)\,P(B)$.

Pf: Say $X = \sum a_i 1_{A_i}$, $\Rightarrow A_i \in \sigma(X)$.

Pick any $F \in \mathcal{G}$ $\int_F X\,dP = \int_F \left(\sum a_i 1_{A_i}\right) dP$

$$= \sum a_i P(A_i \cap F) = P(F) \overbrace{\sum a_i P(A_i)}^{EX} = \int_F (EX)\,dP$$

$$\Rightarrow EX = E(X \mid \mathcal{G}).$$

(2) If X is not simple $\to$ approximate.

Properties: (1) Linearity: $E(X + \alpha Y \mid \mathcal{G}) = E(X \mid \mathcal{G}) + \alpha E(Y \mid \mathcal{G})$.

$(X, Y \to$ RV's, & $\alpha \in \mathbb{R})$.

(2) If $X \leq Y$ a.s. then $E(X \mid \mathcal{G}) \leq E(Y \mid \mathcal{G})$ a.s.

(3) Say $X \to \mathcal{G}$-meas., $Y \to \mathcal{G}$-meas.

$$E(XY \mid \mathcal{G}) = \cancel{X\, E(X \mid \mathcal{G})}\ X\, E(Y \mid \mathcal{G})$$

(4) (Tower):

7 (Tower): $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$ then

$$E(X|\mathcal{E}) = E\big(E(X|\mathcal{F}) \,|\, \mathcal{E}\big).$$

Filterations: X a stochastic process. ($X(t)$ is a RV).

Def: Let $t \in [0,\infty)$. Define $\mathcal{F}_t^X = \sigma\left(\bigcup_{s\le t} \sigma(X_s)\right)$.

$\mathcal{F}_t^X \approx$ all info obsened up to time t.

Note: $\forall t$, $\mathcal{F}_t^X$ is a σ-alg & $s<t$, $\mathcal{F}_s^X \subseteq \mathcal{F}_t^X$

Def: A filtration $\{\mathcal{F}_t\}$ is a family of σ-alg

+ ① $\forall t$, $\mathcal{F}_t$ is a σ-alg.

2 ② If $s < t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$.

Def: $\mathcal{F}_t^X$ is called the filtration generated by X.

Mostly: Choose $\{\mathcal{F}_t\} = \{\mathcal{F}_t^W\}$ $W \to$ BM.

<u>Def:</u> Given a filtration $\{\mathcal{F}_t\}$. We say a process X is <u>adapted</u> to $\{\mathcal{F}_t\}$ if $\forall t$, $X(t)$ is $\mathcal{F}_t$-meas.

Martingale: We say a process M is a <u>Martingale</u> wrt the filtration $\{\mathcal{F}_t\}$ if.

1. M is adapted. ($M(t)$ is $\mathcal{F}_t$ - meas).

& 2. $E(M(t) \mid \mathcal{F}_s) = M(s)$. for $s \leq t$.

Sub Mg: ~~M~~ . $E(M(t) \mid \mathcal{F}_s) \geq M(s)$.

Super Mg: $E(M(t) \mid \mathcal{F}) \leq M(s)$.

Random Walks are Mg.

<u>Prop:</u> Brownian Motion is a Mg. (wrt the Brownian Filtration).

Pf: W a BM. $\{\mathcal{F}_t\} = \{\mathcal{F}_t^W\}$.

~~NB~~ NTS $E(W(t) \mid \mathcal{F}_s)$. for $s \leq t$.

Compute $E(W(t) \mid \mathcal{F}_s) = E(W(t) - W(s) + W(s) \mid \mathcal{F}_s)$.

$$= \underbrace{E(W(t) - W(s) \mid \mathcal{F}_s)}_{0} + \underbrace{E(W(s) \mid \mathcal{F}_s)}_{W(s).} = W(s).$$

Since: (1) $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

(2) $E(W(t) - W(s) \mid \mathcal{F}_s) = E(W(t) - W(s)) = 0$

QED.

Eg 2: Let $\mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right)$. Let X be $\mathcal{F}_\infty$ meas.

Define $M(t) = E(X \mid \mathcal{F}_t)$. Claim: M is a Mg.

<u>Check:</u> $E(M(t) \mid \mathcal{F}_s) = E(E(X \mid \mathcal{F}_t) \mid \mathcal{F}_s)$.

$$= E(X \mid \mathcal{F}_s) = M(s).$$

~~Theorem:~~ $W \to$ Std BM. $W(0) = 0$

$W(t) - W(s) \sim N(0, t-s)$.

$f \to$ Any fn. $\cancel{f(x) = x^3 + 944}$

Compute : $E\left(f(W(t)) \mid \mathcal{F}_s\right) = E\left(f\left(\boxed{W(t) - W(s)} + W(s)\right) \mid \mathcal{F}_s\right)$.

$N(0, t-s)$ & Ind of $\mathcal{F}_s$.

$$= \int_{-\infty}^{\infty} f(y + W(s)) \, \underline{G(t-s, y)} \, dy$$

Where $G(t-s, y) =$ Density of $N(0, t-s)$.

$$= \frac{1}{\sqrt{2\pi(t-s)}} e^{-y^2/2(t-s)}$$