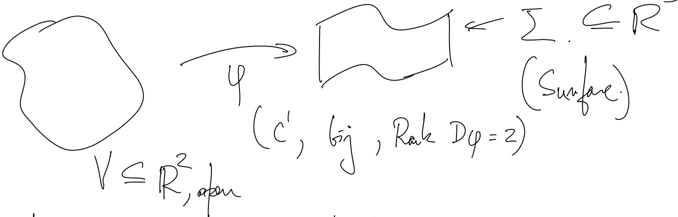
PARAMETRIZATION INVARIANCE OF SURFACE INTEGRALS

Recall:



$\Sigma \subseteq \mathbb{R}^3$ a surface, $\varphi : V \longrightarrow \Sigma$ C^1, bij & Rank $D\varphi = 2$

Say $f : \Sigma \longrightarrow \mathbb{R}$ cts.

Def: $$\int_\Sigma f \, dS \stackrel{\text{def}}{=} \int_V f \circ \varphi \, |\partial_1 \varphi \times \partial_2 \varphi| \, dA$$

Prop: (Param inv) Say $\tilde{\varphi} : \tilde{V} \longrightarrow \Sigma$ is any param of Σ $(\tilde{V} \subseteq \mathbb{R}^2 \text{ open})$.

(i.e, $\tilde{\varphi}$ is C^1, bij & Rank $D\tilde{\varphi} = 2$)

Then
$$\int_V f\circ\varphi \, |\partial_1\varphi \times \partial_2\varphi| \, dA = \int_{\tilde{V}} f\circ\tilde{\varphi} \, |\partial_1\tilde{\varphi} \times \partial_2\tilde{\varphi}| \, dA.$$

Pf: ① Let $g = \tilde{\varphi}^{-1}\varphi : V \longrightarrow \tilde{V}$, bij.

Claim: g is C^1. (You check: local flattening $\Longrightarrow g$ is C^1).

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "g"] & \Sigma \subseteq \mathbb{R}^3 \arrow[d, "f"] \\
\tilde{V} \arrow[ur, "\tilde{\varphi}"'] & \mathbb{R}
\end{tikzcd}$$

$$\boxed{\tilde{\varphi}\circ g = \varphi}$$

② Compute

$$\int_V f\circ\varphi \ |\partial_1\varphi \times \partial_2 \varphi| \, dA = \int_V f\circ\tilde{\varphi}\circ g \ |\partial_1(\tilde{\varphi}\circ g) \times \partial_2(\tilde{\varphi}\circ g)| \, dA$$

Note: $\partial_1(\tilde{\varphi}\circ g) = D(\tilde{\varphi}\circ g)\, e_1 = D\tilde{\varphi}_g \, \partial_1 g$

$$= \partial_1 g_1 \ \partial_1 \tilde{\varphi}_g + \partial_1 g_2 \ \partial_2 \tilde{\varphi}_g$$

$$\Rightarrow \int_V f\circ\varphi \ |\partial_1\varphi \times \partial_2\varphi| \, dA = \int_V f\circ\tilde{\varphi}\circ g \ \left| (\partial_1 g_1 \, \partial_1\tilde{\varphi}_g + \partial_1 g_2 \, \partial_2 \tilde{\varphi}_g) \times (\partial_2 g_1 \, \partial_1 \tilde{\varphi}_g + \partial_2 g_2 \, \partial_2 \tilde{\varphi}_g) \right| dA$$

$$= \int_V f\circ\tilde{\varphi}\circ g \ \left| 0 + \partial_1 g_1 \partial_2 g_2 \, \partial_1\tilde{\varphi}_g \times \partial_2 \tilde{\varphi}_g + \partial_1 g_2 \partial_2 g_1 \, \partial_2 \tilde{\varphi}_g \times \partial_1 \tilde{\varphi}_g + 0 \right| dA$$

$$= \int_V f\circ\tilde{\varphi}\circ g \left| 0 + \partial_1 g_1 \partial_2 g_2 \, \partial_1 \tilde{\varphi}_g \times \partial_2 \tilde{\varphi}_g + \partial_1 g_2 \partial_2 g_1 \, \partial_2 \tilde{\varphi}_g \times \partial_1 \tilde{\varphi}_g + 0 \right| dA$$

$$= \int_V f\circ\tilde{\varphi}\circ g \left| \left(\partial_1 g_1 \, \partial_2 g_2 - \partial_1 g_2 \partial_2 g_1\right) \partial_1 \tilde{\varphi} \times \partial_2 \tilde{\varphi} \right| dA$$

$$= \int_V f\circ\tilde{\varphi}\circ g \; |\partial_1 \tilde{\varphi}_g \times \partial_2 \tilde{\varphi}_g| \; |\det(Dg)| \; dA$$

$$= \int_V \left[(f\circ\tilde{\varphi}) \, |\partial_1 \tilde{\varphi} \times \partial_2 \tilde{\varphi}| \right] \circ g \; |\det(Dg)| \; dA$$

Change of var $$= \int_{\tilde{V}} f\circ\tilde{\varphi} \; |\partial_1 \tilde{\varphi} \times \partial_2 \tilde{\varphi}| \; dA.$$

QED.