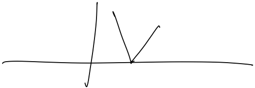
GREENS THEOREM.

① Piecewise C^1 fn:

Defⁿ: $f: (0,1) \longrightarrow \mathbb{R}^d$ is piecewise C^1 if $\exists$ a finite set $S \subseteq (0,1)$ & $f: (0,1) - S \longrightarrow \mathbb{R}^d$ is C^1 & $f: (0,1) \longrightarrow \mathbb{R}^d$ is cts & $\forall s \in S$, $\lim\limits_{x \to s^-} f'(x)$ & $\lim\limits_{x \to s^+} f'(x)$ both exist. (need not be equal).

Typical Eg:  (piecewise C^1, not C^1).

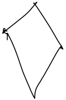
② $C \subseteq \mathbb{R}^d$ is a closed curve if ~~① C is connected~~ ← not always required.

& ② C is a C^1 curve & ③ C is closed & bdd.

③ If C is a closed curve, then we denote line integrals by

$$\int_C F \cdot dl = \oint_C F \cdot dl.$$

④ Will henceforth assume all curves are piecewise C^1.

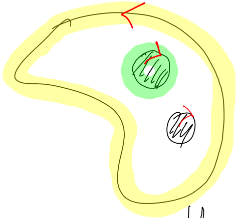
(i.e.  ← is a piecewise C^1 closed curve

is not a C^1 closed curve).

Thm: (Greens Theorem) $U \subseteq \mathbb{R}^d$ is open, $\boxed{\text{connected}}$, ∂U is a piecewise C^1 closed curve ($\Rightarrow U$ is bdd!).

Let $F : \overline{U} \longrightarrow \mathbb{R}^2$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

Then
$$\oint_{\partial U} F \cdot dl = \int_U (\partial_1 F_2 - \partial_2 F_1)\, dA$$



U

Convention: ① Outer boundaries of U are traversed counter clockwise (CCW)
② Inner " of " " " clockwise (CW).

Idea of Proof:

① Prove Green's thm on the unit square $S = (0,1)^2$

② Prove Green's thm assuming $\exists$ C^1 bij $\varphi : S \to U$.

③ Inner boundaries etc.

Step I: $S = (0,1)^2$. $F : S \to \mathbb{R}^2$

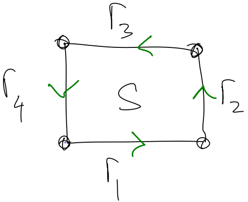
NTS $\displaystyle \int_S \underbrace{(\partial_1 F_2 - \partial_2 F_1)}_{\text{Compute.}} \, dA = \int_{\partial S} F \cdot dl.$

$$\int_S (\partial_1 F_2 - \partial_2 F_1)\, dA = \int_{x_1=0}^{1} \int_{x_2=0}^{1} (\partial_1 F_2 - \partial_2 F_1)\, dx_2\, dx_1$$

$$= \int_{x_2=0}^{1} \underbrace{\int_{x_1=0}^{1} \partial_1 F_2\, dx_1}\, dx_2 - \int_{x_1=0}^{1} \int_{x_2=0}^{1} \partial_2 F_1\, dx_2\, dx_1$$

$$= \int_{x_2=0}^{1} \Big(\underbrace{F_2(1, x_2)}_{\int_{\Gamma_2} F \cdot dl} - \underbrace{F_2(0, x_2)}_{\int_{\Gamma_4} F \cdot dl} \Big)\, dx_2 - \int_{x_1=0}^{1} \Big(\underbrace{F_1(x_1, 1)}_{\int_{\Gamma_3} F \cdot dl} - \underbrace{F_1(x_1, 0)}_{\int_{\Gamma_1} F \cdot dl} \Big)\, dx_1$$

① $\int_{\Gamma_1} F \cdot dl$: Param Γ_1.



$$\gamma_1(x_1) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}. \quad x_1 \in (0,1)$$

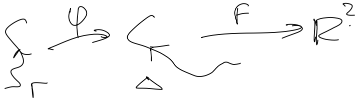
$$\int_{\Gamma_1} F \cdot dl = \int_0^1 F(x_1, 0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx_1$$

$$= \int_0^1 F_1(x_1, 0)\, dx_1$$

$$\Rightarrow \int_S (\partial_1 F_2 - \partial_2 F_1)\, dA = \int_{\Gamma_1} F \cdot dl + \int_{\Gamma_2} F \cdot dl + \int_{\Gamma_3} F \cdot dl + \int_{\Gamma_4} F \cdot dl = \oint_{\partial S} F \cdot dl$$

QED ①

Step 2: Coordinate change.



Lemma: Say $\Gamma \subseteq \mathbb{R}^2$ is a pw C^1 curve. $U \subseteq \mathbb{R}^2$ open $U \supseteq \Gamma$.

Say $\varphi : U \longrightarrow V$ is C^1 (bij). Let $\Delta = \varphi(\Gamma)$.

$V \subseteq \mathbb{R}^2$.

Say $F : \Delta \longrightarrow \mathbb{R}^2$ is C^1. $\displaystyle\int_\Delta F \cdot dl = \int_\Gamma (D\varphi)^T F \circ \varphi \cdot dl$

Proof: Let $\gamma : [0,1] \longrightarrow \Gamma$ be a param of $\Gamma \Rightarrow \varphi \circ \gamma$ is a param of Δ.

$$\Rightarrow \int_{\Delta} F \cdot d\ell = \int_{t=0}^{1} F \circ (\varphi \circ \gamma) \cdot (\varphi \circ \gamma)' \, dt$$

$$= \int_{t=0}^{1} (F \circ \varphi) \circ \gamma \cdot \left(D\varphi_{\gamma} \, \gamma' \right) dt \qquad \left(Au \cdot v = u \cdot A^{T} v \right).$$

$$= \int_{t=0}^{1} \left((D\varphi)^{T} F \circ \varphi \right) \circ \gamma \cdot \gamma' \, dt$$

$$= \int_{\Gamma} (D\varphi)^{T} F \circ \varphi \cdot d\ell \qquad \text{QED.}$$

Part II of Greens theorem..

$S = (0,1)^2$ (Know Greens thm on S).

Say $\exists$ a C^1 bij $\varphi : S \longrightarrow U$ (preserves the orientation of $\partial U \Rightarrow \det D\varphi > 0$)

Let $F : \overline{U} \longrightarrow \mathbb{R}^2$ be C^1

NTS $\displaystyle\oint_{\partial U} F \cdot dl = \int_U (\partial_1 F_2 - \partial_2 F_1)\, dA$

$\partial U = \varphi(\partial S)$: $\displaystyle\int_{\partial U} F \cdot dl = \int_{\partial S} (D\varphi)^T F\circ\varphi \cdot dl$

Let $G = (D\varphi)^T F \circ \varphi \Rightarrow \displaystyle\int_{\partial U} F \cdot dl = \int_{\partial S} G \cdot dl$

Part I

$$\overset{\downarrow}{=} \int_S (\partial_1 G_2 - \partial_2 G_1)\, dA \qquad (G = (D\varphi)^T\, F \circ \varphi).$$

You check: $\partial_1 G_2 - \partial_2 G_1 = \left((\partial_1 F_2 - \partial_2 F_1) \circ \varphi \right) \det(D\varphi) \quad \cdots \quad *$

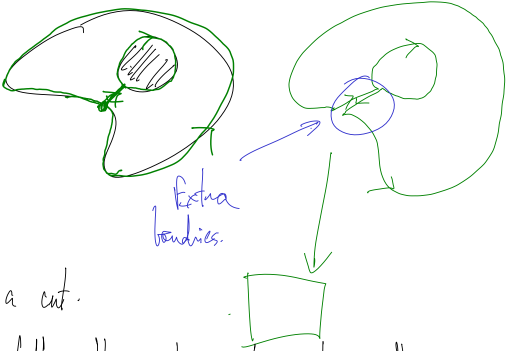
Check.

Key: $\det(D\varphi) > 0 \Rightarrow \det(D\varphi) = |\det(D\varphi)|$.

$$\int_S (\partial_1 G_2 - \partial_2 G_1)\, dA = \int_S (\partial_1 F_2 - \partial_2 F_1) \circ \varphi\, |\det(D\varphi)|\, dA$$

$$= \int_U (\partial_1 F_2 - \partial_2 F_1)\, dA \qquad \text{QED.}$$

③ Say U =



If U has holes, make a cut.

now transversing the boundary, follow the cut inside, trace the inner boundary CW follow the cut outside. The line integral over the boundary now involves two line integrals over the cut, which cancel. ... QED