Monday, January 12, 2015

1 Real Numbers

There are two ways to introduce the real numbers. The first is to give them in an axiomatic way, the second is to construct them starting from the natural numbers. We will use the first method.

The *real numbers* are a set $\mathbb{R}$ with two binary operations, *addition*

$$\begin{aligned} + : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$

and *multiplication*

$$\begin{aligned} \cdot : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

and a relation $\leq$ such that $(\mathbb{R}, +, ., \leq)$ is an *ordered field*, that is,

(A) $(\mathbb{R}, +)$ is an *commutative group*, that is,

- (A_1) for every $a, b \in \mathbb{R}$, $a + b = b + a$,
- (A_2) for every $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$,
- (A_3) there exists a unique element in $\mathbb{R}$, called *zero* and denoted 0, such that $0 + a = a + 0 = a$ for every $a \in \mathbb{R}$,
- (A_4) for every $a \in \mathbb{R}$ there exists a unique element in $\mathbb{R}$, called the *opposite* of a and denoted $-a$, such that $(-a) + a = a + (-a) = 0$,

(M)

- (M_1) for every $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$,
- (M_2) for every $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- (M_3) there exists a unique element in $\mathbb{R}$, called *one* and denoted 1, such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$ for every $a \in \mathbb{R}$ with $a \neq 0$,
- (M_4) for every $a \in \mathbb{R}$ with $a \neq 0$ there exists a unique element in $\mathbb{R}$, called the *inverse* of a and denoted a^{-1}, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$,

(O) $\leq$ is a *total order relation*, that is,

- (O_1) for every $a, b \in \mathbb{R}$ either $a \leq b$ or $b \leq a$,
- (O_2) for every $a, b, c \in \mathbb{R}$ if $a \leq b$ and $b \leq c$, then $a \leq c$,
- (O_3) for every $a, b \in \mathbb{R}$ if $a \leq b$ and $b \leq a$, then $a = b$,
- (O_4) for every $a \in \mathbb{R}$ we have $a \leq a$,

(AM) for every $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

(AO) for every $a, b, c \in \mathbb{R}$ if $a \leq b$, $a + c \leq b + c$,

(MO) for every $a, b \in \mathbb{R}$ if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$,

(S) **(supremum property)**

Remark 1 *Properties (A), (M), (O), (AM), (AO), (MO), and (S) completely characterize the real numbers in the sense that if $(\mathbb{R}', \oplus, \odot, \preccurlyeq)$ satisfies the same properties, then there exists a bijection $T : \mathbb{R} \to \mathbb{R}'$ such that T is an isomorphism between the two fields, that is,*

$$T(a + b) = T(a) \oplus T(b), \qquad T(a \cdot b) = T(a) \odot T(b)$$

for all $a, b \in \mathbb{R}$, and $a \leq b$ if and only if $T(a) \preccurlyeq T(b)$. Hence, for all practical purposes, we cannot distinguish $\mathbb{R}$ from $\mathbb{R}'$.

If $a \leq b$ and $a \neq b$, we write $a < b$.

Exercise 2 *Using **only** the axioms (A), (M), (O), (AO), (AM) and (MO) of $\mathbb{R}$, prove the following properties of $\mathbb{R}$:*

(i) *if $a \cdot b = 0$ then either $a = 0$ or $b = 0$,*

(ii) *if $a \geq 0$ then $-a \leq 0$,*

(iii) *if $a \leq b$ and $c < 0$ then $ac \geq bc$,*

(iv) *for every $a \in \mathbb{R}$ we have $a^2 \geq 0$,*

(v) $1 > 0$.

Definition 3 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) *An element $L \in \mathbb{R}$ is called an* upper bound *of E if $x \leq L$ for all $x \in E$;*

(ii) *E is said to be* bounded from above *if it has at least an upper bound;*

(iii) *if E is bounded from above, the least of all its upper bounds, if it exists, is called the* supremum *of E and is denoted $\sup E$.*

(iv) *E has a* maximum *if there exists $L \in E$ such that $x \leq L$ for all $x \in E$. We write $L = \max E$.*

We are now ready to state the supremum property.

(S) **(supremum property)** every nonempty set $E \subseteq \mathbb{R}$ bounded from above has a supremum in $\mathbb{R}$.

The supremum property says that in $\mathbb{R}$ the supremum of a nonempty set bounded from above always exists in $\mathbb{R}$. We will see that this is not the case for the rationals numbers.

Remark 4 *(i) Note that if a set has a maximum L, then L is also the supremum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from above, to prove that a number $L \in \mathbb{R}$ is the supremum of E, we need to show that L is an upper bound of E, that is, that $x \leq L$ for every $x \in E$, and that any number $s < L$ cannot be an upper bound of E, that is, that there exists $x \in E$ such that $s < x$.

Example 5 *Let $E := \{x \in \mathbb{R} : \ x < 1\}$. Then 1 is an upper bound of the set E and so E is bounded from above. We claim that 1 is the supremum of the set E. To see this, let $y \in \mathbb{R}$ with $y < 1$. We need to prove that y is not an upper bound of the set E, that is, we need to show that there are elements in the set E that are larger than y. Take $x := \frac{1+y}{2}$. Since $y < 1$, we have that $1 + y < 1 + 1$, and so $\frac{1+y}{2} < 1$. Thus x belongs to E. On the other hand, $x = \frac{1+y}{2} > y$, and so y is not an upper bound of E. This shows that $1 = \sup E$. Note that 1 does not belong to the set E and so the set E has no maximum.*

Definition 6 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) An element $\ell \in \mathbb{R}$ is called a lower bound *of E if $\ell \leq x$ for all $x \in E$;*

(ii) E is said to be bounded from below *if it has at least a lower bound;*

(iii) if E is bounded from below, the greatest of all its lower bounds, if it exists, is called the infimum *of E and is denoted $\inf E$;*

(iv) E has a minimum *if there exists $\ell \in E$ such that $\ell \leq x$ for all $x \in E$. We write $\ell = \min E$.*

Remark 7 *(i) Note that if a set has a minimum ℓ, then ℓ is also the infimum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from below, to prove that a number $\ell \in \mathbb{R}$ is the infimum of E, we need to show that ℓ is a lower bound of E, that is, that $\ell \leq x$ for every $x \in E$, and that any number $\ell < s$ cannot be a lower bound of E, that is, that there exists $x \in E$ such that $x < s$.

2 Natural Numbers

Definition 8 *A set $E \subseteq \mathbb{R}$ is called an* inductive set *if it has the following properties*

(i) the number 1 belongs to E,

(ii) if a number x belongs to E, then $x + 1$ also belongs to E.

Example 9 *The sets $[0, \infty) = \{x \in \mathbb{R} : \ 0 \leq x\}$, $[1, \infty) = \{x \in \mathbb{R} : \ 1 \leq x\}$, and $\mathbb{R}$ are all inductive sets.*

Definition 10 *The set of the* natural numbers $\mathbb{N}$ *is defined as the intersection of all inductive sets of* $\mathbb{R}$.

Note that $\mathbb{N}$ is nonempty, since 1 belongs to every inductive set, and so also to $\mathbb{N}$. We also define

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Example 11 *The number $\frac{1}{2}$ is not a natural number. Indeed, $[1,\infty)$ is an inductive set and $\frac{1}{2}$ does not belong to E, so $\frac{1}{2}$ cannot belong to $\mathbb{N}$. Also $\frac{3}{2}$ is not a natural number. Indeed, the set $E = \{1\} \cup \{n \in \mathbb{N} : \ n \geq 2\}$ is an inductive set that does not contain $\frac{3}{2}$. Hence, $\frac{3}{2}$ cannot be a natural number.*

Proposition 12 *The set $\mathbb{N}$ is an inductive set.*

Proof. We already know that 1 belongs to $\mathbb{N}$. If x belongs to $\mathbb{N}$, then it belongs to every inductive set E but then, since E is an inductive set, it follows that $x+1$ belongs E. Hence, $x+1$ belongs to every inductive set, and so by definition of $\mathbb{N}$, we have that $x+1$ also belongs to $\mathbb{N}$. ■

The next result is very important.

Theorem 13 (Principle of mathematical induction) *Let $\{p_n\}$, $n \in \mathbb{N}$, be a family of propositions such that*

(i) p_1 is true,

(ii) if p_n is true for some $n \in \mathbb{N}$, then p_{n+1} is also true.

Then p_n is true for every *$n \in \mathbb{N}$.*

Proof. Let $E := \{n \in \mathbb{N} \text{ such that } p_n \text{ is true}\}$. Note that $E \subseteq \mathbb{N}$. It follows by (i) and (ii) that E is an inductive set, and so E contains $\mathbb{N}$ (since $\mathbb{N}$ is the intersection of all inductive sets). Hence, $E = \mathbb{N}$. ■

Wednesday, January 14, 2015

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

If $x \neq 0$, we define $x^0 := 1$. We do not define 0^0.

The following will be used later on.

Exercise 14 *Let $x \geq -1$. Prove that*

$$(1+x)^n \geq 1 + nx \tag{1}$$

for every $n \in \mathbb{N}$.

Exercise 15 *Prove that*

$$1 + \cdots + n = \frac{n(n+1)}{2} \tag{2}$$

for every $n \in \mathbb{N}$

Exercise 16 *Let $x \neq 1$. Prove that*

$$1 + x \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for all $n \in \mathbb{N}$.

In what follows $0! := 1$, $1! := 1$ and $n! := 1 \cdot 2 \cdot \cdots \cdot n$ for all $n \in \mathbb{N}$. The number $n!$ is called the *factorial* of n. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}.$$

Exercise 17 *Let $j, k \in \mathbb{N}$ and $a \in \mathbb{R}$. Given the function $f(x) = (x+a)^j$, prove that*

$$\frac{d^k f}{dx^k}(x) = \begin{cases} 0 & \text{if } k > j, \\ j(j-1)\cdots(j-k+1)(x+a)^{j-k} & \text{if } k < j, \\ k! & \text{if } k = j. \end{cases}$$

Exercise 18 *Let $x, y \in \mathbb{R} \setminus \{0\}$ and let $n \in \mathbb{N}$.*

(i) Prove that for every $1 \leq k \leq n$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

(ii) Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Remark 19 *If in Theorem 13 we replace property (i) with*

(i)′ if p_{n_0} is true for some $n_0 \in \mathbb{N}$,

then we can conclude that p_n is true for all $n \in \mathbb{N}$ with $n \geq n_0$. To see this, it is enough to define

$$E := \{n \in \mathbb{N} \text{ such that } p_{n+n_0-1} \text{ is true}\},$$

which is still an inductive set.

Exercise 20 *Prove that*

$$n^n > 2^n n!$$

for all $n > 6$. Hint: Use the binomial theorem.

Proposition 21 (Archimedean Property) *If $a, b \in \mathbb{R}$ with $a > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. If $b \leq 0$, then $n = 1$ will do. Thus, assume that $b > 0$. Assume by contradiction that $na \leq b$ for all $n \in \mathbb{N}$ and define the set

$$E = \{na : n \in \mathbb{N}\}.$$

Then the set E is nonempty and has an upper bound, b. By the supremum property, there exists $L = \sup E$. Hence, for every $m \in \mathbb{N}$, we have that $(m+1)\,a \leq L$, or, equivalently, $ma \leq L - a$ for all $m \in \mathbb{N}$. But this shows that $L - a$ is an upper bound of E, which contradicts the fact that L is the least upper bound. ■

In the previous section we have defined the natural numbers. Note that $(\mathbb{N}, +, \cdot, \leq)$ does not satisfy properties (A_3), (A_4), and (M_4). In particular, we cannot subtract two numbers $a, b \in \mathbb{N}$ unless, $a \geq b + 1$. For this reason, we define the *set of integers* $\mathbb{Z}$ as follows

$$\mathbb{Z} := \{\pm n : n \in \mathbb{N}\} \cup \{0\}.$$

The next result is left as an exercise.

Corollary 22 (The integer part) *Given a real number $x \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \leq x < k + 1$.*

Definition 23 *Given a real number $x \in \mathbb{R}$, the integer k given by the previous corollary is called the* integer part *of x and is denoted $\lfloor x \rfloor$. The number $x - \lfloor x \rfloor$ is called the* fractional part *of x and is denoted $\operatorname{frac} x$ (or $\{x\}$). Note that $0 \leq \operatorname{frac} x < 1$.*

Exercise 24 *Prove that every nonempty subset of the natural numbers has a minimum.*

3 The Rationals Numbers and the Supremum Property

Now $(\mathbb{Z}, +, \cdot, \leq)$ satisfies properties (A_3), (A_4), but not (M_4). To resolve this issue, we introduce the *set of rational numbers* $\mathbb{Q}$ defined by

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\, q \neq 0 \right\},$$

where $\frac{p}{q} := p \cdot q^{-1}$. Then $(\mathbb{Q}, +, \cdot, \leq)$ satisfies properties (A), (M), (O), (AM), (AO), (MO). So, what's wrong? We will see that the rational numbers do not satisfy the supremum property.

Theorem 25 *There does not exist a rational number r such that $r^2 = 2$.*

Proof. Exercise. ■

Thus in the set of rational numbers the square root $\sqrt{r}$ is not defined, in general.

This follows from the Archimedean Property.

Corollary 26 (Density of the rationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. By the Archimedean property (applied with 1 and $\frac{1}{b-a}$ in place of a and b) there exists $q \in \mathbb{N}$ such that $0 < \frac{1}{b-a} < q$. By the previous corollary there exists an integer $p \in \mathbb{Z}$ such that

$$p \leq qa < p + 1. \tag{3}$$

Note that since $1 < q(b-a)$,

$$p + 1 \leq qa + 1 < qa + q(b-a) = qb. \tag{4}$$

It follows by (3) and (4) that

$$qa < p + 1 < qb.$$

Multiplying by $\frac{1}{q} > 0$ gives

$$a < \frac{p+1}{q} < b.$$

It suffices to define $r := \frac{p+1}{q}$. ■

Corollary 27 (Density of the irrationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.*

Proof. Since $a < b$, we have that $\sqrt{2}a < \sqrt{2}b$. By the density of the rationals, there exists $r \in \mathbb{Q}$ such that $\sqrt{2}a < r < \sqrt{2}b$. Without loss of generality, we may assume that $r \neq 0$ (why?). Hence, $a < \frac{r}{\sqrt{2}} < b$. Since $\frac{r}{\sqrt{2}}$ is irrational (why?), the result is proved. ■

Theorem 28 *The rational numbers do not satisfy the supremum property.*

Proof. We need a nonempty set $E \subseteq \mathbb{Q}$ bounded from below but for which there exists no supremum in $\mathbb{Q}$. Define

$$E := \left\{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\right\}.$$

Then E is nonempty, since $1 \in E$. Moreover, E is bounded from below, since 2 is an upper bound.

Let's prove that if $y \in \mathbb{Q} \setminus E$ and $y > 0$, then y is an upper bound of E. Indeed, let $x \in E$. If $x > 0$, then $x^2 < 2 < y^2$, which, since $y > 0$, implies that $x < y$ (why?).

Assume by contradiction that there exists $L \in \mathbb{Q}$ such that $L = \sup E$. It cannot be $L \leq 0$, since $1 \in E$ and $1 > 0$. Hence, $L > 0$. Let's prove that it cannot be $L^2 < 2$.

Choose $n \in \mathbb{N}$ so large that $n > \frac{2L+1}{2-L^2}$. Then

$$\left(L + \frac{1}{n}\right)^2 = L^2 + \frac{1}{n^2} + \frac{2L}{n} < L^2 + \frac{1}{n} + \frac{2L}{n} = L^2 + \frac{2L+1}{n} < 2,$$

by the choice of n. Hence, $L + \frac{1}{n}$ belongs to E, which contradicts the fact that L is an upper bound of E. Similarly, taking $L - \frac{1}{n}$, for n large, we can show that $\left(L - \frac{1}{n}\right)^2 > 2$ and $L - \frac{1}{n} > 0$, which, by what we proved before, shows that $L - \frac{1}{n}$ is an upper bound of E. This contradicts the fact that L is the least upper bound of E. Hence, it cannot be $L^2 > 2$, Thus, $L^2 = 2$, which is again a contradiction by Theorem 25. ■

The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational numbers*.

Theorem 29 *The set of irrational numbers is nonempty.*

Proof. Take

$$E := \left\{x \in \mathbb{R} : \, 0 < x \text{ and } x^2 < 2\right\}.$$

Exactly as in the previous proof, we have that E is nonempty and bounded from above. Hence, by the supremum property there exists $L \in \mathbb{R}$ such that $L = \sup E$. It follows as in the previous proof that $L^2 = 2$, and so L belongs to $\mathbb{R} \setminus \mathbb{Q}$. ■

The number L is denoted $\sqrt{2}$ and called *square root* of 2.

Friday, January 16, 2015

Similarly, for every $n \in \mathbb{N}$ with n even and every $x \in \mathbb{R}$ with $x \geq 0$, we can show that there exists a unique $y \in \mathbb{R}$ with $y \geq 0$ such that $x^n = y$. On the other hand, for every $n \in \mathbb{N}$ with n odd and every $x \in \mathbb{R}$, we can show that there exists a unique $y \in \mathbb{R}$ such that $x^n = y$.

The number y is denoted $\sqrt[n]{x}$ and called *n-th root* of x.

Exercise 30 (The n-th root of a) *Given $x > 0$ and $n \in \mathbb{N}$, with $n \geq 2$, we want to define the n-th root of x.*

(i) Prove that if $r, s \in \mathbb{Q}$ with $r < s$, then $r^n < s^n$.

(ii) Prove that the set

$$E := \{r^n : \, r \in \mathbb{Q}, \, r > 1\}$$

does not have a minimum and that $\inf E = 1$.

(iii) Given $x > 0$ consider the set

$$F := \{y \in \mathbb{R} : \, y > 0, \; y^n \leq x\}.$$

Prove that F is bounded from above and nonempty. Let $\ell := \sup F$. Prove that $\ell^n = x$.

4 Powers with Real Exponents

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

But what does it mean $x^{\sqrt{2}}$? Or more generally, x^a if $a \in \mathbb{R}$? To define this, we will assume that $x > 0$ (this is needed to preserve the properties of powers). If a is positive and rational, say $a = \frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{\frac{n}{m}} := \left(\sqrt[m]{x}\right)^n.$$

Remark 31 *Note that $x^{\frac{n}{m}} = \sqrt[m]{x^n}$. Indeed, let $y = \sqrt[m]{x}$. Then*

$$(y^n)^m = (y^m)^n = x^n,$$

and so $y^n = \sqrt[m]{x^n}$, that is, $\left(\sqrt[m]{x}\right)^n = \sqrt[m]{x^n}$.

If a is rational and negative, say $a = -\frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{-\frac{n}{m}} := \left(x^{-1}\right)^{\frac{n}{m}}.$$

Exercise 32 *Prove that if $x > 0$ and $r, q \in \mathbb{Q}$, then*

$$x^r \cdot x^s = x^{r+s},$$
$$(x^r)^s = (x^s)^r = x^{rs}.$$

Exercise 33 *Let $x > 1$ and $r, q \in \mathbb{Q}$.*

(i) Prove that if $r > 0$, then $x^r > 1$.

(ii) Prove that if $r < s$, then $x^r < x^s$.

Define

$$\mathbb{Q}^+ := \{r \in \mathbb{Q} : \ r > 0\}.$$

We are now ready to define x^a for a real. Assume that $x > 1$ and $a > 0$. Consider the set

$$E_a := \{x^r : \ r \in \mathbb{Q}^+, \ r < a\}.$$

The set E_a is bounded from above and nonempty. We define $x^a := \sup E_a$.

Theorem 34 *Let $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ and let $x \in \mathbb{R}$ with $x > 1$. Then*

$$x^a \cdot x^b = x^{a+b}$$

Proof. Consider the three sets

$$E_a := \{x^r : \ r \in \mathbb{Q}^+, \ r < a\},$$
$$E_b := \{x^s : \ s \in \mathbb{Q}^+, \ s < b\},$$
$$E_{a+b} := \{x^t : \ t \in \mathbb{Q}^+, \ t < a + b\},$$

and let $\ell_a = \sup E_a$, $\ell_b = \sup E_b$, and $\ell_{a+b} = \sup E_{a+b}$. Let's prove that

$$\ell_a \ell_b \leq \ell_{a+b}.$$

If $r \in \mathbb{Q}^+$ is such that $r < a$ and $s \in \mathbb{Q}^+$ is such that $s < b$, then $r + s \in \mathbb{Q}^+$ and $r + s < a + b$. Hence,

$$x^r x^s = x^{r+s} \leq \ell_{a+b}.$$

Fix $s \in \mathbb{Q}^+$ with $s < b$ and divide by x^s. Then

$$x^r \leq \frac{\ell_{a+b}}{x^s}$$

for all $r \in \mathbb{Q}^+$ with $r < a$. This shows that the number $\frac{\ell_{a+b}}{x^s}$ is an upper bound for the set E_a. Hence,

$$\ell_a \leq \frac{\ell_{a+b}}{x^s}.$$

Now rewrite this inequality as

$$x^s \leq \frac{\ell_{a+b}}{\ell_a}.$$

Recall that $s \in \mathbb{Q}^+$ with $s < b$. Since the previous inequality is true for all such s, it shows that the number $\frac{\ell_{a+b}}{\ell_a}$ is an upper bound for the set E_a. Hence,

$$\ell_b \leq \frac{\ell_{a+b}}{\ell_a}.$$

Thus, we have proved that

$$\ell_a \ell_b \leq \ell_{a+b}.$$

Next let's prove that

$$\ell_{a+b} \leq \ell_a \ell_b.$$

Consider $t \in \mathbb{Q}^+$ with $t < a + b$. We want to find $p, q \in \mathbb{Q}^+$ with $t < p + q$, $p < a$ and $q < b$. Since $t - a < b$, by the density of the rationals there exists $q \in \mathbb{Q}$ such that $t - a < q < b$. Since $b > 0$ we can assume that $q > 0$ (if not apply the density of the rationals once more). Since $t - a < q$ we have that $t - q < a$ and so again by the density of the rationals there exists $p \in \mathbb{Q}$ such that $t - q < p < a$. Again, since $a > 0$ we can assume that $p > 0$ (if not apply the density of the rationals once more). Thus, $t < p + q$ and so by Exercises 32 and 33,

$$x^t < x^{p+q} = x^p \cdot x^q \leq \ell_a \ell_b.$$

Since this is true for all $t \in \mathbb{Q}^+$ with $t < a + b$ we have that $\ell_a \ell_b$ is an upper bound of the set E_{a+b} and so $\ell_{a+b} \leq \ell_a \ell_b$. ■

If $0 < x < 1$, we set

$$x^a := \left(x^{-1}\right)^{-a}.$$

Exercise 35 *Let $x > 0$ and $a, b \in \mathbb{R}$. Prove that*

$$(x^a)^b = \left(x^b\right)^a = x^{ab}.$$

Hint: It is enough to show $(x^a)^b = x^{ab}$. Consider first the case in which a is real and b is rational.

Monday, January 19, 2015

MLK day, no classes.

Wednesday, January 21, 2015

Given a number $x \in \mathbb{R}$, the *absolute value* of x is the number

$$|x| := \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value satisfies the following properties, which are left as as exercise.

Theorem 36 *Let $x, y, z \in \mathbb{R}$. Then the following properties hold.*

(i) $|x| \geq 0$ *for all* $x \in \mathbb{R}$, *with* $|x| = 0$ *if and only if* $x = 0$,

(ii) $|-x| = |x|$ *for all* $x \in \mathbb{R}$,

(iii) *if* $y \geq 0$ *and* $x \in \mathbb{R}$, *then* $|x| \leq y$ *if and only if* $-y \leq x \leq y$,

(iv) $-|x| \leq x \leq |x|$ *for all* $x \in \mathbb{R}$,

(iii) $|xy| = |x|\,|y|$ *for all* $x, y \in \mathbb{R}$,

(iv) $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

5 Inner Products, Norms, Distances

Definition 37 *A* vector space*, or* linear space*, over* $\mathbb{R}$ *is a nonempty set* X*, whose elements are called* vectors*, together with two operations,* addition *and* multiplication by scalars,

$$\begin{array}{c} X \times X \to X \\ (x, y) \mapsto x + y \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{R} \times X \to X \\ (t, x) \mapsto tx \end{array}$$

with the properties that

(i) $(X, +)$ *is a commutative group, that is,*

  (a) $x + y = y + x$ *for all* $x, y \in X$ *(commutative property),*

  (b) $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in X$ *(associative property),*

  (c) there is a vector $0 \in X$*, called* zero*, such that* $x + 0 = 0 + x = x$ *for all* $x \in X$*,*

  (d) for every $x \in X$ *there exists a vector in* X*, called the* opposite *of* x *and denoted* $-x$*, such that* $x + (-x) = 0$*,*

(ii) for all $x, y \in X$ *and* $s, t \in \mathbb{R}$*,*

  (a) $s\,(tx) = (st)\,x$*,*

  (b) $1x = x$*,*

(c) $s(x+y) = (sx) + (sy)$,

(d) $(s+t)x = (sx) + (tx)$.

Remark 38 *Instead of using real numbers, one can use a field F. For most or our purposes the real numbers will suffice.* From now on, whenever we don't specify, it is understood that a vector space is over $\mathbb{R}$.

Example 39 *Some important examples of vector spaces over $\mathbb{R}$ are the following.*

(i) The Euclidean space $\mathbb{R}^N$ *is the space of all N-tuples $\mathbf{x} = (x_1, \ldots, x_N)$ of real numbers. The elements of $\mathbb{R}^N$ are called* vectors *or* points*. The Euclidean space is a vector space with the following operations*

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_N + y_N), \quad t\mathbf{x} := (tx_1, \ldots, tx_N)$$

for every $t \in \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ in $\mathbb{R}^N$.

(ii) The collection of all polynomials $p : \mathbb{R} \to \mathbb{R}$.

(iii) The space of continuous functions $f : [a,b] \to \mathbb{R}$.

Definition 40 *Given a set E and a function $f : E \to \mathbb{R}$, we say that f is* bounded from above *if the set*

$$f(E) := \{y \in \mathbb{R} : y = f(x), x \in E\}$$

is bounded from above. We say that f is bounded from below *if the set $f(E)$ is bounded from below. Finally, we say that f is* bounded *if the set $f(E)$ is bounded. We write*

$$\sup_E f := \sup f(E), \quad \inf_E f := \inf f(E).$$

Exercise 41 *Given a set E, consider the vector space $X := \{f : E \to \mathbb{R} \text{ bounded}\}$. Prove that X is a vector space.*

Given $a < b$, consider the interval $[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$. A *partition* of $[a,b]$ is a finite set $P := \{x_0, \ldots, x_n\} \subset [a,b]$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a function $f : [a,b] \to \mathbb{R}$, the *pointwise variation* of f on the interval $[a,b]$ is

$$\operatorname{Var} f := \sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\},$$

where the supremum is taken over all partitions $P := \{x_0, \ldots, x_n\}$ of $[a,b]$, and all $n \in \mathbb{N}$. A function $f : [a,b] \to \mathbb{R}$ has *finite* or *bounded pointwise variation* if $\operatorname{Var} f < \infty$. The space of all functions $f : [a,b] \to \mathbb{R}$ of bounded pointwise variation is denoted by $BPV([a,b])$.

Exercise 42 *Prove that* $BPV([a,b])$ *is a vector space.*

Definition 43 *An* inner product*, or* scalar product*, on a vector space* X *is a function*

$$(\cdot,\cdot) : X \times X \to \mathbb{R}$$

such that

(i) $(x,x) \geq 0$ *for every* $x \in X$*,* $(x,x) = 0$ *if and only if* $x = 0$ *(positivity);*

(ii) $(x,y) = (y,x)$ *for all* $x, y \in X$ *(symmetry);*

(iii) $(sx + ty, z) = s(x,z) + t(y,z)$ *for all* $x, y, z \in X$ *and* $s, t \in \mathbb{R}$ *(bilinearity).*

An *inner product space* $(X, (\cdot,\cdot))$ is a vector space X endowed with an inner product $(\cdot,\cdot)$.

Example 44 *Some important examples of inner products are the following.*

(i) Consider the Euclidean space $\mathbb{R}^N$*, then*

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_N y_N,$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ *and* $\mathbf{y} = (y_1, \ldots, y_N)$*, is an inner product.*

(ii) Consider the space of X *of all integrable functions* $f : [a,b] \to \mathbb{R}$*. Then*

$$(f,g) := \int_a^b f(x)\, g(x)\, dx$$

is not an inner product. Indeed, if $f(x) = 0$ *for all* $x \in [a,b]$*,* $x \neq \frac{a+b}{2}$ *and* $f\left(\frac{a+b}{2}\right) = 1$*, then* $\int_a^b f^2(x)\, dx = 0$ *but* f *is not* 0*.*

To fix this problem one can take X *to be the space of all continuous functions* $f : [a,b] \to \mathbb{R}$*. Then*

$$(f,g) := \int_a^b f(x)\, g(x)\, dx$$

is an inner product.

Friday, January 23, 2015

Definition 45 *A* norm *on a vector space* X *is a map*

$$\|\cdot\| : X \to [0,\infty)$$

such that

(i) $\|x\| = 0$ *implies* $x = 0$*;*

(ii) $\|tx\| = |t| \|x\|$ *for all* $x \in X$ *and* $t \in \mathbb{R}$*;*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$*.*

A *normed space* $(X, \|\cdot\|)$ is a vector space X endowed with a norm $\|\cdot\|$. For simplicity, we often say that X is a normed space.

Example 46 *Some important examples of norms are the following.*

(i) Consider the space $\mathbb{R}$*. By Theorem 36, the absolute value is a norm.*

(ii) Consider the Euclidean space $\mathbb{R}^N$*, then*

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{(x_1)^2 + \cdots + (x_N)^2},$$

where $\mathbf{x} = (x_1, \ldots, x_N)$*, is a norm. We will prove this below.*

Exercise 47 *Given a set* E*, consider the vector space* $X := \{f : E \to \mathbb{R} \text{ bounded}\}$*. For* $f \in X$*, define*

$$\|f\| := \sup_E |f| .$$

Prove that $\|\cdot\|$ *is a norm.*

Given an inner product $(\cdot, \cdot) : X \times X \to \mathbb{R}$ on a vector space X, it turns out that the function

$$\|x\| := \sqrt{(x, x)}, \quad x \in X, \tag{5}$$

is a norm. This follows from the following result.

Proposition 48 (Cauchy–Schwarz's inequality) *Given an inner product* $(\cdot, \cdot) : X \times X \to \mathbb{R}$ *on a vector space* X*,*

$$|(x, y)| \leq \|x\| \|y\|$$

for all $x, y \in X$*.*

Proof. If $y = 0$, then both sides of the previous inequality are zeros, and so there is nothing to prove. Thus, assume that $y \neq 0$ and let $t \in \mathbb{R}$. By properties (i)-(iii),

$$0 \leq (x + ty, x + ty) = \|x\|^2 + t^2 \|y\|^2 + 2t (x, y) . \tag{6}$$

Taking

$$t := -\frac{(x, y)}{\|y\|^2}$$

in the previous inequality gives

$$0 \leq \|x\|^2 + \frac{(x, y)^2}{\|y\|^4} \|y\|^2 - 2\frac{(x, y)^2}{\|y\|^2},$$

or, equivalently,

$$(x, y)^2 \leq \|x\|^2 \|y\|^2 .$$

It now suffices to take the square root on both sides. ■

Remark 49 *It follows from the proof that equality holds in the Cauchy–Schwarz inequality if and only you have equality in (6), that is, if* $x + ty = 0$ *for some* $t \in \mathbb{R}$ *or* $y = 0$.

Corollary 50 *Given a scalar product* $(\cdot, \cdot) : X \times X \to \mathbb{R}$ *on a vector space* X, *the function*

$$\|x\| := \sqrt{(x, x)}, \quad x \in X,$$

is a norm.

Proof. By property (i), $\|\cdot\|$ is well-defined and $\|x\| = 0$ if and only if $x = 0$. Taking $t = 1$ in (6) and using the Cauchy–Schwarz inequality gives

$$\begin{aligned} 0 &\leq \|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2(x, y) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2, \end{aligned}$$

which is the triangle inequality for the norm. Moreover, by properties (ii) and (iii) for every $t \in \mathbb{R}$,

$$\|tx\| = \sqrt{(tx, tx)} = \sqrt{t(x, tx)} = \sqrt{t(tx, x)} = \sqrt{t^2(x, x)} = |t|\|x\|.$$

Thus $\|\cdot\|$ is a norm. ■

Example 51 *Other important norms that one can put in* $\mathbb{R}^N$ *are*

$$\begin{aligned} \|\mathbf{x}\|_\infty &:= \max\{|x_1|, \dots, |x_N|\}, \\ \|\mathbf{x}\|_1 &:= |x_1| + \cdots + |x_N|, \\ \|\mathbf{x}\|_p &:= (|x_1|^p + \cdots + |x_N|^p)^{1/p}, \end{aligned}$$

for $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ *and where* $1 \leq p < \infty$.

Theorem 52 *Let* $(X, \|\cdot\|)$ *be a normed space. Then there exists an inner product* $(\cdot, \cdot) : X \times X \to \mathbb{R}$ *such that* $\|x\| = \sqrt{(x, x)}$ *for all* $x \in X$ *if and only if* $\|\cdot\|$ *satisfies the parallelogram law*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Exercise 53 *Consider the vector space* $X := \{f : [0, 1] \to \mathbb{R}$ *bounded*$\}$ *with*

$$\|f\| := \sup_{[0,1]} |f|.$$

Let's prove that $\|\cdot\|$ does not satsify the parallelogram law. Take $f(x) = -x^2$ and $g(x) = x$. Then

$$\begin{aligned}
\sup_{[0,1]} |f+g| &= \sup_{[0,1]} |-x^2+x| = \max_{[0,1]}(x-x^2) = \frac{1}{4}, \\
\sup_{[0,1]} |f-g| &= \sup_{[0,1]} |-x^2-x| = \max_{[0,1]}(x^2+x) = 2, \\
\sup_{[0,1]} |f| &= \sup_{[0,1]} |-x^2| = \max_{[0,1]} x^2 = 1, \\
\sup_{[0,1]} |g| &= \sup_{[0,1]} |g| = \max_{[0,1]} x = 1,
\end{aligned}$$

and so

$$\begin{aligned}
\left(\sup_{[0,1]} |f+g|\right)^2 + \left(\sup_{[0,1]} |f-g|\right)^2 &= \frac{1}{16} + 4 = \\
&\neq 2\left(\sup_{[0,1]} |f|\right)^2 + 2\left(\sup_{[0,1]} |g|\right)^2 \\
&= 2+2.
\end{aligned}$$

Example 54 *In $\mathbb{R}^N$ the norm $\|\cdot\|_\infty$ does not satsify the parallelogram law. Take $\mathbf{x} = (1,1,0,\ldots)$, $\mathbf{y} = (1,-1,0,\ldots)$. Then $\mathbf{x}+\mathbf{y} = (2,0,\ldots)$, $\mathbf{x}-\mathbf{y} = (0,2,0,\ldots)$. Hence,*

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_\infty^2 + \|\mathbf{x}-\mathbf{y}\|_\infty^2 &= 4+4 = 8 \\
&\neq 2\|\mathbf{x}\|_\infty^2 + 2\|\mathbf{y}\|_\infty^2 \\
&= 2+2.
\end{aligned}$$

Example 55 *In $\mathbb{R}^N$ the norm $\|\cdot\|_p$ for $p \neq 2$ does not satsify the parallelogram law. Take $\mathbf{x} = (1,1,0,\ldots)$, $\mathbf{y} = (1,-1,0,\ldots)$. Then $\mathbf{x}+\mathbf{y} = (2,0,\ldots)$, $\mathbf{x}-\mathbf{y} = (0,2,0,\ldots)$. Hence,*

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_{\ell^p}^2 + \|\mathbf{x}-\mathbf{y}\|_{\ell^p}^2 &= (2^p)^{\frac{2}{p}} + (2^p)^{\frac{2}{p}} = 8 \\
&\neq 2\|\mathbf{x}\|_{\ell^p}^2 + 2\|\mathbf{y}\|_{\ell^p}^2 \\
&= 2(1^p+1^p)^{\frac{2}{p}} + 2(1^p+1^p)^{\frac{2}{p}} = 2^{2+\frac{2}{p}}.
\end{aligned}$$

Monday, January 26, 2015

Proposition 56 (Parallelogram law) *Given an inner product $(\cdot,\cdot) : X \times X \to \mathbb{R}$ on a vector space X,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Proof. Taking $t = \pm 1$ in (6), we get

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2(x,y),$$
$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2(x,y).$$

By adding these identities, we obtain the desired result. ■

Theorem 57 *Let $(X, \|\cdot\|)$ be a normed space. Then there exists an inner product $(\cdot,\cdot) : X \times X \to \mathbb{R}$ such that $\|x\| = \sqrt{(x,x)}$ for all $x \in X$ if and only if $\|\cdot\|$ satisfies the parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Proof. In view of the previous two propositions, we only need to prove one direction of the theorem. Assume that $\|\cdot\|$ satisfies the parallelogram law. Define

$$(x,y) := \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$$

for all $x, y \in X$. We claim that $(\cdot,\cdot) : X \times X \to \mathbb{R}$ is an inner product.

Indeed,

$$(x,x) := \frac{1}{4}\left[\|2x\|^2 - \|0\|^2\right] = \|x\| \geq 0$$

with the equality holding if and only if $x = 0$. Hence, property (i) holds. Concerning property (ii), using the fact that $\|-z\| = |-1|\,\|z\| = \|z\|$, we have that

$$(y,x) = \frac{1}{4}\left[\|y+x\|^2 - \|y-x\|^2\right] = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right] = (x,y).$$

Finally, by the parallelogram law,

$$\|u+v+w\|^2 + \|u+v-w\|^2 = 2\|u+v\|^2 + 2\|w\|^2,$$
$$\|u-v+w\|^2 + \|u-v-w\|^2 = 2\|u-v\|^2 + 2\|w\|^2.$$

Subtracting these identities and rewriting some of the terms, we get

$$\|(u+w)+v\|^2 - \|(u+w)-v\|^2 + \|(u-w)+v\|^2$$
$$- \|(u-w)-v\|^2 = 2\|u+v\|^2 - 2\|u-v\|^2,$$

or, equivalently,

$$(u+w,v) + (u-w,v) = 2(u,v).$$

Taking $u = w$ gives $(2u,v) = 2(u,v)$. Given $x, y, z \in X$, let $u = \frac{x+y}{2}$ and $w = \frac{x-y}{2}$ and $v = z$. Then

$$(x,z) + (y,z) = 2\left(\frac{x+y}{2}, z\right) = (x+y,z).$$

To prove that $(tu,v) = t(u,v)$ for $t \in \mathbb{R}$, we need first to prove it for $t \in \mathbb{N}$, then for rationals, and then for reals. We leave it as an exercise. ■

Definition 58 *A* metric *on a set X is a map $d : X \times X \to [0, \infty)$ such that*

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry),

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

A metric space *(X, d) is a set X endowed with a metric d. When there is no possibility of confusion, we abbreviate by saying that X is a metric space.*

Proposition 59 *Let $(X, \|\cdot\|)$ be a normed space. Then*

$$d(x, y) := \|x - y\|$$

is a metric.

Proof. By property (i) in Definition 45, we have that $0 = d(x, y) = \|x - y\|$ if and only if $x - y = 0$, that is, $x = y$.

By property (ii) in Definition 45, we obtain that

$$d(y, x) = \|y - x\| = \|-1(x - y)\| = |-1| \cdot \|x - y\| = \|x - y\| = d(x, y).$$

Finally, by property (ii) in Definition 45,

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y).$$

■

Exercise 60 *Prove that in $\mathbb{R}$ the function*

$$d_1(x, y) := \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right| \tag{7}$$

is a metric.

Wednesday, January 28, 2015

Definition 61 *Given a metric space (X, d), a point $x_0 \in X$, and $r > 0$, the* ball *centered at x_0 and of radius r is the set*

$$B(x_0, r) := \{x \in X : d(x, x_0) < r\}.$$

Definition 62 *Given a metric space (X, d), and a nonempty set $E \subseteq X$, a point $x \in E$ is called an* interior point *of E if there exists $r > 0$ such that $B(x, r) \subseteq E$. The* interior *E° of a set $E \subseteq \mathbb{R}^N$ is the union of all its interior points. A subset $U \subseteq X$ is* open *if every $x \in U$ is an interior point of U.*

Example 63 *Given a metric space (X,d), the ball $B(x_0,r)$ is open. To see this, let $x \in B(x_0,r)$. Then $B(x, r-d(x,x_0))$ is contained in $B(x_0,r)$. Indeed, if $y \in B(x_0, r-d(x,x_0))$, then*

$$d(y,x_0) \leq d(y,x) + d(x,x_0) < r - d(x,x_0) + +d(x,x_0) = r,$$

and so $y \in B(x_0,r)$.

Example 64 *Some simple examples of sets that are open and of some that are not.*

(i) The set $(a,\infty) = \{x \in \mathbb{R} : x > a\}$ is open. Indeed, if $x > a$, take $r := x - a > 0$. Then $B(x,r) \subset (a,\infty)$. Similarly, the set $(-\infty, a)$ is open.

(ii) The set $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ is open. Indeed, given $a < x < b$, take $r := \min\{b-x, x-a\} > 0$. Then $B(x,r) \subseteq (a,b)$.

(iii) The set $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ is not open, since b belongs to the set but there is no ball $B(b,r)$ contained in $(a,b]$.

Example 65 *Consider the set*

$$U = \mathbb{R} \setminus \left(\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right).$$

Let's prove that U is open. If $x < 0$, take $r = -x > 0$, then $B(x,r) = (-2x, 0) \subseteq U$. If $x > 1$, take $r = x - 1$, then $B(x,r) = (1, 2x-1) \subseteq U$. If $\frac{1}{n+1} < x < \frac{1}{n}$, take $r = \min\left\{\frac{1}{n} - x, x - \frac{1}{n+1}\right\} = \frac{1}{n+1}$, then $B(x,r) \subseteq U$. Hence, U is open.

Example 66 *Consider the set*

$$E = \mathbb{R} \setminus \left(\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right).$$

Let's prove that E is not open. The point $x = 0$ belongs to E, but for every $r > 0$, by the Archimedean principle we can find $n \in \mathbb{N}$ such that $n > \frac{1}{r}$, and so $0 < \frac{1}{n} < r$, which shows that $\frac{1}{n} \in (-r,r)$. Since $\frac{1}{n}$ does not belong to E, the ball $(-r,r)$ is not contained in E for any *$r > 0$. Hence, E is not open.*

The main properties of open sets are given in the next proposition.

In what follows by an arbitrary family of sets of $\mathbb{R}^N$ we mean that there exists a set I and a function

$$f : I \to \mathcal{P}\left(\mathbb{R}^N\right)$$
$$\alpha \in I \mapsto f(\alpha) = U_\alpha$$

We write $\{U_\alpha\}$ or $\{U_\alpha\}_I$ or $\{U_\alpha\}_{\alpha \in I}$ to denote the set $\{f(\alpha) : \alpha \in I\}$.

Proposition 67 *Given a metric space (X,d), the following properties hold:*

(i) $\emptyset$ *and* X *are open.*

(ii) If $U_i \subseteq X$*,* $i = 1, \ldots, n$*, is a finite family of open sets of* X*, then* $U_1 \cap \cdots \cap U_n$ *is open.*

(iii) If $\{U_\alpha\}_\alpha$ *is an arbitrary collection of open sets of* X*, then* $\bigcup_\alpha U_\alpha$ *is open.*

Proof. To prove (ii), let $x \in U_1 \cap \cdots \cap U_M$. Then $x \in U_i$ for every $i = 1, \ldots, n$, and since U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Take $r := \min\{r_1, \ldots, r_n\} > 0$. Then

$$B(x, r) \subseteq U_1 \cap \cdots \cap U_n,$$

which shows that $U_1 \cap \cdots \cap U_n$ is open.

To prove (iii), let $x \in U := \bigcup_\alpha U_\alpha$. Then there is α such that $x \in U_\alpha$ and since U_α is open, there exists $r > 0$ such that $B(x, r) \subseteq U_\alpha \subseteq U$. This shows that U is open. ■

Properties (i)–(iii) are used to define topological spaces.

Definition 68 *Let* X *be a nonempty set and let* τ *be a family of sets of* X*. The pair* (X, τ) *is called a* topological space *if the following hold.*

(i) $\emptyset, X \in \tau$*.*

(ii) If $U_i \in \tau$ *for* $i = 1, \ldots, M$*, then* $U_1 \cap \ldots \cap U_M \in \tau$*.*

(iii) If $\{U_\alpha\}_\alpha$ *is an arbitrary collection of elements of* τ*, then* $\bigcup_\alpha U_\alpha \in \tau$*.*

The elements of the family τ *are called* open sets*.*

Remark 69 *The intersection of infinitely many open sets is not open in general. Take* $U_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ *for* $n \in \mathbb{N}$*. Then*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

but $\{0\}$ *is not open. Indeed, for every* $r > 0$*, the ball* $(-r, r)$ *is not contained in* $\{0\}$*.*

Remark 70 *Proposition 67 shows that the family of open sets in* $\mathbb{R}^N$ *defined in Definition 62 is a a topology, called the* Euclidean topology*. Unless specified, in* $\mathbb{R}^N$ *we will always consider the Euclidean topology.*

Example 71 *Given a nonempty set* X*, there are always at least two topologies on* X*, namely,*

$$\tau_1 = \{\emptyset, X\}$$

(so according to τ_1*, the only open sets are the empty set and* X*) and*

$$\tau_2 = \{\textit{all subsets of } X\}$$

(so according to τ_2 *every set* $E \subseteq X$ *is open).*

Exercise 72 *Prove that in $\mathbb{R}^N$ the norms*

$$\|\mathbf{x}\|_{\ell^\infty} := \max\{|x_1|, \ldots, |x_N|\},$$
$$\|\mathbf{x}\|_{\ell^1} := |x_1| + \cdots + |x_N|,$$
$$\|\mathbf{x}\|_{\ell^p} := (|x_1|^p + \cdots + |x_N|^p)^{1/p},$$

generate the same topology.

Remark 73 *For a topological space (X, τ), given a point $x \in X$, a* neighborhood *of x is an open set containing x. Neighborhoods play the role of balls in metric spaces. Thus, given a set $E \subseteq X$, a point $x \in E$ is called an* interior point *of E if there exists a neighborhood U of x such that $U \subseteq E$.*

The proof of following proposition is left as an exercise.

Proposition 74 *Given a metric space (X, d). Then*

(i) *E° is an open subset of E,*

(ii) *E° is given by the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,*

(iii) *E is open if and only if $E = E^\circ$,*

(iv) *$(E^\circ)^\circ = E^\circ$.*

Example 75 *Consider the set $E = [0, 1)$. Then 0 is not an in interior point of E, so $E^\circ \subseteq (0, 1)$. On the other hand, since $(0, 1)$ is open and contained in E, by part (ii) of the previous proposition, $E^\circ \supseteq (0, 1)$, which shows that $E^\circ = (0, 1)$.*

Exercise 76 *Some properties of the interior.*

(i) *Prove that if E, F are subsets of $\mathbb{R}^N$, then*

$$E^\circ \cap F^\circ = (E \cap F)^\circ,$$
$$E^\circ \cup F^\circ \subseteq (E \cup F)^\circ.$$

(ii) *Show that in general $E^\circ \cup F^\circ \neq (E \cup F)^\circ$.*

(iii) *Let $\{E_\alpha\}_\alpha$ be an arbitrary collection of sets of $\mathbb{R}^N$. What is the relation, if any, between $\bigcap_\alpha (U_\alpha)^\circ$ and $(\bigcap_\alpha U_\alpha)^\circ$? And between $\bigcup_\alpha (U_\alpha)^\circ$ and $(\bigcup_\alpha U_\alpha)^\circ$?*

Definition 77 *Given a metric space (X, d), A subset $C \subseteq X$ is* closed *if its complement $X \setminus C$.*

The main properties of closed sets are given in the next proposition.

Proposition 78 *Given a metric space (X, d), the following properties hold:*

(i) $\emptyset$ *and* X *are closed.*

(ii) If $C_i \subseteq X$*,* $i = 1, \ldots, n$*, is a finite family of closed sets of* X*, then* $C_1 \cup \cdots \cup C_n$ *is closed.*

(iii) If $\{C_\alpha\}_\alpha$ *is an arbitrary collection of closed sets of* X*, then* $\bigcap_\alpha C_\alpha$ *is closed.*

The proof follows from Proposition 67 and De Morgan's laws. If $\{E_\alpha\}_\alpha$ is an arbitrary collection of subsets of a set $\mathbb{R}^N$, then *De Morgan's laws* are

$$X \setminus \left(\bigcup_\alpha E_\alpha \right) = \bigcap_\alpha (X \setminus E_\alpha),$$
$$X \setminus \left(\bigcap_\alpha E_\alpha \right) = \bigcup_\alpha (X \setminus E_\alpha).$$

Remark 79 *Note that the majority of sets are neither open nor closed. The set* $E = (0, 1]$ *is neither open nor closed.*

Definition 80 *Given a metric space (X, d) and a set $E \subseteq X$, the* closure *of E, denoted $\overline{E}$, is the intersection of all closed sets that contain E*

In other words, the closure of E is the smallest (with respect to inclusion) closed set that contains E. It follows by Proposition 78 that $\overline{E}$ is closed.

The proof of following proposition is left as an exercise.

Proposition 81 *Given a metric space (X, d), let $C \subseteq X$. Then C is closed if and only if $C = \overline{C}$.*

Friday, January 30, 2015

Proposition 82 *Given a metric space (X, d), let $E \subseteq X$, and let $x \in X$. Then $x \in \overline{E}$ if and only if $B(x, r) \cap E \neq \emptyset$ for every $r > 0$.*

Proof. Let $x \in \overline{E}$ and assume by contradiction that there exists $r > 0$ such that $B(x, r) \cap E = \emptyset$. Since $B(x, r)$ is open and $B(x, r) \cap E = \emptyset$, it follows that $X \setminus B(x, r)$ is closed and contains E. By the definition of $\overline{E}$ we have that $\overline{E} \subseteq X \setminus B(x, r)$, which contradicts the fact that $x \in \overline{E}$.

Conversely, let $x \in X$ and assume that $B(x, r) \cap E \neq \emptyset$ for every $r > 0$. We claim that $x \in \overline{E}$. Indeed, if not, then $x \in X \setminus \overline{E}$, which is open. Thus, there exists $B(x, r) \subseteq X \setminus \overline{E}$, which contradicts the fact that $B(x, r) \cap E \neq \emptyset$. ■

The previous proposition leads us to the definition of accumulation points.

Definition 83 *Given a metric space (X, d) and a set $E \subseteq X$, a point $x \in X$ is an* accumulation point*, or* cluster point *of E if for every $r > 0$ the ball $B(x, r)$ contains at least one point of E different from x. The set of all accumulation points of E is denoted* $\operatorname{acc} E$*.*

Note that x does not necessarily belong to the set E.

Remark 84 *Note take if $\mathbf{x} \in \mathbb{R}^N$ is an accumulation point of E, then by taking $r = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a sequence $\{\mathbf{x}_n\} \subseteq E$ with $\mathbf{x}_n \neq \mathbf{x}$ for all $n \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \frac{1}{n} \to 0$. Thus $\{x_n\}$ converges to x. Conversely, if there exists $\{\mathbf{x}_n\} \subseteq E$ with $\mathbf{x}_n \neq \mathbf{x}$ for all $n \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}\| \to 0$, then $\mathbf{x}$ is an accumulation point of E.*

It turns out that the closure of a set is given by the set and all its accumulations points.

Proposition 85 *Given a metric space (X, d) and a set $E \subseteq X$, then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, a set $C \subseteq X$ is closed if and only if C contains all its accumulation points.

Proof. Exercise. ■

Exercise 86 *(i) Prove that if $E_1, \ldots, E_n$ are subsets of $\mathbb{R}^N$, then*

$$\overline{E_1} \cap \cdots \cap \overline{E_n} \supseteq \overline{E_1 \cap \cdots \cap E_n},$$
$$\overline{E_1} \cup \cdots \cup \overline{E_n} = \overline{E_1 \cup \cdots \cup E_n}.$$

(ii) Show that in general $\overline{E_1} \cap \cdots \cap \overline{E_n} \neq \overline{E_1 \cap \cdots \cap E_n}$.

(iii) Let $\{E_\alpha\}_\alpha$ be an arbitrary collection of sets of $\mathbb{R}^N$. What is the relation, if any, between between $\bigcap_\alpha \overline{U_\alpha}$ and $\overline{\bigcap_\alpha U_\alpha}$? And between $\bigcup_\alpha \overline{U_\alpha}$ and $\overline{\bigcup_\alpha U_\alpha}$?

Definition 87 *Given a metric space (X, d), a set $E \subseteq X$ is* bounded *if it is contained in a ball.*

Theorem 88 (Bolzano–Weierstrass) *Every bounded set $E \subseteq \mathbb{R}^N$ with infinitely many elements has at least one accumulation point.*

The proof relies on a few preliminary results, which are of interest in themselves.

Lemma 89 *Let $\{[a_n, b_n]\}_n$ be a sequence of closed bounded intervals such that $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for all $n \in \mathbb{N}$. Then the intersection*

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is nonempty.

Proof. Since

$$\cdots \subseteq [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \subseteq \cdots \subseteq [a_1, b_1]\,,$$

we have that

$$a_1 \le \cdots \le a_n \le a_{n+1} \le \cdots\,, \tag{8}$$
$$b_1 \ge \cdots \ge b_n \ge b_{n+1} \ge \cdots\,. \tag{9}$$

Let

$$A := \{a_1, \ldots, a_n, \ldots\}\,.$$

By (8) and (9), for $n \in \mathbb{N}$,

$$a_n \le b_n \le b_1.$$

Hence, A is bounded from above, and so by the supremum property, there exists $x := \sup A \in \mathbb{R}$ and

$$a_n \le x$$

for all $n \in \mathbb{N}$. We claim that $x \le b_n$ for all $n \in \mathbb{N}$. If not, then there exists $m \in \mathbb{N}$ such that $b_m < x$. Since x is the least upper bound of A, there exists $n \in \mathbb{N}$ such that $b_m < a_n$. Find $k \ge m, n$. Then by (8) and (9),

$$a_n \le a_k \le b_k \le b_m,$$

which is a contradiction. This proves the claim. Hence, $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$, and so $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. ■

Given N bounded intervals $I_1, \ldots, I_N \subset \mathbb{R}$, a *rectangle* in $\mathbb{R}^N$ is a set of the form

$$R := I_1 \times \cdots \times I_N.$$

If all the intervals have the same length, we call R a cube.

Lemma 90 *Let $\{R_n\}_n$ be a sequence of closed bounded rectangles in $\mathbb{R}^N$ such that $R_n \supseteq R_{n+1}$ for all $n \in \mathbb{N}$. Then the intersection*

$$\bigcap_{n=1}^{\infty} R_n$$

is nonempty.

Proof. Each rectangle R_n has the form

$$R_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,N}, b_{n,N}]\,.$$

Since $R_n \supseteq R_{n+1}$ for all $n \in \mathbb{N}$, for every fixed $k = 1, \ldots, N$, we have that $[a_{n,k}, b_{n;k}] \supseteq [a_{n+1,k}, b_{n+1,k}]$ for all $n \in \mathbb{N}$, and so by the previous lemma there exists $x_k \in \bigcap_{n=1}^{\infty} [a_{n,k}, b_{n;k}]$. Define $\mathbf{x} = (x_1, \ldots, x_N)$. Then $\mathbf{x} = (x_1, \ldots, x_N) \in [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,N}, b_{n,N}] = R_n$ for every $n \in \mathbb{N}$, and so $\mathbf{x} \in \bigcap_{n=1}^{\infty} R_n$. ■

Monday, February 02, 2015

We are now ready to prove the Bolzano–Weierstrass theorem.

Proof of the Bolzano–Weierstrass theorem. Since E is bounded, it is contained in ball, and in turn a ball is contained in a cube Q_1 of side-length ℓ. Divide Q_1 into 2^N two closed cubes of side-length $\frac{\ell}{2}$. Since E has infinitely many elements, at least one of these 2^N closed cubes contains infinitely many elements of E. Let's call this closed interval Q_2. Then $Q_2 \subset Q_1$, and Q_2 contains infinitely many elements of E.

Divide Q_2 into into 2^N two closed cubes of side-length $\frac{\ell}{2^2}$. Since E has infinitely many elements, at least one of these 2^N closed cubes contains infinitely many elements of E. Let's call this closed interval Q_3. By induction, we construct a sequence of closed cubes Q_n, $n \in \mathbb{N}$, with $Q_n \supseteq Q_{n+1}$, such that the side-length of Q_n is $\frac{\ell}{2^{n-1}}$ and Q_n contains infinitely many elements of E. By the previous lemma, there exists $\mathbf{x} \in \bigcap_{n=1}^{\infty} Q_n$. We claim that $\mathbf{x}$ is an accumulation point of E.

Fix $r > 0$ and consider the ball $B(\mathbf{x}, r)$. We claim that for n sufficiently large, $Q_n \subset B(\mathbf{x}, r)$. To see this, let $\mathbf{y} \in Q_n$. Then

$$\|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_1 - x_1)^2} < \sqrt{N\left(\frac{\ell}{2^{n-1}}\right)^2} = \frac{2\ell}{2^n}\sqrt{N}$$

By the Archimedean property, there exists $n \in \mathbb{N}$ such that

$$\frac{2\ell\sqrt{N}}{r} < 1 + n \leq 2^n,$$

and so $r > \frac{2\ell}{2^n}\sqrt{N}$, which proves the claim. Since Q_n contains infinitely many elements of E, the same holds for $B(\mathbf{x}, r)$ and so $\mathbf{x}$ is an accumulation point of E. ■

Definition 91 *Given a metric space (X, d) and a set $E \subseteq X$, a point $x \in$ is a* boundary point *of E if for every $r > 0$ the ball $B(x, r)$ contains at least one point of E and one point of $X \setminus E$. The set of boundary points of E is denoted ∂E.*

The following theorem is left as an exercise.

Theorem 92 *Let $E \subseteq \mathbb{R}^N$. Then*

(i) $\overline{E} = E \cup \partial E$,

(ii) E is closed if and only if it contains all its boundary points,

(iii) $\partial E = \partial\left(\mathbb{R}^N \setminus E\right)$,

(iv) $\partial E = \overline{(\mathbb{R}^N \setminus E)} \cap \overline{E}$.

6 Compactness

Exercise 93 *Let $K \subseteq \mathbb{R}^N$ be closed and bounded. Prove that if $E \subseteq K$ has infinitely many elements, then E has an accumulation point that belongs to K.*

Exercise 94 *Let $K_n \subset \mathbb{R}^N$ be nonempty, bounded, and closed. Assume that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n$ is nonempty.*

Definition 95 *Given a metric space (X, d), a set $K \subseteq X$ is* compact *if for every open cover of K, i.e., for every collection $\{U_\alpha\}_\alpha$ of open sets such that $\bigcup_\alpha U_\alpha \supseteq K$, there exists a finite subcover (i.e., a finite subcollection of $\{U_\alpha\}_\alpha$ whose union still contains K).*

Example 96 *The set $(0, 1]$ is not compact, since taking $U_n := \left(\frac{1}{n}, 2\right)$, a finite number of U_n does not cover $(0, 1]$.*

Here the problem is that 0 does not belong to E. But what if E is closed?

Wednesday, February 04, 2015

Example 97 *The set $[0, \infty)$ is not compact, since taking $U_n := (-1, n)$, a finite number of U_n does not cover $[0, \infty)$.*

Here E is closed but the problem is that E is not bounded.

Theorem 98 *Given a metric space (X, d), a compact set $K \subseteq X$ is closed and bounded.*

Proof. To prove that K is closed, we show that $X \setminus K$ is open. Fix $x \in X \setminus K$. For every $y \in K$ consider the balls $B(y, r_y)$ and $B(x, r_y)$, where $r := \frac{d(x,y)}{4}$. These two balls do not intersect each other (why?). Then $\{B(y, r_y)\}_{y \in K}$ is an open cover of K, and so there exist $y_1, \dots, y_m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} B(y_i, r_{y_i}).$$

Let $r := \min\{r_{y_1}, \dots, r_{y_m}\} > 0$. Then $x \in B(x, r)$ and the ball $B(x, r)$ does not intersect $B(y_i, r_{y_i})$ for any $i = 1, \dots, m$. Hence, $B(x, r)$ is contained in $X \setminus K$. This shows that every point x of $X \setminus K$ is an interior point, and so $X \setminus K$ is open.

To prove that K is bounded, consider a point $x_0 \in X$ and $B(x_0, n)$. The family of balls $\{B(x_0, n)\}_{n \in \mathbb{N}}$ covers the entire space X and in particular K. By compactness K is contained in a finite number of balls. Since the balls are one contained into the other, we have that K is contained in the ball of largest radius. Hence, K is bounded. ■

Remark 99 *For a topological space (X, τ) we can still prove that a compact set $K \subseteq X$ is closed, provided the topological space X is a* Hausdorff space*, that is,* *for every x and $y \in X$, with $x \neq y$, there exist disjoint neighborhoods of x and y.*

A very simple example of a space that is not Hausdorff can be obtained by considering a nonempty set X and taking as topology $\tau := \{\emptyset, X\}$. If X has at least two elements, then any singleton $\{x\}$ is compact but not closed.

There is a way to define a notion of boundedness for special topological spaces, called topological vector spaces.

Theorem 100 *A closed and bounded set $K \subset \mathbb{R}^N$ is compact.*

Proof. Let $\{U_\alpha\}_\alpha$ be a family of open sets such that $\bigcup_\alpha U_\alpha \supseteq K$ and assume by contradiction that no finite subcover covers K. Since K is bounded, it is contained in ball, and in turn a ball is contained in a cube Q_1 of side-length ℓ. Divide Q_1 into 2^N two closed cubes of side-length $\frac{\ell}{2}$. If $K \cap Q'$ is contained in a finite subcover for every such subcube, then K would be contained in a finite subcover. Hence, there exists at least one subcube Q_1 such that $K \cap Q_1$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$.

By induction, we construct a sequence of closed cubes Q_n, $n \in \mathbb{N}$, with $Q_n \supseteq Q_{n+1}$, such that the side-length of Q_n is $\frac{\ell}{2^{n-1}}$ and $K \cap Q_n$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$. By Exercise 94, there exists $\mathbf{x} \in \bigcap_{n=1}^\infty Q_n \cap K$. Since $\{U_\alpha\}_\alpha$ covers K, there exists β such that $\mathbf{x} \in U_\beta$. On the other hand, U_β is open, and so there is a ball $B(\mathbf{x}, r)$ contained in U_β. As in the proof of the Bolzano–Weierstrass theorem, we have that for n sufficiently large, $Q_n \subset B(\mathbf{x}, r) \subseteq U_\beta$, which contradicts the fact that $K \cap Q_n$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$. ■

Remark 101 *The previous theorem fails for infinite dimensional normed spaces, and so, in general, for infinite dimensional metric spaces.*

Friday, February 06, 2015

7 Functions

Given two sets X and Y consider a function $f : E \to Y$, where $E \subseteq X$. The set E is called the *domain* of f. When $X = \mathbb{R}^M$, if E is not specified, then E should be taken to be the largest set of x for which $f(x)$ makes sense. This means that:

If there are even roots, their arguments should be nonnegative. If there are logarithms, their arguments should be strictly positive. Denominators should be different from zero. If a function is raised to an irrational number, then the function should be nonnegative.

Given a set $F \subseteq E$, the set $f(F) = \{y \in Y : y = f(x) \text{ for some } x \in F\}$ is called the *image* of F through f.

Given a set $G \subseteq \mathbb{R}$, the set $f^{-1}(G) = \{x \in E : f(x) \in G\}$ is called the *inverse image* or *preimage* of F through f. It has NOTHING to do with the inverse function. It is just one of those unfortunate cases in which we use the same symbol for two different objects.

The *graph* of a function is the set of $X \times Y$ defined by

$$\operatorname{gr} f = \{(x, f(x)) : x \in E\}.$$

A function f is said to be

- *one-to-one* or *injective* if $f(x) \neq f(z)$ for all $x, z \in E$ with $x \neq z$.
- *onto* or *surjective* if $f(E) = F$,
- *bijective* or *invertible* if it is one-to-one and onto. The function $f^{-1} : F \to E$, which assigns to each $y \in F = f(E)$ the unique $x \in E$ such that $f(x) = y$, is called the *inverse* function of f.

8 Limits of Functions

Definition 102 *If (X, d_X) and (Y, d_Y) are two metric spaces, $E \subseteq X$, $x_0 \in X$ is an accumulation point of E and $f : E \to Y$, we say that $\ell \in Y$ is the* limit *of $f(x)$ as x approaches x_0 if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ with the property that*

$$d_Y(f(x), \ell) < \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. We write

$$\lim_{x \to x_0} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to x_0.$$

Remark 103 *Note that even when $x \in E$, we cannot take $x = x_0$ since in the definition we require $0 < d_X(x, x_0)$.*

Remark 104 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{x}_0 \in \mathbb{R}^N$ be an accumulation point of E, and let $\mathbf{f} : E \to \mathbb{R}^N$. We say that a number $\boldsymbol{\ell} \in \mathbb{R}^M$ is the* limit *of $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, \mathbf{x}_0) > 0$ with the property that*

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. We write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell} \quad \text{or} \quad \mathbf{f}(\mathbf{x}) \to \boldsymbol{\ell} \text{ as } \mathbf{x} \to \mathbf{x}_0.$$

Remark 105 *If (X, τ_X) and (Y, τ_Y) are two topological spaces, $E \subseteq X$, $x_0 \in X$ is an accumulation point of E and $f : E \to Y$, we say that $\ell \in Y$ is the* limit *of* $f(x)$ *as* x *approaches* x_0 *if for every neighborhood* V *of* ℓ *there exists a neighborhood* U *of* x_0 *with the property that*

$$f(x) \in V$$

for all $x \in E$ *with* $x \in U \setminus \{x_0\}$*. We write*

$$\lim_{x \to x_0} f(x) = \ell.$$

Note that unless the space Y *is Hausdorff, the limit may not be unique.*

Definition 106 *If* (X, d_X) *and* (Y, d_Y) *are two metric spaces,* $E \subseteq X$*, and* $f : E \to Y$*, given a subset* $F \subseteq E$ *we denote by* $f|_F$ *the* restriction *of the function* f *to the set* F*, that is, the function* $f : F \to Y$*.*

Remark 107 *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces, let* $E \subseteq X$*, let* $x_0 \in X$ *be an accumulation point of* E *and* $f : E \to Y$*. Assume that there exists*

$$\lim_{x \to x_0} f(x) = \ell.$$

Then for every $\varepsilon > 0$ *there exists a real number* $\delta = \delta(\varepsilon, x_0) > 0$ *with the property that*

$$d_Y(f(x), \ell) < \varepsilon \tag{10}$$

for all $x \in E$ *with* $0 < d_X(x, x_0) < \delta$*. if* $F \subseteq E$ *is a subset such that* x_0 *is an accumulation point of* F*, then by restricting (10) we have that*

$$d_Y(f(x), \ell) < \varepsilon$$

for all $x \in F$ *with* $0 < d_X(x, x_0) < \delta$*. Hence, there exists*

$$\lim_{x \to x_0} f|_F(x) = \ell.$$

It follows that if we can find two sets $F \subseteq E$ *and* $G \subseteq E$ *such that* $x_0 \in \operatorname{acc} F$ *and* $x_0 \in \operatorname{acc} G$

$$\lim_{x \to x_0} f|_F(x) = \ell_1 \neq \ell_2 = \lim_{x \to x_0} f|_G(x),$$

then by the uniqueness of the limit (which we will prove later), it follows that the limit over E *cannot exist.*

Example 108 *Let's study the limit*

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2},$$

where $m \in \mathbb{N}$*. In this case* $f(x, y) = \frac{xy}{x^2+y^2}$ *and the domain is* $E = \mathbb{R}^2 \setminus \{(0, 0)\}$*. Note that* $(0, 0)$ *is an accumulation point of* E*.*

Taking $F = \{(x,x) : \; x \in \mathbb{R} \setminus \{0\}\}$, we have that $(0,0)$ is an accumulation point of F. For $(x,x) \in F$ we have

$$f(x,x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2} \to \frac{1}{2}$$

as $x \to 0$, while taking $G = \{(x,0) : \; x \in \mathbb{R} \setminus \{0\}\}$, we have that $(0,0)$ is an accumulation point of G. For $(x,0) \in F$ we have

$$f(x,0) = \frac{0}{x^2 + 0} = 0 \to 0,$$

and so the limit does not exist.

Monday, February 09, 2015

Example 109 *Let's study the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^m y}{x^2 + y^2},$$

where $m \in \mathbb{N}$. In this case $f(x,y) = \frac{x^m y}{x^2+y^2}$ and the domain is $\mathbb{R}^2 \setminus \{(0,0)\}$.

For $m \geq 2$, we have that the limit is 0. Indeed, using the fact that $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$, we have

$$\left| \frac{x^m y}{x^2 + y^2} - 0 \right| = \frac{|x|^m |y|}{x^2 + y^2} \leq \frac{\left(x^2 + y^2\right)^{m/2} \left(x^2 + y^2\right)^{1/2}}{x^2 + y^2} = \left(x^2 + y^2\right)^{(m-1)/2} \to 0$$

as $(x,y) \to (0,0)$.

Remark 110 *Note that the degree of the numerator is $m+1$ and the degree of the denominator is 2, so that in this particular example the limit exists if the degree of the numerator is higher than the degree of the denominator, that is, if $m + 1 > 2$.*

The next example shows that checking the limit on every line passing through $\mathbf{x}_0$ is not enough to gaurantee the existence of the limit.

Example 111 *Let*

$$f(x,y) := \begin{cases} 1 & \text{if } y = x^2,\, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given the line $y = mx$, the line intersects the parabola $y = x^2$ only in $\mathbf{0}$ and in at most one point. Hence, if x is very small,

$$f(x, mx) = 0 \to 0$$

as $x \to 0$. However, since $f\left(x, x^2\right) = 1 \to 1$ as $x \to 0$, the limit does not exists.

Theorem 112 *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces, let* $E \subseteq X$*, let* $x_0 \in X$ *be an accumulation point of* E *and* $f : E \to Y$*. If the limit*

$$\lim_{x \to x_0} f(x)$$

exists, it is unique.

Proof. Assume by contradiction that there exist

$$\lim_{x \to x_0} f(x) = \ell \quad \text{and} \quad \lim_{x \to x_0} f(x) = L$$

with $\ell \neq L$. Then $d_Y(\ell, L) > 0$. Fix $0 < \varepsilon = \frac{1}{2} d_Y(\ell, L)$. Since $\lim_{x \to x_0} f(x) = \ell$, there exists $\delta_1 > 0$ with the property that

$$d_Y(f(x), \ell) < \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_1$, while, since $\lim_{x \to x_0} f(x) = L$, there exists $\delta_2 > 0$ with the property that

$$d_Y(f(x), L) < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < d_X(x, x_0) < \delta_2$.

Take $\delta = \min\{\delta_1, \delta_2\} > 0$ and take $x \in E$ with $0 < d_X(x, x_0) < \delta$. Note that such x exists because x_0 is an accumulation point of E. Then by the properties of the distance,

$$\begin{aligned} d_Y(\ell, L) &\leq d_Y(f(x), \ell) + d_Y(f(x), L) \\ &< \varepsilon + \varepsilon = d_Y(\ell, L), \end{aligned}$$

which implies that $d_Y(\ell, L) < d_Y(\ell, L)$. This contradiction proves the theorem. ■

Remark 113 *For topological spaces in general the limit is not unique. Given* (X, τ_X) *and* (Y, τ_Y) *are two topological spaces,* $E \subseteq X$*,* $x_0 \in X$ *is an accumulation point of* E *and* $f : E \to Y$*, it can be shown that the limit is unique if the space* Y *is Hausdorff. A topological space* Y *is a* Hausdorff space*, if for every* x *and* $y \in Y$*, with* $x \neq y$*, there exist disjoint neighborhoods of* x *and* y*.*

Exercise 114 *Study the limit*

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}.$$

We now list some important operations for limits.

Theorem 115 *Let* (X, d) *be a metric space, let* $E \subseteq X$*, let* $x_0 \in X$ *be an accumulation point of* E*. Given two functions* $\boldsymbol{f}, \boldsymbol{g} : E \to \mathbb{R}^M$*, assume that there exist*

$$\lim_{x \to x_0} \boldsymbol{f}(x) = \boldsymbol{\ell}_1 \in \mathbb{R}^M, \qquad \lim_{x \to x_0} \boldsymbol{g}(x) = \boldsymbol{\ell}_2 \in \mathbb{R}^M.$$

Then

(i) there exists $\lim_{x \to x_0} (\boldsymbol{f} + \boldsymbol{g})(x) = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$*,*

(ii) there exists $\lim_{x \to x_0} (\boldsymbol{f} \cdot \boldsymbol{g})(x) = \boldsymbol{\ell}_1 \cdot \boldsymbol{\ell}_2$*,*

(iii) if $M = 1$*,* $\ell_2 \neq 0$ *and* $g(x) \neq 0$ *for all* $x \in E$*, then there exists* $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\ell_1}{\ell_2}$.

Wednesday, February 11, 2015

Proof. Parts (i) and (ii) are left as an exercise. We prove part (iii). Write

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| &= \left|\frac{f(x)\ell_2 - g(x)\ell_1}{g(x)\ell_2}\right| = \left|\frac{f(x)\ell_2 \pm \ell_1\ell_2 - g(x)\ell_1}{g(x)\ell_2}\right| \\ &= \frac{1}{|g(x)|\,|\ell_2|} \left|\ell_2 (f(x) - \ell_1) + \ell_1 (\ell_2 - g(x))\right| \\ &\leq \frac{1}{|g(x)|} |f(x) - \ell_1| + \frac{1}{|g(x)|} \frac{|\ell_1|}{|\ell_2|} |g(x) - \ell_2| .\end{aligned}$$

Thus we need to bound $\frac{1}{|g(x)|}$ from above, or, equivalently, we need $|g(x)|$ to stay away from zero. Since $\ell_2 \neq 0$, taking $\varepsilon = \frac{|\ell_2|}{2} > 0$, there exist $\delta_1 > 0$ such that

$$|g(x) - \ell_2| \leq \frac{|\ell_2|}{2}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_1$. Hence, using the inequality $|a| \geq |b| - |b - a|$,

$$|g(x)| \geq |\ell_2| - |g(x) - \ell_2| \geq |\ell_2| - \frac{|\ell_2|}{2} = \frac{|\ell_2|}{2}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_1$. It follows that

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| &\leq \frac{1}{|g(x)|} |f(x) - \ell_1| + \frac{1}{|g(x)|} \frac{|\ell_1|}{|\ell_2|} |g(x) - \ell_2| \\ &\leq \frac{2}{|\ell_2|} |f(x) - \ell_1| + \frac{2}{|\ell_2|} \frac{|\ell_1|}{|\ell_2|} |g(x) - \ell_2| .\end{aligned}$$

Given $\varepsilon > 0$ there exist $\delta_2 > 0$ such that

$$|f(x) - \ell_1| \leq \frac{\varepsilon\,|\ell_2|}{4}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_2$ and $\delta_3 > 0$ such that

$$|g(x) - \ell_2| \leq \frac{\varepsilon\,|\ell_2|^2}{4(1 + |\ell_1|)}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_3$. Then for all $x \in E$ with $0 < d_X(x, x_0) < \delta := \min\{\delta_1, \delta_2, \delta_3\}$, we have that

$$\left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| \le \frac{2}{|\ell_2|}|f(x) - \ell_1| + \frac{2}{|\ell_2|}\frac{|\ell_1|}{|\ell_2|}|g(x) - \ell_2|$$
$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{|\ell_1|}{1 + |\ell_1|} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}1 = \varepsilon.$$

This completes the proof. ■

Remark 116 *The previous theorem continues to hold if $\ell_1, \ell_2 \in [-\infty, \infty]$, provided we avoid the cases $\infty - \infty$, 0∞, $\frac{0}{0}$, $\frac{\infty}{\infty}$.*

Exercise 117 (Important) *Let (X, d_X) and (Y, d_Y) be two metric spaces, let $E \subseteq X$, let $x_0 \in X$ be an accumulation point of E and let $f : E \to Y$. Assume that there exists $\ell \in Y$ and a function $g : [0, \infty) \to \mathbb{R}^+$ with*

$$\lim_{s \to 0^+} g(s) = 0,$$

such that for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ with the property that

$$d_Y(f(x), \ell) < g(\varepsilon)$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. Prove that there exists

$$\lim_{x \to x_0} f(x) = \ell.$$

Remark 118 (Important) *The previous exercise says that we do not have to be very precise when applying the definition of limit, in the sense that, if we can prove that for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ with the property that*

$$d_Y(f(x), \ell) < 4\varepsilon^{1/2}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$ or

$$d_Y(f(x), \ell) < 4\varepsilon^3 + 16\varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$ or anything like that, then we know that we can conclude that there exists

$$\lim_{x \to x_0} f(x) = \ell.$$

This is very useful when proving theorems about limits.

Theorem 119 (Squeeze Theorem) *Let (X, d) be a metric space, let $E \subseteq X$, let $x_0 \in X$ be an accumulation point of E. Given three functions $f, g, h : E \to \mathbb{R}$, assume that there exist*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \ell.$$

and that $f(x) \le h(x) \le g(x)$ for every $x \in E$. Then there exists $\lim_{x \to x_0} h(x) = \ell$.

Proof. Given $\varepsilon > 0$ there exist $\delta_1 > 0$ such that

$$|f(x) - \ell| \leq \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_1$ and $\delta_2 > 0$ such that

$$|g(x) - \ell| \leq \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta_2$. Then for all $x \in E$ with $0 < d_X(x, x_0) < \delta = \min\{\delta_1, \delta_2\}$, we have that

$$\ell - \varepsilon \leq f(x) \leq h(x) \leq g(x) \leq \ell + \varepsilon.$$

Hence,

$$|h(x) - \ell| \leq \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$, which shows that $\lim_{x \to x_0} h(x) = \ell$. ■

Example 120 *The previous theorem can be used for example to show that for* $a > 0$

$$\lim_{x \to 0} |x|^a \sin \frac{1}{x} = 0.$$

Indeed,

$$0 \leq \left| |x|^a \sin \frac{1}{x} \right| = |x|^a \left| \sin \frac{1}{x} \right| \leq |x|^a$$

and since $|x|^a \to 0$ *as* $x \to 0$ *we can apply the squeeze theorem. We could also used the following Exercise.*

Theorem 121 *Let* (X, d) *be a metric space, let* $E \subseteq X$*, let* $x_0 \in X$ *be an accumulation point of* E*. Given two functions* $\boldsymbol{f}, \boldsymbol{g} : E \to \mathbb{R}^M$*, assume that there exists*

$$\lim_{x \to x_0} \boldsymbol{f}(x) = \mathbf{0},$$

and that $\boldsymbol{g}$ *is bounded, that is,* $\|\boldsymbol{g}(x)\| \leq L$ *for all* $x \in E$ *and for some* $L > 0$*. Then there exists* $\lim_{x \to x_0} (\boldsymbol{f} \cdot \boldsymbol{g})(x) = 0$*.*

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x) - \mathbf{0}\| < \frac{\varepsilon}{1 + L}$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. Hence, by the Cauchy–Schwarz's inequality

$$|(\boldsymbol{f} \cdot \boldsymbol{g})(x) - 0| = |(\boldsymbol{f} \cdot \boldsymbol{g})(x)| \leq \|f(x)\| \|g(x)\| < \frac{\varepsilon}{1 + L} L < \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. ■

Exercise 122 *Let (X, d) be a metric space, let $E \subseteq X$, let $x_0 \in X$ be an accumulation point of E. Given two functions $f, g : E \to \mathbb{R}$, assume that there exist*

$$\lim_{x \to x_0} f(x) = \ell, \quad \lim_{x \to x_0} g(x) = L$$

and that $f(x) \leq g(x)$ for every $x \in E$. Then $\ell \leq L$.

We next study the limit of composite functions.

Definition 123 *Let (X, d_X), (Y, d_Y) be metric spaces, let $E \subseteq X$, and let $f : E \to Y$. We say that f is* continuous *at $x_0 \in E$ if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ such that for all $x \in E$ with $d_X(x, x_0) < \delta$ we have*

$$d_Y(f(x), f(x_0)) < \varepsilon.$$

Theorem 124 *Let (X, d_X), (Y, d_Y), (Z, d_Z) be three metric spaces, let $E \subseteq X$, let $x_0 \in E$ be an accumulation point of E, and let $F \subseteq Y$. Given two functions $f : E \to F$ and $g : F \to Y$, assume that there exist*

$$\lim_{x \to x_0} f(x) = \ell \in Y,$$

and that either ℓ is an accumulation point of F, there exists $\delta_1 > 0$ such that $f(x) \neq \ell$ for all $x \in E$ with $0 < d_X(x, x_0) < \delta_1$, and there exists

$$\lim_{y \to \ell} g(y) = L \in Z.$$

or that $\ell \in F$ and g is continuous at ℓ. Then there exists $\lim_{x \to x_0} g(f(x)) = L$.

Friday, February 13, 2015

Proof. Fix $\varepsilon > 0$ and find $\eta = \eta(\varepsilon, \boldsymbol{\ell}) > 0$ such that

$$\|\mathbf{g}(\mathbf{y}) - \mathbf{L}\| < \varepsilon \tag{11}$$

for all $\mathbf{y} \in F$ with $0 < \|\mathbf{y} - \boldsymbol{\ell}\| < \eta$.

Since $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell}$, there exists $\delta_2 = \delta_2(\mathbf{x}_0, \eta) > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \eta$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$.

We now distinguish two cases.

Case 1: Assume that $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$ for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. Then taking $\delta = \min\{\delta_1, \delta_2\}$, we have that for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,

$$0 < \|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \eta.$$

Hence, taking $\mathbf{y} = \mathbf{f}(\mathbf{x})$, by (11), it follows that

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. This shows that there exists $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{L}$.

Case 2: Assume that $\boldsymbol{\ell} \in F$ and $\mathbf{g}(\boldsymbol{\ell}) = \mathbf{L}$. Let $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. If $\mathbf{f}(\mathbf{x}) = \boldsymbol{\ell}$, then $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{L}$, and so

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| = 0 < \varepsilon,$$

while if $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$, then taking $\mathbf{y} = \mathbf{f}(\mathbf{x})$, by (11), it follows that

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| < \varepsilon.$$

■

Example 125 *Let's prove that the previous theorem fails without the hypotheses that either $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$ for all $\mathbf{x} \in E$ near $\mathbf{x}_0$ or $\boldsymbol{\ell} \in F$, $L \in \mathbb{R}$ and $\mathbf{g}(\boldsymbol{\ell}) = L$. Consider the function*

$$g(y) := \begin{cases} 1 & \text{if } y \neq 0, \\ 2 & \text{if } y = 0. \end{cases}$$

Then there exists

$$\lim_{y \to 0} g(y) = 1.$$

So $L = 1$. Consider the function $f(x) := 0$ for all $x \in \mathbb{R}$. Then for every $x_0 \in \mathbb{R}$, we have that

$$\lim_{x \to x_0} f(x) = 0.$$

So $\ell = 0$. However, $g(f(x)) = g(0) = 2$ for all $x \in \mathbb{R}$. Hence,

$$\lim_{x \to x_0} g(f(x)) = \lim_{x \to x_0} 2 = 2 \neq 1,$$

which shows that the conclusion of the theorem is violated.

Remark 126 *One can use Theorem 124 to prove Theorem 115. Indeed, $(\boldsymbol{f} + \boldsymbol{g})(x)$ is the composition of the function $\boldsymbol{h}_1 : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ given by*

$$\boldsymbol{h}_1(\boldsymbol{s}, \boldsymbol{t}) := \boldsymbol{s} + \boldsymbol{t}$$

with the function $\boldsymbol{F} : E \to \mathbb{R}^M \times \mathbb{R}^M$ given by $\boldsymbol{F}(x) = (\boldsymbol{f}(x), \boldsymbol{g}(x))$, while $(\boldsymbol{f} \cdot \boldsymbol{g})(x)$ is the composition of the function $h_2 : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ given by

$$\boldsymbol{h}_2(\boldsymbol{s}, \boldsymbol{t}) := \boldsymbol{s} \cdot \boldsymbol{t}$$

with the function $\boldsymbol{F}$, while $\dfrac{f(x)}{g(x)}$ is the composition of the function $h_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$h_3(s, t) := \frac{s}{t}$$

with the function $\boldsymbol{F}$ where $M = 1$. Thus, to apply the theorem it is enough to verify the continuity of the functions $\boldsymbol{h}_1$, $\boldsymbol{h}_2$, and h_3. This is left as an exercise.

Example 127 *We list below some important limits.*

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2},\qquad \lim_{x\to 0}\frac{\log(1+x)}{x}=1,$$
$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a\quad \textit{for } a\in\mathbb{R},\qquad \lim_{x\to 0}\frac{e^x-1}{x}=1.$$

Note that the previous theorem can be used to change variables in limits.

Example 128 *Let's try to calculate*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{x}.$$

For $\sin x\neq 0$*, we have*

$$\frac{\log(1+\sin x)}{x}=\frac{\log(1+\sin x)}{x}\frac{\sin x}{\sin x}=\frac{\log(1+\sin x)}{\sin x}\frac{\sin x}{x}.$$

Since $\lim_{x\to 0}\frac{\sin x}{x}=1$*, it remains to study*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}.$$

Consider the function $g(y)=\frac{\log(1+y)}{y}$ *and the function* $f(x)=\sin x$*. As* $x\to 0$*, we have that* $\sin x\to 0=\ell$*, while*

$$\lim_{y\to 0}\frac{\log(1+y)}{y}=1.$$

Moreover $\sin x\neq 0$ *for all* $x\in E:=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$*. Hence, we can apply Theorem 124 to conclude that*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}=1.$$

In turn,

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{x}=1.$$

9 Limits of Monotone Functions

Let $E\subseteq\mathbb{R}$ and let $f:E\to\mathbb{R}$. Then f is said to be

- *increasing* if $f(x)\leq f(y)$ for all $x,y\in E$ with $x<y$,
- *strictly increasing* if $f(x)<f(y)$ for all $x,y\in E$ with $x<y$,
- *decreasing* if $f(x)\geq f(y)$ for all $x,y\in E$ with $x<y$,
- *strictly decreasing* if $f(x)>f(y)$ for all $x,y\in E$ with $x<y$,

- *monotone* if one of the four property above holds.

Monday, February 16, 2015

Midterm solutions.

Wednesday, February 18, 2015

Theorem 129 *Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$ be a monotone function. If $x_0 \in \mathbb{R}$ is an accumulation point of $E \cap (-\infty, x_0)$, then there exists*

$$\lim_{x \to x_0^-} f(x) = \begin{cases} \sup_{E\cap(-\infty,x_0)} f & \text{if } f \text{ is increasing,} \\ \inf_{E\cap(-\infty,x_0)} f & \text{if } f \text{ is decreasing,} \end{cases}$$

while if $x_0 \in \mathbb{R}$ is an accumulation point of $E \cap (x_0, \infty)$ then there exists

$$\lim_{x \to x_0^+} f(x) = \begin{cases} \sup_{E\cap(x_0,\infty)} f & \text{if } f \text{ is decreasing,} \\ \inf_{E\cap(x_0,\infty)} f & \text{if } f \text{ is increasing.} \end{cases}$$

Proof. Assume that $x_0 \in \mathbb{R}$ is an accumulation point of $E \cap (-\infty, x_0)$ and that f is increasing (the other cases are similar). There are two cases.

Case 1: The function f is bounded from above in $E \cap (-\infty, x_0)$. Hence, there exists

$$\sup_{E\cap(-\infty,x_0)} f = \ell \in \mathbb{R}.$$

We need to prove that there exists

$$\lim_{x \to x_0^-} f(x) = \ell. \tag{12}$$

Let $\varepsilon > 0$. Since ℓ is the supremum of $f(E \cap (-\infty, x_0))$, we have that $f(x) \le \ell$ for all $x \in E$ with $x < x_0$. On the other hand, since $\ell - \varepsilon$ is not an upper bound for the set $f(E \cap (-\infty, x_0))$ there exists $x_1 \in E \cap (-\infty, x_0)$ such that $\ell - \varepsilon < f(x_1)$. But since f is increasing, for all $x \in E$ with $x_1 < x < x_0$ we have that $\ell - \varepsilon < f(x_1) \le f(x)$. Thus,

$$\ell - \varepsilon < f(x) \le \ell < \ell + \varepsilon$$

for all $x \in E \cap (-\infty, x_0)$ with $x_1 < x < x_0$. Take $\delta := x_0 - x_1 > 0$. Then $|f(x) - \ell| < \varepsilon$ for all $x \in E \cap (-\infty, x_0)$ with $0 < |x - x_0| < \delta$. This proves (12).

Case 2: The function f is not bounded from above in $E \cap (-\infty, x_0)$. We need to prove that there exists

$$\lim_{x \to x_0^-} f(x) = \infty. \tag{13}$$

Let $M > 0$. Since the set $f(E \cap (-\infty, x_0))$ is not bounded from above there exists $x_1 \in E \cap (-\infty, x_0)$ such that $f(x_1) > M$. But since f is increasing, for all $x \in E$ with $x_1 < x < x_0$ we have that $M < f(x_1) \le f(x)$. Thus,

$$M < f(x)$$

for all $x \in E \cap (-\infty, x_0)$ with $x_1 < x < x_0$. Take $\delta := x_0 - x_1 > 0$. Then $f(x) > M$ for all $x \in E \cap (-\infty, x_0)$ with $0 < |x - x_0| < \delta$. This proves (13). ∎

Remark 130 *A similar result holds if* $E \subseteq \overline{\mathbb{R}}$ *and if* $f : E \to \overline{\mathbb{R}}$.

Definition 131 *A set* $E \subseteq \mathbb{R}^N$ *is* countable *if there exists a one-to-one function* $f : E \to \mathbb{N}$.

Remark 132 *It can be shown that* $\mathbb{Q}$ *is countable and that if* $E_n \subseteq \mathbb{R}$, $n \in \mathbb{N}$, *is countable, then*

$$E = \bigcup_{n=1}^{\infty} E_n$$

is countable. It can also be shown that $\mathbb{R}$ *and the irrationals are NOT countable.*

Definition 133 *A set* $I \subseteq \mathbb{R}$ *is an* interval *if for every* $x, y \in I$, *with* $x < y$, *we have that the interval* $[x, y]$ *is contained in* I.

Definition 134 *Given a set* X *and a function* $f : X \to [0, \infty]$ *the* infinite sum

$$\sum_{x \in X} f(x)$$

is defined as

$$\sum_{x \in X} f(x) := \sup \left\{ \sum_{x \in Y} f(x) : \, Y \subset X, \, Y \textit{ finite} \right\}.$$

Proposition 135 *Given a set* X *and a function* $f : X \to [0, \infty]$, *if*

$$\sum_{x \in X} f(x) < \infty,$$

then the set $\{x \in X : \, f(x) > 0\}$ *is countable. Moreover,* f *does not take the value* ∞.

Proof. Define

$$M := \sum_{x \in X} f(x) < \infty.$$

For $k \in \mathbb{N}$ set $X_k := \left\{ x \in X : \, f(x) > \frac{1}{k} \right\}$ and let Y be a finite subset of X_k. Then

$$\frac{1}{k} \text{number of elements of } Y \leq \sum_{x \in Y} f(x) \leq M,$$

which shows that Y cannot have more than $\lfloor kM \rfloor$ elements, where $\lfloor \cdot \rfloor$ is the integer part. In turn, X_k has a finite number of elements, and so

$$\{x \in X : \, f(x) > 0\} = \bigcup_{k=1}^{\infty} X_k$$

is countable. ■

Exercise 136 *Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Prove that the set of discontinuity points of f is countable.*

Exercise 137 *Given a nonempty set X and two functions $f, g : X \to [0, \infty]$.*

(i) Prove that

$$\sum_{x \in X} (f(x) + g(x)) \leq \sum_{x \in X} f(x) + \sum_{x \in X} g(x).$$

(ii) If $f \leq g$, then

$$\sum_{x \in X} f(x) \leq \sum_{x \in X} g(x).$$

Theorem 138 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotone function. Then there exists*

$$\lim_{y \to x} f(y) = f(x)$$

for all $x \in I$ except at most for countably many.

Proof. Step 1: Assume that $I = [a, b]$ and, without loss of generality, that f is increasing. For every $x \in (a, b)$ there exist

$$\lim_{y \to x^+} f(y) =: f_+(x), \quad \lim_{y \to x^-} f(y) =: f_-(x).$$

Let $S(x) := f_+(x) - f_-(x) \geq 0$ be the jump of f at x. Then f is continuous at x if and only if $S(x) = 0$. Let $J \subseteq [a, b]$ be any finite subset, and write

$$J = \{x_1, \ldots, x_k\}, \quad \text{where } x_1 < \cdots < x_k.$$

Since f is increasing, we have that

$$\begin{aligned} f(a) &\leq f_-(x_1) \leq f_+(x_1) \leq f_-(x_2) \leq f_+(x_2) \\ &\leq \cdots \leq f_-(x_k) \leq f_+(x_k) \leq f(b), \end{aligned}$$

and so,

$$\sum_{x \in J} S(x) = \sum_{i=1}^{k} (f_+(x_i) - f_-(x_i)) \leq f(b) - f(a),$$

which implies that

$$\sum_{x \in (a,b)} S(x) \leq f(b) - f(a).$$

By the previous proposition, it follows that the set of discontinuity points of f is at most countable.

Step 2: If I is an arbitrary interval, construct an increasing sequence of intervals $[a_n, b_n]$ such that

$$a_n \searrow \inf I, \quad b_n \nearrow \sup I.$$

Since the union of countable sets is countable and on each interval $[a_n, b_n]$ the set of discontinuity points of f is at most countable, by the previous step it follows that the set of discontinuity points of f in I is at most countable. ■

Friday, February 20, 2015

10 Liminf and Limsup

Let (X, d) be a metric space, let $E \subseteq X$ and let $f : E \to \mathbb{R}$. Assume that $x_0 \in X$ is an accumulation point of E. For every $r > 0$ define

$$g(r) := \inf_{E \cap (B(x_0, r) \setminus \{x_0\})} f.$$

Note that $g(r)$ is $-\infty$ if f is not bounded from below in $E \cap (B(x_0, r) \setminus \{x_0\})$. If $r_1 < r_2$ then $g(r_1) \geq g(r_2)$. Hence the function $g : (0, \infty)$ is decreasing. It follows by Theorem 129 that there exists

$$\lim_{r \to 0^+} g(r) = \sup_{(0,\infty)} g = \ell \in \overline{\mathbb{R}}.$$

This limit is called the *limit inferior of f as x approaches x_0* and is denoted

$$\liminf_{x \to x_0} f(x) \quad \text{or} \quad \underline{\lim}_{x \to x_0} f(x).$$

On the other hand, for every $r > 0$ define

$$h(r) := \sup_{E \cap (B(x_0, r) \setminus \{x_0\})} f.$$

Note that $h(r)$ is ∞ if f is not bounded from above in $E \cap (B(x_0, r) \setminus \{x_0\})$. If $r_1 < r_2$ then $h(r_1) \leq h(r_2)$. Hence the function $h : (0, \infty)$ is increasing. It follows by Theorem 129 that there exists

$$\lim_{r \to 0^+} h(r) = \inf_{(0,\infty)} h = L \in \overline{\mathbb{R}}.$$

This limit is called the *limit superior of f as x approaches x_0* and is denoted

$$\limsup_{x \to x_0} f(x) \quad \text{or} \quad \overline{\lim}_{x \to x_0} f(x).$$

Theorem 139 *Let (X, d) be a metric space, let $E \subseteq X$ and let $f : E \to \mathbb{R}$. Assume that $x_0 \in X$ is an accumulation point of E. Then*

$$\liminf_{x \to x_0} f(x) \leq \limsup_{x \to x_0} f(x). \tag{14}$$

Moreover, there exists $\lim_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if and only if

$$\liminf_{x \to x_0} f(x) = \limsup_{x \to x_0} f(x) = \ell. \tag{15}$$

Proof. Step 1: For every $r > 0$ we have

$$g(r) = \inf_{E\cap(B(x_0,r)\setminus\{x_0\})} f \le \sup_{E\cap(B(x_0,r)\setminus\{x_0\})} f = h(r).$$

Hence, $g \le h$. As we already observed, since g is decreasing and h is increasing, there exist

$$\lim_{r\to 0^+} g(r) = \ell, \quad \lim_{r\to 0^+} h(r) = L.$$

It follows by Exercise 122 that $\ell \le L$, and so (14).

Step 2: Next we prove the second part of the statement. Assume that there exists $\lim_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$. We consider here the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. By the definition of limit, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon$$

for all $x \in E$ with $0 < d(x, x_0) < \delta$, that is,

$$\ell - \varepsilon < f(x) < \ell + \varepsilon$$

for all $x \in E$ with $0 < d(x, x_0) < \delta$. The inequality $\ell - \varepsilon < f(x)$ for all $x \in E$ with $0 < d(x, x_0) < \delta$ implies that $\ell - \varepsilon$ is a lower bound for the set $f(E \cap (B(x_0, \delta) \setminus \{x_0\}))$ and so

$$\ell - \varepsilon \le g(\delta) = \inf_{E\cap(B(x_0,\delta)\setminus\{x_0\})} f.$$

Since g is decreasing, we have that

$$\ell - \varepsilon \le \lim_{r\to 0^+} g(r) = \liminf_{x\to x_0} f(x).$$

On the other hand, the inequality $f(x) < \ell+\varepsilon$ for all $x \in E$ with $0 < d(x, x_0) < \delta$ implies that $\ell + \varepsilon$ is an upper bound for the set $f(E \cap (B(x_0, \delta) \setminus \{x_0\}))$ and so

$$h(\delta) = \sup_{E\cap(B(x_0,\delta)\setminus\{x_0\})} f \le \ell + \varepsilon.$$

Since h is increasing, we have that

$$\lim_{r\to 0^+} h(r) = \limsup_{x\to x_0} f(x) \le \ell + \varepsilon.$$

Hence, also by the first part

$$\ell - \varepsilon \le \liminf_{x\to x_0} f(x) \le \limsup_{x\to x_0} f(x) \le \ell + \varepsilon.$$

But this is true for all $\varepsilon > 0$. Hence, we can send $\varepsilon \to 0^+$ to get

$$\ell \le \liminf_{x\to x_0} f(x) \le \limsup_{x\to x_0} f(x) \le \ell,$$

which shows that (15) holds.

Step 3: Next assume that (15) holds. We need to prove that there exists $\lim_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$. We consider here the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. Let $\varepsilon > 0$. Since

$$\ell = \liminf_{x\to x_0} f(x) = \lim_{r\to 0^+} g(r) = \sup_{(0,\infty)} g,$$

we have that $\ell - \varepsilon$ is not an upper bound for the function g. Hence, there exists $r_1 > 0$ such that

$$\ell - \varepsilon < g(r_1) = \inf_{E\cap(B(x_0,r_1)\setminus\{x_0\})} f.$$

In turn,

$$\ell - \varepsilon < f(x)$$

for all $x \in E \cap (B(x_0, r_1) \setminus \{x_0\})$, that is, for all $x \in E$ with $0 < d(x, x_0) < r_1$.

On the other hand, since

$$\ell = \limsup_{x\to x_0} f(x) = \lim_{r\to 0^+} h(r) = \inf_{(0,\infty)} h,$$

we have that $\ell + \varepsilon$ is not a lower bound for the function h. Hence, there exists $r_2 > 0$ such that

$$\sup_{E\cap(B(x_0,r_2)\setminus\{x_0\})} f = h(r_2) < \ell + \varepsilon.$$

In turn,

$$f(x) < \ell + \varepsilon$$

for all $x \in E \cap (B(x_0, r_2) \setminus \{x_0\})$, that is, for all $x \in E$ with $0 < d(x, x_0) < r_2$. It is enough to take $\delta := \min\{r_1, r_2\}$ to conclude that for all $x \in E$ with $0 < d(x, x_0) < \delta$,

$$|f(x) - \ell| < \varepsilon,$$

which shows that there exists $\lim_{x\to x_0} f(x) = \ell$. ■

Monday, February 23, 2015

The next theorems are very useful in exercises.

Theorem 140 *Let (X, d) be a metric space, let $E \subseteq X$ and let $f : E \to \mathbb{R}$. Assume that $x_0 \in X$ is an accumulation point of E and that f is bounded from above. Given $\ell \in \mathbb{R}$ we have that*

$$\limsup_{x\to x_0} f(x) = \ell$$

if and only if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$f(x) \le \ell + \varepsilon$$

for all $x \in E$ with $0 < d(x, x_0) < \delta_\varepsilon$ and for every $0 < \delta \le \delta_\varepsilon$ there is some $x_1 \in E$ with $0 < d(x_1, x_0) < \delta$ such that

$$\ell - \varepsilon \le f(x_1).$$

Proof. Exercise. ∎

Exercise 141 *Give a similar characterization for the case* $\limsup_{x \to x_0} f(x) = \infty$.

Similarly we have the following.

Theorem 142 *Let* (X, d) *be a metric space, let* $E \subseteq X$ *and let* $f : E \to \mathbb{R}$. *Assume that* $x_0 \in X$ *is an accumulation point of* E *and that* f *is bounded from below. Given* $\ell \in \mathbb{R}$ *we have that*

$$\liminf_{x \to x_0} f(x) = \ell$$

if and only if for every $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that*

$$\ell - \varepsilon \leq f(x)$$

for all $x \in E$ *with* $0 < d(x, x_0) < \delta_\varepsilon$ *and and for every* $0 < \delta \leq \delta_\varepsilon$ *there is some* $x_1 \in E$ *with* $0 < d(x_1, x_0) < \delta$ *such that*

$$f(x_1) \leq \ell + \varepsilon.$$

Proof. Exercise. ∎

Exercise 143 *Give a similar characterization in the case* $\liminf_{x \to x_0} f(x) = -\infty$.

Theorem 144 *Let* (X, d) *be a metric space, let* $E \subseteq X$, *let* $f : E \to \mathbb{R}$ *and let* $g : E \to \mathbb{R}$. *Assume that* $x_0 \in X$ *is an accumulation point of* E *and that* f *and* g *are bounded. Then*

$$\begin{aligned}\liminf_{x \to x_0} f(x) + \liminf_{x \to x_0} g(x) &\leq \liminf_{x \to x_0} (f(x) + g(x)) \\ &\leq \liminf_{x \to x_0} f(x) + \limsup_{x \to x_0} g(x) \\ &\leq \limsup_{x \to x_0} (f(x) + g(x)) \leq \limsup_{x \to x_0} f(x) + \limsup_{x \to x_0} g(x).\end{aligned}$$

In particular, if there exist $\lim_{x \to x_0} g(x)$ *then*

$$\liminf_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = \liminf_{x \to x_0} (f(x) + g(x))$$

and

$$\limsup_{x \to x_0} (f(x) + g(x)) = \limsup_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

Proof. Exercise. ∎

Remark 145 *The previous theorem continues to hold when* f *and* g *are not bounded, provided we exclude the cases* $-\infty + \infty$.

Example 146 *In general all the inequalities in the previous theorem can be strict. For example, if we take*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

then

$$\liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) = 0 + 0 < \liminf_{x\to x_0}(f(x) + g(x)) = 1,$$
$$\limsup_{x\to x_0}(f(x) + g(x)) = 1 < 1 + 1 = \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x).$$

Theorem 147 *Let (X, d) be a metric space, let $E \subseteq X$, let $f : E \to \mathbb{R}$ and let $g : E \to \mathbb{R}$. Assume that $x_0 \in X$ is an accumulation point of E, that f and g are bounded and that there exists $\lim_{x\to x_0} g(x) = \ell \in \mathbb{R}$. If $\ell > 0$, then*

$$\liminf_{x\to x_0}(f(x)g(x)) = \liminf_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$$

and

$$\limsup_{x\to x_0}(f(x)g(x)) = \limsup_{x\to x_0} f(x) \lim_{x\to x_0} g(x).$$

On the other hand, if $\ell < 0$, then

$$\liminf_{x\to x_0}(f(x)g(x)) = \limsup_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$$

and

$$\limsup_{x\to x_0}(f(x)g(x)) = \liminf_{x\to x_0} f(x) \lim_{x\to x_0} g(x).$$

Remark 148 *Note that if $\ell = 0$, then there we can apply Exercise* **??** *to conclude that there exists*

$$\lim_{x\to x_0}(f(x)g(x)) = 0.$$

Definition 149 *Let (X, d) be a metric space and let $E \subseteq X$. A point $x_0 \in E$ is called an* isolated point *of E if there exists $\delta > 0$ such that*

$$B(x_0, \delta) \cap E = \{x_0\}.$$

Note that if a point of the set E is not an isolated point of E then it is an accumulation point of E.

Definition 150 *Let (X, d) be a metric space, let $E \subseteq X$, and let $f : E \to \mathbb{R}$. We say that f is* lower semicontinuous *at $x_0 \in E \cap \operatorname{acc} E$ if*

$$\liminf_{x\to x_0} f(x) \geq f(x_0).$$

We say that f is lower semicontinuous *if it is lower semicontinuous at every point in $E \cap \operatorname{acc} E$.*

Remark 151 *If* $x_0 \in E$ *is not an accumulation point of* E*, then it is an isolated point of* E *and so we cannot approach it with other points in* E*, thus it makes no sense to talk about* $\liminf_{x \to x_0} f(x)$*.*

Example 152 *Let* $E \subseteq \mathbb{R}$ *and let*

$$f(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbb{R} \setminus E. \end{cases}$$

Prove that f *is lower semicontinuous if and only if* E *is open.*

Wednesday, February 25, 2015

Definition 153 *Let* (X, d) *be a metric space and let* $E \subseteq X$*. A set* $F \subseteq E$ *is said to be* relatively open *in* E *if there exists an open set* $U \subseteq X$ *such that* $F = U \cap E$*. A set* $G \subseteq E$ *is said to be* relatively closed *in* E *if there exists a closed set* $C \subseteq X$ *such that* $G = C \cap E$*.*

Theorem 154 (Weierstrass) *Let* (X, d) *be a metric space, let* $K \subseteq X$ *be compact and let* $f : K \to \mathbb{R}$ *be lower semicontinuous. Then there exists* $x_0 \in K$ *such that*

$$f(x_0) = \min_{x \in K} f(x).$$

Proof. Step 1: Let's prove that for every $t \in \mathbb{R}$ the set $K_t := \{x \in K : f(x) > t\}$ is relatively open in K. Let $x_0 \in K_t$. If x_0 is an isolated point of K, then there exists $B(x_0, r)$ such that $K \cap B(x_0, r) = \{x_0\}$. In turn, $K_t \cap B(x_0, r) = K \cap B(x_0, r)$.

On the other hand, if x_0 is an accumulation point of K then

$$\ell := \liminf_{x \to x_0} f(x) = \lim_{r \to 0^+} \inf_{K \cap B(x_0, r) \setminus \{x_0\}} f \geq f(x_0) > t.$$

If $\ell < \infty$ take $\varepsilon = \ell - t > 0$. Then by the definition of limit there exists $\delta > 0$ such that

$$t = \ell - \varepsilon < \inf_{K \cap B(x_0, r) \setminus \{x_0\}} f < \ell + \varepsilon$$

for all $r > 0$ with $0 < r < \delta$. If $\ell = \infty$, then exists $\delta > 0$ such that

$$t < \inf_{K \cap B(x_0, r) \setminus \{x_0\}} f$$

for all $r > 0$ with $0 < r < \delta$.

Hence, in both cases there exists $r > 0$ such that

$$\inf_{K \cap B(x_0, r) \setminus \{x_0\}} f > t,$$

which implies that for every $x \in K \cap B(x_0, r)$, $f(x) > t$, and so x belongs to K_t, thus $K_t \cap B(x_0, r) = K \cap B(x_0, r)$.

Step 2: Let

$$t := \inf_{x \in K} f(x).$$

If the infimum is not attained, then for every $x \in K$ we may find $t < t_x < f(x)$. By the previous step the set $\{y \in K : f(y) > t_x\}$ is relatively open and so there exists an open set U_x such that

$$U_x \cap K = \{y \in K : f(y) > t_x\}, \quad x \in K.$$

Note that $x \in U_x$. Hence, the family of open sets $\{U_x\}_{x \in K}$ is an open cover for the compact set K, and so we may find a finite cover $U_{x_1}, \ldots, U_{x_l}$ of the set K. But then for all $x \in K$,

$$f(x) \geq \min_{i=1,\ldots,l} t_{x_i} > t = \inf_{w \in K} f(w),$$

which contradicts the definition of t. ■

Example 155 *A lower semicontinuous function in general has no maximum. To see this, take the function*

$$f(x) = \begin{cases} -x^2 & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Then f is lower semicontinuous, $\sup f = 0$ *but there is no maximum.*

Definition 156 *Let (X, d) be a metric space, let $E \subseteq X$, and let $f : E \to \mathbb{R}$. We say that f is* upper semicontinuous *at $x_0 \in E \cap \operatorname{acc} E$ if*

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

We say that f is upper semicontinuous *if it is upper semicontinuou at every point in E.*

Exercise 157 *Let (X, d) be a metric space, let $K \subseteq X$ be compact and let $f : K \to \mathbb{R}$ be upper semicontinuous. Prove that there exists $x_0 \in K$ such that*

$$f(x_0) = \max_{x \in K} f(x).$$

11 Continuity

We recall that

Definition 158 *Let (X, d_X), (Y, d_Y) be metric spaces, let $E \subseteq X$, and let $f : E \to Y$. We say that f is* continuous *at $x_0 \in E \cap \operatorname{acc} E$ if there exists*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If f is continuous at every point of $E \cap \operatorname{acc} E$ we say that f is continuous *on E and we write $f \in C(E)$ or $f \in C^0(E)$.*

Remark 159 *If* (X, τ_X) *and* (Y, τ_Y) *are two topological spaces,* $E \subseteq X$*,* $x_0 \in E \cap \operatorname{acc} E$*, and* $f : E \to Y$*, we say that* f *is* continuous *at* x_0 *if for every neighborhood* V *of* $f(x_0)$ *there exists a neighborhood* U *of* x_0 *with the property that*

$$f(x) \in V$$

for all $x \in E$ *with* $x \in U$*.*

Exercise 160 *Prove that the functions* $\sin x$*,* $\cos x$*,* x^n*, where* $n \in \mathbb{N}$*, are continuous.*

The following theorems follows from the analogous results for limits.

Theorem 161 *Let* (X, d)*, be a metric space, let* $E \subseteq X$*, and let* $x_0 \in E$*. Given two functions* f*,* $g : E \to \mathbb{R}$ *assume that* f *and* g *are continuous at* x_0*. Then*

(i) $f + g$ *and* fg *are continuous at* x_0*;*

(ii) if $g(x) \neq 0$ *for all* $x \in E$*, then* $\dfrac{f}{g}$ *is continuous at* x_0*.*

Example 162 *In view of Exercise 160 and the previous theorem, the functions* $\tan x = \frac{\sin x}{\cos x}$ *and* $\cot x = \frac{\cos x}{\sin x}$ *are continuous in their domain of definition.*

Theorem 163 *Let* (X, d_X)*,* (Y, d_Y)*,* (Z, d_Z) *be metric spaces, let* $E \subseteq X$*, let* $F \subseteq Y$*, and let* $f : E \to F$ *and* $g : F \to Z$*. Assume that* f *is continuous at* x_0 *and that* g *is continuous at* $f(x_0)$*. Then* $g \circ f : E \to Z$ *is continuous at* x_0*.*

Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{f} : E \to \mathbb{R}^M$. Given $\mathbf{x}_0 \in E$, what happens when $\mathbf{f}$ is discontinuous at $\mathbf{x}_0$? Then $\mathbf{x}_0$ is an accumulation point of E. The following situations can arise. It can happen that there exists

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell} \in \mathbb{R}^M$$

but $\boldsymbol{\ell} \neq \mathbf{f}(\mathbf{x}_0)$. In this case, we say that $\mathbf{x}_0$ is a *removable discontinuity.* Indeed, consider the function $g : E \to \mathbb{R}^M$ defined by

$$g(\mathbf{x}) := \begin{cases} \mathbf{f}(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{x}_0, \\ \boldsymbol{\ell} & \text{if } \mathbf{x} = \mathbf{x}_0. \end{cases}$$

Then there exists

$$\lim_{\mathbf{x} \to \mathbf{x}_0} g(\mathbf{x}) = \boldsymbol{\ell} = g(\mathbf{x}_0),$$

and so the new function g is continuous at $\mathbf{x}_0$.

Another type of discontinuity is when x_0 is an accumulation point of $E^- := E \cap (-\infty, x_0]$ and of $E^+ := E \cap (x_0, \infty)$ and there exist

$$\lim_{x \to x_0^-} \mathbf{f}(x) = \boldsymbol{\ell} \in \mathbb{R}^M, \quad \lim_{x \to x_0^+} \mathbf{f}(x) = L \in \mathbb{R}^M$$

but $\boldsymbol{\ell} \neq L$. In this case the point x_0 is called a *jump discontinuity* of $\mathbf{f}$.

Example 164 *The integer and fractional part of x have jump discontinuity at every integer.*

Finally, the last type of discontinuity is when at least one of the limits $\lim_{x\to x_0^-} \mathbf{f}(x)$ and $\lim_{x\to x_0^+} \mathbf{f}(x)$ is not finite or does not exist. In this case, the point x_0 is called an *essential discontinuity* of $\mathbf{f}$.

Example 165 *The function*

$$f(x) := \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

and

$$g(x) := \begin{cases} \log x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \end{cases}$$

have an essential discontinuity at $x = 0$.

Friday, February 27, 2015

Theorem 166 *Let (X, d_X), (Y, d_Y) be metric spaces, let $E \subseteq X$, and let $f : E \to Y$.*

(i) Then f is continuous if and only if $f^{-1}(V)$ is relatively open for every open set $V \subseteq Y$.

(ii) Then f is continuous if and only if $f^{-1}(C)$ is relatively closed for every closed set $C \subseteq Y$.

Proof. (i) **Step 1:** Let $V \subseteq Y$ be open. Assume that f is continuous. If $f^{-1}(V)$ is empty, then there is nothing to prove. Otherwise, let $x_0 \in f^{-1}(V)$. Since V is open and $f(x_0) \in V$, there exists $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq V$. Since f is continuous at x_0 there exists $\delta_{x_0} > 0$ such that for all $x \in E$ with $d_X(x, x_0) < \delta_{x_0}$, we have

$$d_Y(f(x), f(x_0)) < \varepsilon.$$

Hence, for all $x \in E$ with $d_X(x, x_0) < \delta_{x_0}$,

$$f(x) \in B_Y(f(x_0), \varepsilon) \subseteq V,$$

and so $B_X(x_0, \delta_{x_0}) \cap E \subseteq f^{-1}(V)$.

Take

$$U := \bigcup_{x \in f^{-1}(V)} B_X(x, \delta_x).$$

Then U is open and $f^{-1}(V) \subseteq U$. Hence,

$$U \cap E = f^{-1}(V),$$

which shows that $f^{-1}(V)$ is relatively open.

Step 2: Assume that $f^{-1}(V)$ is relatively open for every open set $V \subseteq Y$. Let $x_0 \in E \cap \operatorname{acc} E$ and let $\varepsilon > 0$. Consider the open set $V = B_Y(f(x_0), \varepsilon)$. Then $f^{-1}(V)$ is relatively open and so there exists an open set $U \subseteq X$ such that $U \cap E = f^{-1}(V)$. Since $x_0 \in f^{-1}(U)$, we have $x_0 \in U$. Hence there exists $B_X(x_0, \delta) \subseteq U$. It follows that for every $x \in U$ with $0 < d_X(x, x_0) < \delta$, then x belongs to $U \cap E = f^{-1}(V)$ and so $f(x) \in V = B_Y(f(x_0), \varepsilon)$, that is,

$$d_Y(f(x), f(x_0)) < \varepsilon,$$

which shows that there exists

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(ii) Exercise. ∎

As a corollary, we get.

Corollary 167 *Let (X, d_X), (Y, d_Y) be metric spaces, let $E \subseteq X$, and let $f : E \to Y$.*

(i) If E is open, then f is continuous if and only if $f^{-1}(V)$ is open for every open set $V \subseteq Y$.

(ii) If E is closed, then f is continuous if and only if $f^{-1}(C)$ is relatively closed for every closed set $C \subseteq Y$.

Remark 168 *The previous characterization of continuous functions is useful to define continuity in a topological space.*

Example 169 *The previous theorem implies in particular that sets of the form*

$$\{x \in \mathbb{R} : 4 \sin x - \log(1 + |x|) > 0\}$$

are open. We used this in the exercises.

Next we show that continuous functions preserve compactness.

Proposition 170 *Let (X, d_X), (Y, d_Y) be metric spaces, let $E \subseteq X$, and let $f : E \to Y$ be continuous. Then $f(K)$ is compact for every compact set $K \subseteq E$.*

Proof. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $f(K)$. By continuity, $f^{-1}(U_\alpha)$ is relatively open for every $\alpha \in \Lambda$, and so there exists W_α open such that $f^{-1}(U_\alpha) = E \cap W_\alpha$. The family $\{W_\alpha\}_{\alpha \in \Lambda}$ is an open cover of K. Since K is compact, we may find $U_{\alpha_1}, \ldots, U_{\alpha_l}$ such that $\{W_{\alpha_i}\}_{i=1}^{l}$ cover K. In turn, $U_{\alpha_1}, \ldots, U_{\alpha_l}$ cover $f(K)$. Indeed, if $y \in f(K)$, then there exists $x \in K$ such that $f(x) = y$. Let $i = 1, \ldots, l$ be such that $x \in f^{-1}(U_{\alpha_i}) = E \cap W_{\alpha_i}$. Then $y = f(x) \in U_{\alpha_i}$. ∎

We now discuss the continuity of inverse functions and of composite functions. If a continuous function f is invertible its inverse function f^{-1} may not be continuous.

Example 171 *Let*

$$f(x) := \begin{cases} x & \text{if } 0 \le x \le 1, \\ x-1 & \text{if } 2 < x \le 3. \end{cases}$$

Then $f^{-1} : [0,2] \to R$ *is given by*

$$f^{-1}(x) := \begin{cases} x & \text{if } 0 \le x \le 1, \\ x+1 & \text{if } 1 < x \le 2, \end{cases}$$

which is not continuous at $x = 1$*.*

We will see that this cannot happen if E is an interval or a compact set.

Theorem 172 *Let* (X, d_X)*,* (Y, d_Y) *be metric spaces, let* $K \subseteq X$ *be compact, and let* $f : K \to Y$ *be one-to-one and continuous. Then the inverse function* $f^{-1} : f(K) \to X$ *is continuous.*

Lemma 173 *Let* (X, d_X)*, let* $K \subset X$ *be a compact set, and let* $C \subseteq K$ *be a closed set. Then* C *is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of C. Since C is closed, the set $U := X \setminus C$ is open. Note that

$$K = (K \setminus C) \cup C \subseteq U \cup \bigcup_\alpha U_\alpha.$$

Since K is compact, there exist $U_{\alpha_1}, \ldots, U_{\alpha_l}$ such that

$$K \subseteq U \cup \bigcup_{i=1}^{l} U_{\alpha_i}.$$

But since $U = X \setminus C$, it follows that

$$C \subseteq \bigcup_{i=1}^{l} U_{\alpha_i},$$

which shows that C is compact. ■

Proof of Theorem 172. Let $C \subseteq X$ be a closed set. By the previous lemma $K \cap C$ is compact. By Proposition 170 we have that $f(K \cap C)$ is compact. In particular, $f(K \cap C)$ is closed by Theorem 98. Let $g := f^{-1}$. Then

$$f(K \cap C) = g^{-1}(C),$$

which shows that $g^{-1}(C)$ is closed for every closed set $C \subseteq X$. Thus, by Theorem 166, g is continuous. ■

Remark 174 *Here we used the fact that a compact set is closed, so to extend this to a function* $f : K \to Y$*, where* $K \subseteq X$ *and* X *and* Y *are topological spaces, we need* Y *to be a Hausdorff topological space (see Remark 99).*

Example 175 *In view of the previous theorem and Exercise 160, the functions* $\arccos x$, $\arcsin x$, $\arctan x$ *are continuous.*

Given $a > 0$*, the function* $\log_a x$ *is continuous for* $x > 0$*, since it is the inverse of* a^x*.*

Given $n \in \mathbb{N}$*, the function* $\sqrt[2n+1]{x}$*,* $x \in \mathbb{R}$*, is continuous, since it is the inverse of* x^{2n+1}*. The function* $\sqrt[2n]{x}$*,* $x \in [0, \infty)$*, is continuous, since it is the inverse of* x^{2n}*.*

Given $a > 0$*, since* e^x *and* $\log x$ *are continuous in* $(0, \infty)$*, by writing*

$$x^a = e^{\log x^a} = e^{a \log x},$$
$$x^x = e^{\log x^x} = e^{x \log x},$$

it follows from Theorems 161 and 163, that x^a *and* x^x *are continuous in* $(0, \infty)$*.*

Monday, March 1, 2015

Correction exercise in the homework on limits with Taylor.

Theorem 176 (Weierstrass) *Let* (X, d) *be a metric space, let* $K \subseteq X$ *be compact and let* $f : K \to \mathbb{R}$ *be continuous. Then there exist* $x_0, x_1 \in K$ *asuch that*

$$f(x_0) = \min_{x \in K} f(x), \quad f(x_1) = \max_{x \in K} f(x)$$

Proof. A continuous function is lower semicontinuous and upper semicontinuous, and so we can apply Theorem 154 and Exercise 157. ■

12 Directional Derivatives and Differentiability

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces, let $E \subseteq X$, let $f : E \to Y$ and let $x_0 \in E$. Given a direction $v \in X \setminus \{0\}$, let L be the line through x_0 in the direction v, that is,

$$L := \{x \in X : x = x_0 + tx, \, t \in \mathbb{R}\},$$

and assume that x_0 is an accumulation point of the set $E \cap L$. The *directional derivative* of f at x_0 in the direction v is defined as

$$\frac{\partial f}{\partial v}(x_0) := \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t},$$

provided the limit exists in Y.

If $X = \mathbb{R}^N$ and $\mathbf{v} = \mathbf{e}_i$, where $\mathbf{e}_i$ is a vector of the canonical basis, the directional derivative $\frac{\partial f}{\partial \mathbf{e}_i}(\mathbf{x}_0)$, if it exists, is called the *partial derivative* of f with respect to x_i and is denoted $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ or $f_{x_i}(\mathbf{x}_0)$ or $D_i f(\mathbf{x}_0)$.

Remark 177 *When* $X = \mathbb{R}$*, taking* $v = 1$*, we get that*

$$\lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t},$$

which is the standard definition of derivative $f'(x_0)$*. It can be shown that if* $f'(x_0)$ *exists in* $\mathbb{R}$*, then* f *is continuous at* x_0*.*

Wednesday, March 3, 2015

In view of the previous remark, one would be tempted to say that if the directional derivatives at x_0 exist and are finite in every direction, then f is continuous at x_0. This is false in general, as the following example shows.

Example 178 *Let*

$$f(x,y) := \begin{cases} 1 & \text{if } y = x^2,\ x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given a direction $\mathbf{v} = (v_1, v_2)$*, the line* L *through* $(0,0)$ *in the direction* $\mathbf{v}$ *intersects the parabola* $y = x^2$ *only in* $(0,0)$ *and in at most one point. Hence, if* t *is very small,*

$$f(0 + tv_1, 0 + tv_2) = 0.$$

It follows that

$$\frac{\partial f}{\partial \mathbf{v}}(0,0) = \lim_{t\to 0} \frac{f(0 + tv_1, 0 + tv_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

However, f *is not continuous in* $(0,0)$*, since* $f\left(x, x^2\right) = 1 \to 1$ *as* $x \to 0$*, while* $f(x,0) = 0 \to 0$ *as* $x \to 0$*.*

Example 179 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Let's find the directional derivatives of f *at* $(0,0)$*. Given a direction* $\mathbf{v} = (v_1, v_2)$*, with* $v_1^2 + v_2^2 = 1$*, we have*

$$f(0 + tv_1, 0 + tv_2) = 0.$$

It follows that

$$\begin{aligned} \frac{f(0 + tv_1, 0 + tv_2) - f(0,0)}{t} &= \frac{\frac{(tv_1)^2 tv_2}{(tv_1)^4 + (tv_2)^2} - 0}{t} \\ &= \frac{t^3 v_1^2 v_2}{t^5 v_1^4 + t^3 v_2^2}. \end{aligned}$$

If $v_2 = 0$ *then*

$$\frac{f(0 + tv_1, 0 + tv_2) - f(0,0)}{t} = \frac{0}{t^5 v_1^4 + 0} = 0 \to 0$$

as $t \to 0$*, so* $\frac{\partial f}{\partial x}(0,0) = 0$*. If* $v_2 \neq 0$*, then,*

$$\frac{f(0 + tv_1, 0 + tv_2) - f(0,0)}{t} = \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \to \frac{v_1^2 v_2}{0 + v_2^2} = \frac{v_1^2}{v_2},$$

so

$$\frac{\partial f}{\partial \mathbf{v}}(0,0)=\frac{v_1^2}{v_2}.$$

In particular, $\frac{\partial f}{\partial y}(0,0)=\frac{0}{1}=0$*. Now let's prove that* f *is not continuous at* $(0,0)$*. We have*

$$f(x,0)=\frac{0}{0+y^2}=0\to 0$$

as $x\to 0$*, while*

$$f\left(x,x^2\right)=\frac{x^2x^2}{x^4+x^4}=\frac{1}{2}\to\frac{1}{2}$$

as $x\to 0$*. Hence, the limit* $\lim_{(x,y)\to(0,0)} f(x,y)$ *does not exists and so* f *is not continuous at* $(0,0)$*. Note that* f *is continuous at all other points* $(x,y)\neq(0,0)$ *by Theorem 161, since* $h(x,y)=x$ *and* $g(x,y)=y$ *are continuous functions in* $\mathbb{R}^2$*.*

The previous examples show that in dimension $N\geq 2$ partial derivatives do not give the same kind of results as in the case $N=1$. To solve this problem, we introduce a stronger notion of derivative, namely, the notion of differentiability.

Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ be normed spaces. We recall that a function $L: X\to Y$ is *linear* if

$$L(x_1+x_2)=L(x_1)+L(x_2)$$

for all $x_1,x_2\in X$ and

$$L(sx)=sL(x)$$

for all $s\in\mathbb{R}$ and $x\in X$.

Remark 180 *If* $X=\mathbb{R}^N$ *and* $Y=\mathbb{R}^M$*, then every linear function* $\boldsymbol{L}:\mathbb{R}^N\to\mathbb{R}^M$ *is continuous. Indeed, Write* $\mathbf{x}=\sum_{i=1}^N x_i\mathbf{e}_i$*. Then by the linearity of* $\boldsymbol{L}$*,*

$$\boldsymbol{L}(\mathbf{x})=\boldsymbol{L}\left(\sum_{i=1}^N x_i\mathbf{e}_i\right)=\sum_{i=1}^N x_i\boldsymbol{L}(\mathbf{e}_i).$$

Define $\mathbf{b}_i:=\boldsymbol{L}(\mathbf{e}_i)\in\mathbb{R}^M$*. Then the previous calculation shows that*

$$\boldsymbol{L}(\mathbf{x})=\sum_{i=1}^N x_i\mathbf{b}_i \qquad \text{for all } \mathbf{x}\in\mathbb{R}^N,$$

which is continuous by Theorem 161.

The following example shows that when X is infinite-dimensional there exist linear functions which are not continuous.

Example 181 *Let* $X:=\{f:[-1,1]\to\mathbb{R}:$ *there exists* $f'(x)\in\mathbb{R}$ *for all* $x\in[-1,1]\}$*. The vector space* X *is a normed space with the norm* $\|f\|:=\max_{x\in[-1,1]}|f(x)|$*. Note that since* f *has a finite derivative at every* x*, it follows that* f *is continuous at every* x*. By the theorem on composition of continuous functions, the* *function $|f(x)|$ is also continuous. Since $[-1,1]$ is compact, by the Weierestrass theorem, there exists* $\max_{x\in[-1,1]}|f(x)|$. *Hence, $\|f\|$ is well-defined. We have already seen in Exercise 47 that it is a norm.*

Consider the linear function $L : X \to \mathbb{R}$ defined by

$$L(f) := f'(0).$$

Then L is linear. To prove that L is not continuous, consider

$$f_n(x) := \frac{1}{n}\sin\left(n^2 x\right).$$

Then

$$\|f_n - 0\| \le \frac{1}{n} \to 0$$

but

$$f_n'(x) = n\cos\left(n^2 x\right)$$

so that

$$L(f_n) = f_n'(0) = n \to \infty$$

and so L is not continuous, since $L(f_n) \nrightarrow L(0) = 0$.

Definition 182 *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces, let $E \subseteq X$, let $f : E \to Y$, and let $x_0 \in E$ be an accumulation point of E. The function f is* differentiable at x_0 *if there exists a continuous linear function $L : X \to Y$ (depending on f and x_0) such that*

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - L(x - x_0)}{\|x - x_0\|_Y} = 0. \tag{16}$$

provided the limit exists. The function L, if it exists, is called the differential *of f at x_0 and is denoted $df(x_0)$ or df_{x_0}.*

Remark 183 *Since f takes values in Y the limit (16) is equivalent to*

$$\lim_{x\to x_0} \left\| \frac{f(x) - f(x_0) - L(x - x_0)}{\|x - x_0\|_Y} \right\|_Y = 0.$$

Exercise 184 *Prove that if $N = 1$, then f is differentiable at x_0 if and only there exists the derivative $f'(x_0) \in \mathbb{R}$.*

The next theorem shows that differentiability in dimension $N \ge 2$ plays the same role of the derivative in dimension $N = 1$.

Theorem 185 *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces, let $E \subseteq X$, let $f : E \to Y$, and let $x_0 \in E$ be an accumulation point of E. If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Let L be the differential of f at x_0. We have

$$\begin{aligned} f(x) - f(x_0) &= f(x) - f(x_0) - L(x - x_0) + L(x - x_0) \\ &= \frac{f(x) - f(x) - L(x - x_0)}{\|x - x_0\|_X} \|x - x_0\| + L(x - x_0). \end{aligned}$$

Hence, by the properties of the norm for $x \in E$, $x \neq x_0$,

$$\begin{aligned} &0 \leq \|f(x) - f(x_0)\|_Y \leq \left\| \frac{f(x) - f(x) - L(x - x_0)}{\|x - x_0\|_X} \right\|_Y \|x - x_0\|_X + \|L(x - x_0)\|_Y \\ &\to \|0\|_Y \|0\|_X + \|L(0)\| = 0 \end{aligned}$$

as $x \to x_0$. It follows that f is continuous at x_0. ■

Next we study the relation between directional derivatives and differentiability. Here we need x_0 to be an interior point of E.

Theorem 186 *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces, let $E \subseteq X$, let $f : E \to Y$, and let $x_0 \in E^\circ$. If f is differentiable at x_0, then all the directional derivatives of f at x_0 exist and*

$$\frac{\partial f}{\partial v}(x_0) = L(v),$$

where L is the differential of f at x_0. In particular, the function

$$v \in X \mapsto \frac{\partial f}{\partial v}(x_0)$$

is linear.

Proof. Since x_0 is an interior point, there exists $B(x_0, r) \subseteq E$. Let $v \in X$ be a direction and take $x = x_0 + tv$. Note that for $|t| < r / \|v\|_X$, we have that

$$\|x - x_0\|_X = \|x_0 + tv - x_0\|_X = \|tv\|_X = |t| \|v\|_X < r$$

and so $x_0 + tv \in B(x_0, r) \subseteq E$. Moreover, $x \to x_0$ as $t \to 0$ and so, since f is differentiable at x_0,

$$\begin{aligned} &0 = \lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x - x_0)}{\|x - x_0\|_X} = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - L(x_0 + tv - x_0)}{\|x_0 + tv - x_0\|} \\ &= \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - tL(v)}{|t|} = \frac{1}{\|v\|_X} \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - tL(v)}{|t|}. \end{aligned}$$

Multiply everything by $\|v\|_X$ to get

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0) - tL(v)}{|t|} = 0.$$

By considering the left and right limits we get

$$0=\lim_{t\to 0^+}\frac{f(x_0+tv)-f(x_0)-tL(v)}{t}=0,$$
$$0=\lim_{t\to 0^-}\frac{f(x_0+tv)-f(x_0)-tL(v)}{-t}=-\lim_{t\to 0^-}\frac{f(x_0+tv)-f(x_0)-tL(v)}{t}$$

and so

$$0=\lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)-tL(v)}{t}=\lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)}{t}-L(v),$$

which shows that there exists $\frac{\partial f}{\partial v}(x_0)=L(v)$. ■

Monday, March 16, 2015

Remark 187 *If in the previous theorem x_0 is not an interior point but for some direction $v\in X$, the point x_0 is an accumulation point of the set $E\cap L$, where L is the line through x_0 in the direction v, then as in the first part of the proof we can show that there exists the directional derivative $\frac{\partial f}{\partial v}(x_0)$ and*

$$\frac{\partial f}{\partial v}(x_0)=T(v).$$

Remark 188 *In particular, if $X=\mathbb{R}^N$, then by the previous theorem*

$$L(\mathbf{e}_i)=\frac{\partial f}{\partial x_i}(\mathbf{x}_0),$$

and so, writing $\mathbf{v}=\sum_{i=1}^N v_i\mathbf{e}_i$, by the linearity of L we have

$$L(\mathbf{v})=L\left(\sum_{i=1}^N v_i\mathbf{e}_i\right)=\sum_{i=1}^N v_iL(\mathbf{e}_i)=\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\,v_i.$$

Thus, only at interior points *of E, to check differentiability it is enough to prove that*

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\frac{f(\mathbf{x})-f(\mathbf{x}_0)-\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i-x_{0i})}{\|\mathbf{x}-\mathbf{x}_0\|}=0. \tag{17}$$

If all the partial derivatives of f at $\mathbf{x}_0$ exist, the vector

$$\left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0),\ldots,\frac{\partial f}{\partial x_N}(\mathbf{x}_0)\right)\in\mathbb{R}^N$$

is called the *gradient* of f at $\mathbf{x}_0$ and is denoted by $\nabla f(\mathbf{x}_0)$ or $\operatorname{grad} f(\mathbf{x}_0)$ or $Df(\mathbf{x}_0)$. Note the previous theorem shows that

$$df_{\mathbf{x}_0}(\mathbf{v})=L(\mathbf{v})=\nabla f(\mathbf{x}_0)\cdot\mathbf{v}=\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\,v_i. \tag{18}$$

for all directions $\mathbf{v}$.

Exercise 189 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at 0, that all directional derivatives of f at 0 exist but that the formula

$$\frac{\partial f}{\partial \mathbf{v}}(0,0) = \frac{\partial f}{\partial x}(0,0)\, v_1 + \frac{\partial f}{\partial y}(0,0)\, v_2$$

fails.

Exercise 190 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all directional derivatives of f at $\mathbf{0}$. Study the continuity and the differentiability of f at $\mathbf{0}$.

Exercise 191 *Let $f : E \to \mathbb{R}$ be Lipschitz and let $\mathbf{x}_0 \in E^\circ$.*

(i) Assume that all the directional derivatives of f at $\mathbf{x}_0$ exist and that $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, v_i$ for every direction $\mathbf{v}$. Prove that f is differentiable at $\mathbf{x}_0$.

(ii) Assume that all the partial derivatives of f at $\mathbf{x}_0$ exist, that the directional derivatives $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$ exist for all $\mathbf{v} \in S$, where S is dense in the unit sphere, and that $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, v_i$ for every direction $\mathbf{v} \in S$. Prove that f is differentiable at $\mathbf{x}_0$.

Example 192 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 |y|}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Let's study continuity, partial derivatives and differentiability. For $(x,y) \neq (0,0)$, we have that f is continuous by Theorem 161, while for $(x,y) = (0,0)$, we need to check that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

We have

$$0 \leq |f(x,y) - f(0,0)| = \left| \frac{x^2 |y|}{x^2+y^2} - 0 \right| = \frac{x^2 |y|}{x^2+y^2} \leq \frac{\left(x^2+y^2\right)|y|}{x^2+y^2} = |y| \to 0$$

as $(x,y) \to (0,0)$. Hence, f is continuous at $(0,0)$.

Next, let's study partial derivatives. For $(x,y) \neq (0,0)$, by the quotient rule, we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x\,|y|\left(x^2+y^2\right) - x^2\,|y|\,(2x+0)}{\left(x^2+y^2\right)^2}, \tag{19}$$

while for $(x,y) = (0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(0+t1, 0+t0) - f(0,0)}{t} = \lim_{t\to 0}\frac{\frac{t^2|0|}{t^2+0} - 0}{t} = \lim_{t\to 0}\frac{0}{t^3} = 0.$$

For $y \neq 0$, by the quotient rule, we have

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^2\frac{y}{|y|}\left(x^2+y^2\right) - x^2\,|y|\,(0+2y)}{\left(x^2+y^2\right)^2}, \tag{20}$$

while at a point $(x_0, 0)$,

$$\frac{\partial f}{\partial y}(x_0,0) = \lim_{t\to 0}\frac{f(x_0+t0, 0+t1) - f(x_0,0)}{t} = \lim_{t\to 0}\frac{\frac{x_0^2|t|}{x_0^2+t^2} - 0}{t} = \lim_{t\to 0}\frac{|t|}{t}\frac{x_0^2}{x_0^2+t^2}.$$

If $x_0 = 0$, then $\frac{|t|}{t}\frac{x_0^2}{x_0^2+t^2} = \frac{|t|}{t}\frac{0}{0+t^2} = 0 \to 0$ as $t \to 0$, so $\frac{\partial f}{\partial y}(0,0) = 0$, while if $x_0 \neq 0$, we have

$$\lim_{t\to 0^+}\frac{|t|}{t}\frac{x_0^2}{x_0^2+t^2} = \lim_{t\to 0^+}\frac{t}{t}\frac{x_0^2}{x_0^2+t^2} = \lim_{t\to 0^+}\frac{x_0^2}{x_0^2+t^2} = \frac{x_0^2}{x_0^2+0} = 1,$$
$$\lim_{t\to 0^-}\frac{|t|}{t}\frac{x_0^2}{x_0^2+t^2} = \lim_{t\to 0^+}\frac{-t}{t}\frac{x_0^2}{x_0^2+t^2} = -\lim_{t\to 0^+}\frac{x_0^2}{x_0^2+t^2} = -\frac{x_0^2}{x_0^2+0} = -1.$$

Hence, $\frac{\partial f}{\partial y}(x_0,0)$ does not exist at $(x_0,0)$ for $x_0 \neq 0$, and so by Theorem 186, f is not differentiable at $(x_0,0)$ for $x_0 \neq 0$.

On the other hand, at points (x,y) with $y \neq 0$, we have that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in a small ball centered at (x,y) (see (19) and (20)) and they are continuous by Theorem 161. Hence, we can apply Theorem 194 below to conclude that f is differentiable at all points (x,y) with $y \neq 0$.

It remains to study differentiability at $(0,0)$. By 17, we need to prove that

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = 0.$$

We have

$$\frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = \frac{\frac{x^2|y|}{x^2+y^2} - 0 - (0,0)\cdot((x,y)-(0,0))}{\sqrt{x^2+y^2}}$$
$$= \frac{x^2\,|y|}{\left(x^2+y^2\right)^{3/2}}.$$

Taking $y = x$, with $x > 0$, we get

$$\frac{x^2 |x|}{(x^2 + x^2)^{3/2}} = \frac{x^2 x^3}{(x^2 + x^2)^{3/2}} = \frac{1}{(2)^{3/2}} \nrightarrow 0.$$

Hence, f is not differentiable at $(0,0)$.

Example 193 *Consider the function*

$$f(x,y) = |x|y.$$

Let's study the differentiability of f in $\mathbb{R}^2$. We have

$$\frac{\partial f}{\partial y}(x,y) = \frac{x}{|x|} y \quad \text{for } x \neq 0,$$
$$\frac{\partial f}{\partial y}(x,y) = |x|.$$

Thus all the points $x = 0$ create problems and must be studied separately. Consider a point $(0, y_0)$. Then

$$\begin{aligned} \frac{\partial f}{\partial x}(0,y_0) &= \lim_{t \to 0} \frac{f(0 + t1, y_0 + t0) - f(0,0)}{t} \\ &= \lim_{t \to 0} \frac{|t| y_0 - 0}{t} = \lim_{t \to 0} \frac{|t|}{t} y_0 = \begin{cases} 0 & \text{if } y_0 = 0, \\ \text{does not exist} & \text{if } y_0 \neq 0. \end{cases} \end{aligned}$$

Thus if $y_0 \neq 0$, $\frac{\partial f}{\partial x}$ does not exists and so by Theorem 186 below, f is not differentiable at $(0, y_0)$. On the other hand, if $y_0 = 0$, then $\frac{\partial f}{\partial x}(0,0) = 0$ and so we must study the differentiability of f using the definition. We haveWe have

$$\begin{aligned} \frac{f(x,y) - f(0,0) - \nabla f(0,0) \cdot ((x,y) - (0,0))}{\|(x,y) - (0,0)\|} &= \frac{|x|y - 0 - (0,0) \cdot ((x,y) - (0,0))}{\sqrt{x^2 + y^2}} \\ &= \frac{|x|y}{\sqrt{x^2 + y^2}}. \end{aligned}$$

Since

$$0 \leq \left| \frac{|x|y}{\sqrt{x^2 + y^2}} \right| = \frac{\sqrt{x^2}\,|y|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x^2 + y^2}\,|y|}{\sqrt{x^2 + y^2}} = |y| \to 0$$

as $(x,y) \to (0,0)$, we have that f is differentiable at $(0,0)$.

Wednesday, March 18, 2015

The next theorem gives an important sufficient condition for differentiability at a point $\mathbf{x}_0$.

Theorem 194 *Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$, let $\mathbf{x}_0 \in E^\circ$, and let $i \in \{1, \ldots, N\}$. Assume that there exists $r > 0$ such that $B(\mathbf{x}_0, r) \subseteq E$ and for all $j \neq i$ and for all $\mathbf{x} \in B(\mathbf{x}_0, r)$ the partial derivative $\frac{\partial f}{\partial x_j}$ exists at $\mathbf{x}$ and is continuous at $\mathbf{x}_0$. Assume also that $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ exists. Then f is differentiable at $\mathbf{x}_0$.*

To prove this theorem we need some preliminary results.

Definition 195 *Let (X, d) be a metric space, let $E \subseteq X$, let $f : E \to \mathbb{R}$, and let $x_0 \in E$. We say that*

(i) f attains a local minimum *at x_0 if there exists $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in E \cap B(x_0, r)$,*

(ii) f attains a global minimum *at x_0 if $f(x) \geq f(x_0)$ for all $x \in E$,*

(iii) f attains a local maximum *at x_0 if there exists $r > 0$ such that $f(x) \leq f(x_0)$ for all $x \in E \cap B(x_0, r)$,*

(iv) f attains a global maximum *at x_0 if $f(x) \leq f(x_0)$ for all $x \in E$.*

Theorem 196 *Let $(X, \|\cdot\|)$ be a normed space, let $E \subseteq X$, let $f : E \to \mathbb{R}$, and let $x_0 \in E$. Assume that f attains a local minimum (or maximum) at x_0 and that there exists the directional derivative $\frac{\partial f}{\partial v}(x_0)$. If x_0 is an accumulation point for both sets $E \cap \{x_0 + tv : \ t > 0\}$ and $E \cap \{x_0 + tv : \ t < 0\}$, then necessarily, $\frac{\partial f}{\partial v}(x_0) = 0$. In particular, if x_0 is an interior point of E and f is differentiable at x_0, then all the directional derivatives of f at x_0 are zero.*

Proof. Assume that f attains a local minimum (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in E \cap B(x_0, r)$. Take $x = x_0 + tv$, where $|t| < r / \|v\|$. Then

$$\|x_0 + tv - x_0\| = \|tv\| = |t| \, \|v\| < r,$$

and so $f(x_0 + tv) \geq f(x_0)$. If $t > 0$, then

$$\frac{f(x_0 + tv) - f(x_0)}{t} \geq 0.$$

Since x_0 is an accumulation point for the set $E \cap \{x_0 + tv : \ t > 0\}$, there are infinitely many $t > 0$ approaching zero. Hence, letting $t \to 0^+$ and using the fact that there exists $\frac{\partial f}{\partial v}(x_0)$, we get that $\frac{\partial f}{\partial v}(x_0) \geq 0$.

If $t < 0$, then $f(x_0 + tv) \geq f(x_0)$ and

$$\frac{f(x_0 + tv) - f(x_0)}{t} \leq 0.$$

Since x_0 is an accumulation point for the set $E \cap \{x_0 + tv : \ t < 0\}$, there are infinitely many $t < 0$ approaching zero. Hence, letting $t \to 0^-$ and using the fact that there exists $\frac{\partial f}{\partial v}(x_0)$, we get that $\frac{\partial f}{\partial v}(x_0) \leq 0$.

This shows that $\frac{\partial f}{\partial v}(x_0) = 0$. ■

Remark 197 *If if x_0 is a point of local minimum and $\frac{\partial f}{\partial v}(x_0)$ exists, then x_0 is an accumulation point for the set $E \cap \{x_0 + tv : \ t \in \mathbb{R}\}$, so x_0 is an accumulation point for $E \cap \{x_0 + tv : \ t > 0\}$, in which case $\frac{\partial f}{\partial v}(x_0) \geq 0$, or x_0 is an accumulation point for $E \cap \{x_0 + tv : \ t < 0\}$, in which case $\frac{\partial f}{\partial v}(x_0) \leq 0$.*

An important application of the previous theorem is given by the following result.

Theorem 198 (Rolle) *Assume that $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and has a derivative in (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since $[a, b]$ is compact, by the Weierstrass theorem, f has a global maximum and a global minimum in $[a, b]$. If

$$\max_{[a,b]} f = \min_{[a,b]} f,$$

then f is constant, and so $f'(x) = 0$ for all $x \in (a, b)$. If $\max_{[a,b]} f > \min_{[a,b]} f$, then since $f(a) = f(b)$, there f admits one of them at some interior point $c \in (a, b)$. By the previous theorem, $f'(c) = 0$. ■

Theorem 199 (Lagrange or Mean Value Theorem) *Assume that $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and has a derivative in (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. The function

$$g(x) = f(x) - (x - a)\frac{f(b) - f(a)}{b - a}$$

is continuous in $[a, b]$, has a derivative in (a, b), and $g(a) = g(b) = f(a)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a, b)$ such that $g'(c) = 0$, or, equivalently,

$$0 = f'(x) - 1\frac{f(b) - f(a)}{b - a}.$$

■

Friday, March 20, 2015

We are now ready to prove Theorem 194

Proof of Theorem 194. Without loss of generality, we may assume that $i = N$. Let $\mathbf{x} \in B(\mathbf{x}_0, r)$. Write $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{x}_0 = (y_1, \ldots, y_N)$. Then

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{x}_0) = &(f(x_1, \ldots, x_N) - f(y_1, x_2, \ldots, x_N)) \\ &+ \cdots + (f(y_1, \ldots, y_{N-1}, x_N) - f(y_1, \ldots, y_N)). \end{aligned}$$

By the mean value theorem applied to the function of one variable $f(\cdot, x_2, \ldots, x_N)$,

$$f(x_1, \ldots, x_N) - f(y_1, x_2, \ldots, x_N) = \frac{\partial f}{\partial x_1}(\mathbf{z}_1)(x_1 - y_1),$$

where $\mathbf{z}_1 := (\theta_1 x_1 + (1-\theta_1) y_1, x_2, \ldots, x_N)$ for some $\theta_1 \in (0,1)$. Note that

$$\|\mathbf{z}_1 - \mathbf{x}_0\| \leq \|\mathbf{x} - \mathbf{x}_0\|.$$

Similarly, for $i = 2, \ldots, N-1$,

$$f(y_1, \ldots, y_{i-1}, x_i, \ldots, x_N) - f(y_1, \ldots, y_{i-1}, y_i, \ldots, x_N) = \frac{\partial f}{\partial x_i}(\mathbf{z}_i)(x_i - y_i),$$

where $\mathbf{z}_i := (y_1, \ldots, y_{i-1}, \theta_i x_i + (1-\theta_i) y_i, x_{i+1}, \ldots, x_N)$ for some $\theta_i \in (0,1)$ and

$$\|\mathbf{z}_i - \mathbf{x}_0\| \leq \|\mathbf{x} - \mathbf{x}_0\|.$$

Write

$$\begin{aligned} & f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\ &= \sum_{i=1}^{N-1} \left(\frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right)(x_i - y_i) \\ &\quad + \left(\frac{f(y_1, \ldots, y_{N-1}, x_N) - f(y_1, \ldots, y_N)}{x_N - y_N} - \frac{\partial f}{\partial x_N}(\mathbf{x}_0) \right)(x_N - y_N). \end{aligned}$$

Then

$$\begin{aligned} \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} &\leq \sum_{i=1}^{N-1} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \frac{|x_i - y_i|}{\|\mathbf{x} - \mathbf{x}_0\|} \\ + \left| \frac{f(y_1, \ldots, y_{N-1}, y_N + (x_N - y_N)) - f(y_1, \ldots, y_N)}{x_N - y_N} - \frac{\partial f}{\partial x_N}(\mathbf{x}_0) \right| &\frac{|x_N - y_N|}{\|\mathbf{x} - \mathbf{x}_0\|}. \end{aligned}$$

Since $\frac{|x_i - y_i|}{\|\mathbf{x} - \mathbf{x}_0\|} \leq 1$, we have that

$$\begin{aligned} 0 \leq \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} &\leq \sum_{i=1}^{N-1} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \\ + \left| \frac{f(y_1, \ldots, y_{N-1}, y_N + (x_N - y_N)) - f(y_1, \ldots, y_N)}{x_N - y_N} - \frac{\partial f}{\partial x_N}(\mathbf{x}_0) \right|&. \end{aligned} \tag{21}$$

Using the fact that $\|\mathbf{z}_i - \mathbf{x}_0\| \leq \|\mathbf{x} - \mathbf{x}_0\| \to 0$ as $\mathbf{x} \to \mathbf{x}_0$, together with the continuity of $\frac{\partial f}{\partial x_i}$ at $\mathbf{x}_0$, gives

$$\left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \to 0,$$

while, since $t := x_N - y_N \to 0$ as $\mathbf{x} \to \mathbf{x}_0$, we have that

$$\begin{aligned} & \frac{f(y_1, \ldots, y_{N-1}, y_N + (x_N - y_N)) - f(y_1, \ldots, y_N)}{x_N - y_N} \\ &= \frac{f(y_1, \ldots, y_{N-1}, y_N + t) - f(y_1, \ldots, y_N)}{t} \to \frac{\partial f}{\partial x_N}(\mathbf{x}_0), \end{aligned}$$

and so the right-hand side of (21) goes to zero as $\mathbf{x} \to \mathbf{x}_0$. ■

Monday, March 23, 2015

We study the differentiability of composite functions.

Theorem 200 (Chain Rule) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be three normed spaces, let $E \subseteq X$, let $x_0 \in X$ be an accumulation point of E, let $F \subseteq Y$, and let $f : E \to F$ and $g : F \to Z$. Assume that there exists the directional derivative $\frac{\partial f}{\partial v}(x_0)$, that $f(x_0) \in F$ and that g is differentiable at $f(x_0)$. Then there exists the directional derivative*

$$\frac{\partial (g \circ f)}{\partial v}(x_0) = dg_{f(x_0)}\left(\frac{\partial f}{\partial v}(x_0)\right). \tag{22}$$

Moreover, if f is differentiable at x_0, then $g \circ f$ is differentiable at x_0 with

$$d(g \circ f)_{x_0} = dg_{f(x_0)} \circ df_{x_0}.$$

Proof. Since g is differentiable at $f(x_0)$, there exists $L : Y \to Z$ linear and continuous such that

$$\lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0)) - L(y - f(x_0))}{\|y - f(x_0)\|_Y} = 0, \tag{23}$$

where L is the differential of g at $f(x_0)$, so $L = dg_{f(x_0)}$.

Since there exists the directional derivative $\frac{\partial f}{\partial v}(x_0)$, we have that there exists

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \frac{\partial f}{\partial v}(x_0). \tag{24}$$

It follows that the function of one variable

$$t \mapsto f(x_0 + tv)$$

is continuous at $t = 0$. Hence, if we take $y = f(x_0 + tv)$, we have that

$$y = f(x_0 + tv) \to f(x_0) \quad \text{as } t \to 0. \tag{25}$$

Case 1: Assume that $f(x_0 + tv) \neq f(x_0)$ for all t small. Then by (23), (24), (25),

$$\begin{aligned}
&\frac{g(f(x_0 + tv)) - g(f(x_0))}{t} - L\left(\frac{\partial f}{\partial v}(x_0)\right) \\
&= \frac{g(f(x_0 + tv)) - g(f(x_0)) - L(f(x_0 + tv) - f(x_0))}{t} \\
&\quad + L\left(\frac{f(x_0 + tv) - f(x_0)}{t} - \frac{\partial f}{\partial v}(x_0)\right) \\
&= \frac{g(f(x_0 + tv)) - g(f(x_0)) - L(f(x_0 + tv) - f(x_0))}{\|f(x_0 + tv) - f(x_0)\|_Y} \left\|\frac{f(x_0 + tv) - f(x_0)}{t}\right\|_Y \frac{|t|}{t} \\
&\quad + L\left(\frac{f(x_0 + tv) - f(x_0)}{t} - \frac{\partial f}{\partial v}(x_0)\right) \\
&\to 0 \left\|\frac{\partial f}{\partial v}(x_0)\right\|_Y (\pm 1) + L(0) = 0.
\end{aligned}$$

This shows that there exists

$$\frac{\partial(g\circ f)}{\partial v}(x_0)=L\left(\frac{\partial f}{\partial v}(x_0)\right).$$

Case 2: There exists countably many t approaching zero such that $f(x_0+tv)=f(x_0)$. Hence, for these t,

$$\frac{f(x_0+tv)-f(x_0)}{t}=0\to 0$$

as $t\to 0$, which implies that $\frac{\partial f}{\partial v}(x_0)=0$.

Let $F:=\{t\in\mathbb{R}: f(x_0+tv)=f(x_0)\}$. For $t\in F$,

$$\frac{g(f(x_0+tv))-g(f(x_0))}{t}=\frac{0}{t}\to 0.$$

On the other hand, if $t\notin F$, then $f(x_0+tv)\neq f(x_0)$ and so

$$\begin{aligned}
&\frac{g(f(x_0+tv))-g(f(x_0))}{t}\\
&=\frac{g(f(x_0+tv))-g(f(x_0))-L(f(x_0+tv)-f(x_0))}{t}\\
&\quad+L\left(\frac{f(x_0+tv)-f(x_0)}{t}\right)\\
&=\frac{g(f(x_0+tv))-g(f(x_0))-L(f(x_0+tv)-f(x_0))}{\|f(x_0+tv)-f(x_0)\|_Y}\left\|\frac{f(x_0+tv)-f(x_0)}{t}\right\|_Y\frac{|t|}{t}\\
&\quad+L\left(\frac{f(x_0+tv)-f(x_0)}{t}\right)\\
&\to 0\left\|\frac{\partial f}{\partial v}(x_0)\right\|_Y(\pm 1)+L(0)=0.
\end{aligned}$$

by (23), (24), (25). This proves the first part of the statement.

The second part of the statement is left as an exercise. ■

Exercise 201 *Prove the second part of the theorem.*

Wednesday, March 25, 2015

Remark 202 *Assume that* $Y=\mathbb{R}^M$ *and* $Z=\mathbb{R}$. *Then* $\boldsymbol{f}: E\to\mathbb{R}^M$. *Let* $\mathbf{y}_0:=\mathbf{f}(x_0)$. *If* $\mathbf{y}_0\in F^\circ$, *then by (17),*

$$dg_{\mathbf{y}_0}(\mathbf{v})=\nabla g(\mathbf{y}_0)\cdot\mathbf{v}=\sum_{i=1}^{M}\frac{\partial g}{\partial y_i}(\mathbf{y}_0)\,v_i.$$

Hence, (22)becomes

$$\begin{aligned}
\frac{\partial(f\circ\mathbf{g})}{\partial v}(x_0)&=\sum_{i=1}^{M}\frac{\partial g}{\partial y_i}(\mathbf{f}(x_0))\frac{\partial f_i}{\partial v}(x_0)\\
&=\nabla g(\mathbf{f}(x_0))\cdot\frac{\partial\mathbf{f}}{\partial v}(x_0).
\end{aligned}$$

Example 203 (Product Rule) *Let $(X, \|\cdot\|_X)$ be a normed space, let $E \subseteq X$ and let $\boldsymbol{f} : E \to \mathbb{R}^2$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(y_1, y_2) = y_1 y_2$. Then (exercise)*

$$\frac{\partial g}{\partial y_1}(y_1, y_2) = y_2, \quad \frac{\partial g}{\partial y_2}(y_1, y_2) = y_1.$$

If there exists $\frac{\partial \boldsymbol{f}}{\partial v}(x_0)$, then by Remark 202,

$$\begin{aligned}\frac{\partial}{\partial v}(f_1 f_2)(x_0) &= \frac{\partial (g \circ \boldsymbol{f})}{\partial v}(x_0) \\ &= \frac{\partial g}{\partial y_1}(\mathbf{f}(x_0)) \frac{\partial f_1}{\partial v}(x_0) + \frac{\partial g}{\partial y_2}(\mathbf{f}(x_0)) \frac{\partial f_2}{\partial v}(x_0) \\ &= f_2(x_0) \frac{\partial f_1}{\partial v}(x_0) + f_1(x_0) \frac{\partial f_2}{\partial v}(x_0),\end{aligned}$$

which is the product rule.

Example 204 (Quotient Rule) *Let $(X, \|\cdot\|_X)$ be a normed space, let $E \subseteq X$ and let $\boldsymbol{f} : E \to \mathbb{R}^2$, with $\boldsymbol{f}(x) = (f_1(x), f_2(x))$, be such that $f_2(x) \neq 0$ for all $x \in E$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(y_1, y_2) = \frac{y_1}{y_2}$. Then (exercise)*

$$\frac{\partial g}{\partial y_1}(y_1, y_2) = \frac{1}{y_2}, \quad \frac{\partial g}{\partial y_2}(y_1, y_2) = -\frac{y_1}{(y_2)^2}.$$

If there exists $\frac{\partial \boldsymbol{f}}{\partial v}(x_0)$, then by Remark 202,

$$\begin{aligned}\frac{\partial}{\partial v}\left(\frac{f_1}{f_2}\right)(x_0) &= \frac{\partial (g \circ \boldsymbol{f})}{\partial v}(x_0) \\ &= \frac{\partial g}{\partial y_1}(\mathbf{f}(x_0)) \frac{\partial f_1}{\partial v}(x_0) + \frac{\partial g}{\partial y_2}(\mathbf{f}(x_0)) \frac{\partial f_2}{\partial v}(x_0) \\ &= \frac{1}{f_2(x_0)} \frac{\partial f_1}{\partial v}(x_0) - \frac{f_1(x_0)}{(f_2(x_0))^2} \frac{\partial f_2}{\partial v}(x_0) \\ &= \frac{\frac{\partial f_1}{\partial v}(x_0) f_2(x_0) - f_1(x_0) \frac{\partial f_1}{\partial v}(x_0)}{(f_2(x_0))^2},\end{aligned}$$

which is the quotient rule.

Example 205 *Sum.....*

Example 206 *Consider the function*

$$g(\mathbf{x}) := f(\|\mathbf{x}\|) = f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}\right),$$

where $f : [0, \infty) \to \mathbb{R}$ is differentiable. Then for all $\mathbf{x} \neq \mathbf{0}$,

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = f'\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}\right) \frac{2x_i}{2\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}\right)}.$$

Next we define the Jacobian of a vectorial function $\mathbf{f} : E \to \mathbb{R}^M$.

Definition 207 *Given a set $E \subseteq \mathbb{R}^N$ and a function $\mathbf{f} : E \to \mathbb{R}^M$, the* Jacobian matrix *of $\mathbf{f} = (f_1, \ldots, f_M)$ at some point $\mathbf{x}_0 \in E$, whenever it exists, is the $M \times N$ matrix*

$$J_{\mathbf{f}}(\mathbf{x}_0) := \begin{pmatrix} \nabla f_1(\mathbf{x}_0) \\ \vdots \\ \nabla f_M(\mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_N}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix}.$$

It is also denoted

$$\frac{\partial(f_1, \ldots, f_M)}{\partial(x_1, \ldots, x_N)}(\mathbf{x}_0).$$

When $M = N$, $J_{\mathbf{f}}(\mathbf{x}_0)$ is an $N \times N$ square matrix and its determinant is called the Jacobian determinant *of $\mathbf{f}$ at $\mathbf{x}_0$. Thus,*

$$\det J_{\mathbf{f}}(\mathbf{x}_0) = \det\left(\frac{\partial f_j}{\partial x_i}(\mathbf{x}_0)\right)_{i,j=1,\ldots,N}.$$

Remark 208 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{f} : E \to \mathbb{R}^M$, and let $\mathbf{x}_0 \in E^\circ$. Assume that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$. Then all its components f_j, $j = 1, \ldots, M$, are differentiable at $\mathbf{x}_0$ with*

$$d\mathbf{f}_{\mathbf{x}_0} = \left(d(f_1)_{\mathbf{x}_0}, \ldots, d(f_M)_{\mathbf{x}_0}\right).$$

Since $\mathbf{x}_0$ is an interior point, it follows from (18) that for every direction $\mathbf{v}$,

$$d(f_j)_{\mathbf{x}_0}(\mathbf{v}) = \nabla f_j(\mathbf{x}_0) \cdot \mathbf{v} = \sum_{i=1}^N \frac{\partial f_j}{\partial x_i}(\mathbf{x}_0)\, v_i.$$

Hence,

$$\begin{aligned} d\mathbf{f}_{\mathbf{x}_0}(\mathbf{v}) &= \left(d(f_1)_{\mathbf{x}_0}(\mathbf{v}), \ldots, d(f_M)_{\mathbf{x}_0}(\mathbf{v})\right) \\ &= J_{\mathbf{f}}(\mathbf{x}_0)\, \mathbf{v}. \end{aligned}$$

As a corollary of Theorem 200, we have the following result.

Corollary 209 *Let $F \subseteq \mathbb{R}^M$, $E \subseteq \mathbb{R}^N$, let $\mathbf{g} : F \to E$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_N)$, and let $\mathbf{f} : E \to \mathbb{R}^P$. Assume that $\mathbf{g}$ is differentiable at some point $\mathbf{y}_0 \in F^\circ$ and that $\mathbf{f}$ is differentiable at the point $\mathbf{g}(\mathbf{y}_0)$ and that $\mathbf{g}(\mathbf{y}_0) \in E^\circ$. Then the composite function $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{y}_0$ and*

$$J_{\mathbf{f} \circ \mathbf{g}}(\mathbf{y}_0) = J_{\mathbf{f}}(\mathbf{g}(\mathbf{y}_0))\, J_{\mathbf{g}}(\mathbf{x}_0).$$

13 Higher Order Derivatives

Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. Let $i \in \{1, \dots, N\}$ and assume that there exists the partial derivatives $\frac{\partial f}{\partial x_i}(\mathbf{x})$ for all $\mathbf{x} \in E$. If $j \in \{1, \dots, N\}$ and $\mathbf{x}_0$ is an accumulation point of $E \cap L$, where L is the line through $\mathbf{x}_0$ in the direction $\mathbf{e}_j$, then we can consider the partial derivative of the function $\frac{\partial f}{\partial x_i}$ with respect to x_j, that is,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) =: \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Note that in general the order in which we take derivatives is important.

Example 210 *Let*

$$f(x, y) := \begin{cases} \frac{x^3 y - x y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

If $(x, y) \neq (0, 0)$, then by Examples 203 and 204,

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}\left(\frac{x^3 y - x y^3}{x^2 + y^2}\right) = \frac{(3x^2 y - 1y^3)(x^2 + y^2) - (x^3 y - x y^3)(2x + 0)}{(x^2 + y^2)^2},$$

and

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(\frac{x^3 y - x y^3}{x^2 + y^2}\right) = \frac{(x^3 1 - x 3 y^2)(x^2 + y^2) - (x^3 y - x y^3)(0 + 2y)}{(x^2 + y^2)^2},$$

while at $(0, 0)$ we have:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} \frac{\frac{0}{t^2 + 0} - 0}{t} = 0,$$
$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \to 0} \frac{f(0, 0 + t) - f(0, 0)}{t} = \lim_{t \to 0} \frac{\frac{0}{0 + t^2} - 0}{t} = 0.$$

Thus,

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \frac{(3x^2 y - 1y^3)(x^2 + y^2) - (x^3 y - x y^3)(2x + 0)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$
$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} \frac{(x^3 1 - x 3 y^2)(x^2 + y^2) - (x^3 y - x y^3)(0 + 2y)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

To find $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$, we calculate

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0, 0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(0, 0 + t) - \frac{\partial f}{\partial x}(0, 0)}{t}$$
$$= \lim_{t \to 0} \frac{\frac{(0 - 1t^3)(0 + t^2) - 0}{(0 + t^2)^2} - 0}{t} = \lim_{t \to 0} -1 = -1,$$

while

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{t\to 0}\frac{\frac{\partial f}{\partial y}(0+t,0) - \frac{\partial f}{\partial y}(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{\frac{(t^3 1-0)(t^2+0)-0}{(t^2+0)^2} - 0}{t} = \lim_{t\to 0} 1 = 1.$$

Hence, $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Exercise 211 *Let*

$$f(x,y) := \begin{cases} y^2 \arctan \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Prove that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

We present an improved version of the Schwartz theorem.

Theorem 212 (Schwartz) *Let* $E \subseteq \mathbb{R}^N$*, let* $f : E \to \mathbb{R}$*, let* $\mathbf{x}_0 \in E^\circ$*, and let* $i, j \in \{1, \ldots, N\}$*. Assume that there exists* $r > 0$ *such that* $B(\mathbf{x}_0, r) \subseteq E$ *and for all* $\mathbf{x} \in B(\mathbf{x}_0, r)$*, the partial derivatives* $\frac{\partial f}{\partial x_i}(\mathbf{x})$*,* $\frac{\partial f}{\partial x_j}(\mathbf{x})$*, and* $\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x})$ *exist. Assume also that* $\frac{\partial^2 f}{\partial x_j \partial x_i}$ *is continuous at* $\mathbf{x}_0$*. Then there exists* $\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)$ *and*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

Lemma 213 *Let* $A : ((-r,r) \setminus \{0\}) \times ((-r,r) \setminus \{0\}) \to \mathbb{R}$*. Assume that the double limit* $\lim_{(s,t)\to(0,0)} A(s,t)$ *exists in* $\mathbb{R}$ *and that the limit* $\lim_{t\to 0} A(s,t)$ *exists in* $\mathbb{R}$ *for all* $s \in (-r,r) \setminus \{0\}$*. Then the iterated limit* $\lim_{s\to 0}\lim_{t\to 0} A(s,t)$ *exists and*

$$\lim_{s\to 0}\lim_{t\to 0} A(s,t) = \lim_{(s,t)\to(0,0)} A(s,t).$$

Proof. Let $\ell = \lim_{(s,t)\to(0,0)} A(s,t)$. Then for every $\varepsilon > 0$ there exists $\delta = \delta((0,0),\varepsilon) > 0$ such that

$$|A(s,t) - \ell| \leq \varepsilon$$

for all $(s,t) \in ((-r,r) \setminus \{0\}) \times ((-r,r) \setminus \{0\})$, with $\sqrt{|s-0|^2 + |t-0|^2} \leq \delta$.

Fix $s \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \setminus \{0\}$. Then for all $t \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \setminus \{0\}$,

$$|A(s,t) - \ell| \leq \varepsilon$$

and so letting $t \to 0$ in the previous inequality (and using the fact that the limit $\lim_{t\to 0} A(s,t)$ exists), we get

$$\left|\lim_{t\to 0} A(s,t) - \ell\right| \leq \varepsilon$$

for all $s \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \backslash \{0\}$. But this implies that there exists $\lim_{s\to 0} \lim_{t\to 0} A(s,t) = \ell$. ■

Friday, March 27, 2015

Proof of Theorem 212. Step 1: Assume that $N = 2$. Let $|t|, |s| < \frac{r}{\sqrt{2}}$. Then the points $(x_0 + s, y_0)$, $(x_0 + s, y_0 + t)$, and $(x_0, y_0 + t)$ belong to $B((x_0, y_0), r)$. Define

$$
\begin{aligned}
A(s,t) &:= \frac{f(x_0 + s, y_0 + t) - f(x_0 + s, y_0) - f(x_0, y_0 + t) + f(x_0, y_0)}{st}, \\
g(x) &:= f(x, y_0 + t) - f(x, y_0).
\end{aligned}
$$

By the mean value theorem

$$
A(s,t) = \frac{g(x_0 + s) - g(x_0)}{st} = \frac{g'(\xi)}{t} = \frac{\frac{\partial f}{\partial x}(\xi_t, y_0 + t) - \frac{\partial f}{\partial x}(\xi_t, y_0)}{t}
$$

where ξ is between x_0 and $x_0 + t$. Fix t and consider the function

$$
h(y) := \frac{\partial f}{\partial x}(\xi_t, y).
$$

By the mean value theorem,

$$
h(b) - h(a) = h'(c)(b - a) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, c)(b - a)
$$

for some c between a and b. Taking $b = t$ and $a = 0$, we get

$$
\frac{\partial f}{\partial x}(\xi_t, y_0 + t) - \frac{\partial f}{\partial x}(\xi_t, y_0) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, \eta_t)\, t
$$

where η_t is between y_0 and $y_0 + t$. Hence,

$$
A(s,t) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, \eta_t) \to \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0),
$$

where we have used the fact that $(\xi, \eta) \to (x_0, y_0)$ as $(s,t) \to (0,0)$ together with the continuity of $\frac{\partial^2 f}{\partial y \partial x}$ at (x_0, y_0). Note that this shows that there exists the limit

$$
\lim_{(s,t)\to(0,0)} A(s,t) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).
$$

On the other hand, for all $s \neq 0$,

$$
\begin{aligned}
\lim_{t\to 0} A(s,t) &= \frac{1}{s} \lim_{t\to 0} \left[\frac{f(x_0 + s, y_0 + t) - f(x_0 + s, y_0)}{t} - \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t} \right] \\
&= \frac{\frac{\partial f}{\partial y}(x_0 + s, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{s}.
\end{aligned}
$$

Hence, we are in a position to apply the previous lemma to obtain

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) &= \lim_{(s,t)\to(0,0)} A(s,t) = \lim_{s\to 0}\lim_{t\to 0} A(s,t) \\ &= \lim_{s\to 0} \frac{\frac{\partial f}{\partial y}(x_0+s, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{s} = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0).\end{aligned}$$

Step 2: In the case $N \geq 2$ let $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbf{x}_0 = (c_1, \ldots, c_N)$. Assume that $1 < i < j < N$ (the cases $i = 1$ and $j = N$ are similar) and apply Step 1 to the function of two variables

$$F(x_i, x_j) := f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_{j-1}, x_j, c_{j+1}, \ldots, c_N)$$

■

Next we prove Taylor's formula in higher dimensions. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A *multi-index* $\boldsymbol{\alpha}$ is a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N}_0)^N$. The *length* of a multi-index is defined as

$$|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_N.$$

Given a multi-index $\boldsymbol{\alpha}$, the partial derivative $\frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}$ is defined as

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}},$$

where $\mathbf{x} = (x_1, \ldots, x_N)$. If $\boldsymbol{\alpha} = \mathbf{0}$, we set $\frac{\partial^{\mathbf{0}} f}{\partial \mathbf{x}^{\mathbf{0}}} := f$.

Example 214 *If $N = 3$ and $\boldsymbol{\alpha} = (2, 1, 0)$, then*

$$\frac{\partial^{(2,1,0)}}{\partial (x,y,z)^{(2,1,0)}} = \frac{\partial^3}{\partial x^2 \partial y}.$$

Given a multi-index $\boldsymbol{\alpha}$ and $\mathbf{x} \in \mathbb{R}^N$, we set

$$\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_N!, \quad \mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

If $\boldsymbol{\alpha} = \mathbf{0}$, we set $\mathbf{x}^{\mathbf{0}} := 1$.

Using this notation, we can extend the binomial theorem.

Theorem 215 (Multinomial Theorem) *Let $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and let $n \in \mathbb{N}$. Then*

$$(x_1 + \cdots + x_N)^n = \sum_{\boldsymbol{\alpha} \text{ multi-index},\ |\boldsymbol{\alpha}| = n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{x}^{\boldsymbol{\alpha}}.$$

Proof. Exercise. ■

Definition 216 *Given an open set $U \subseteq \mathbb{R}^N$, for every nonnegative integer $m \in \mathbb{N}_0$, we denote by $C^m(U)$ the space of all functions that are continuous together with their partial derivatives up to order m. We set $C^\infty(U) := \bigcap_{m=0}^{\infty} C^m(U)$.*

Monday, March 30, 2015

Theorem 217 (Taylor's Formula) *Let $U \subseteq \mathbb{R}^N$ be an open set, let $f \in C^m(U)$, $m \in \mathbb{N}$, and let $\mathbf{x}_0 \in U$. Then for every $\mathbf{x} \in U$,*

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{\boldsymbol{\alpha} \text{ multi-index},\ 1 \le |\boldsymbol{\alpha}| \le m} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + R_m(\mathbf{x}),$$

where

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_m(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^m} = 0.$$

Definition 218 *Given a metric space (X, d_X), a set $E \subseteq X$, and two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ and a point $x_0 \in \operatorname{acc} E$, we say that the function f is a* little o *of g as $x \to x_0$, and we write $f = \mathrm{o}(g)$, if $g \neq 0$ in E and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Hence, a little o of g is simply a function that goes to zero faster than g as $x \to x_0$. Hence, Taylor's formula can be written as

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \text{ multi-index},\ 0 \le |\boldsymbol{\alpha}| \le m} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + o(\|\mathbf{x} - \mathbf{x}_0\|^m)$$

as $\mathbf{x} \to \mathbf{x}_0$.

To prove Taylor's formula we need some preliminary results.

Exercise 219 *Let (X, d_X) and (Y, d_Y) be two metric spaces, let $E \subseteq X$, let $x_0 \in E$ be an accumulation point of E and let $g : E \to \mathbb{R}$. For $r > 0$ define*

$$\omega_g(r) := \sup \{ d_Y(g(x), g(x_0)) : x \in E,\ d_X(x, x_0) \le r \}.$$

Prove that g is continuous at x_0 if and only if

$$\lim_{r \to 0^+} \omega_g(r) = 0.$$

Theorem 220 (Cauchy) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. The function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

is continuous in $[a, b]$, differentiable in (a, b), and $h(a) = h(b) = f(a)g(b) - g(a)f(b)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a, b)$ such that $h'(c) = 0$, or, equivalently,

$$0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)).$$

■

Remark 221 *Why is $g(b) \neq g(a)$?*

Theorem 222 (De l'Hôpital) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and assume that $f(x_0) = g(x_0) = 0$ for some $x_0 \in [a,b]$. Assume that f and g are differentiable in $(a,b) \setminus \{x_0\}$ with $g(x), g'(x) \neq 0$ for all $x \in (a,b) \setminus \{x_0\}$. If there exists*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \in [-\infty, \infty],$$

then there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

Proof. Let $x \in (a,b) \setminus \{x_0\}$. Apply Cauchy's theorem in the interval of endpoints x and x_0 to find c_x between x and x_0 such that

$$\frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)}.$$

As $x \to x_0$, we have that $c_x \to x_0$, and so by hypothesis

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)} \to \ell.$$

It follows that there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

■

Exercise 223 *Calculate*

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Remark 224 *The converse of De l'Hôpital's theorem does not hold, that is, if there exists $\lim_{x \to x_0} \frac{f(x)}{g(x)}$, we cannot conclude that there exists the limit $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$. To see this, take*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $g(x) = x$. Then f and g are continuous and for $x \neq 0$, have that $f'(x) = 2x \sin \frac{1}{x} - x^2 \left(-\frac{1}{x^2}\right) \cos \frac{1}{x}$ and $g'(x) = 1$. Then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0,$$

since $0 \leq \left| x \sin \frac{1}{x} \right| \leq |x| \to 0$, but the limit

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2x \sin \frac{1}{x} + \cos \frac{1}{x}}{1}$$

does not exist (it oscillates between -1 and 1).

Exercise 225 *Let* $f : [a, b] \setminus \{x_0\} \to \mathbb{R}$ *and* $g : [a, b] \setminus \{x_0\} \to \mathbb{R}$ *be continuous in* $[a, b] \setminus \{x_0\}$ *and assume that*

$$\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = \infty$$

and that f *and* g *are differentiable in* $(a, b) \setminus \{x_0\}$ *with* $g(x), g'(x) \neq 0$ *for all* $x \in (a, b) \setminus \{x_0\}$*. Prove that if there exists*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \in [-\infty, \infty],$$

then there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

Exercise 226 *Prove that*

$$\lim_{x \to 0^+} x^a \log x,$$

where $a > 0$*.*

Exercise 227 *Calculate*

$$\lim_{x \to 0^+} x^{\sin x}.$$

Exercise 228 *Let* $f : [a, \infty) \to \mathbb{R}$ *and* $g : [a, \infty) \to \mathbb{R}$ *be continuous in* $[a, \infty)$ *and differentiable in* (a, ∞)*. Assume that*

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$$

and that $g(x), g'(x) \neq 0$ *for all* $x \in (a, \infty)$*. Prove that if there exists*

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \ell \in [-\infty, \infty],$$

then there exists

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \ell.$$

Exercise 229 *Prove that a similar result as in the previous exercise holds if* $\lim_{x \to \infty} |f(x)| = \lim_{x \to \infty} |g(x)| = \infty$*.*

Exercise 230 *Prove the following:*

$$\lim_{x \to \infty} \frac{\log^b x}{x^a} = 0,$$

where $a > 0$ *and* $b \in \mathbb{R}$*,*

$$\lim_{x \to \infty} \frac{x^a}{b^x} = 0,$$

where $a \in \mathbb{R}$ *and* $b > 1$*.*

Exercise 231 *Calculate*

$$\lim_{x\to\infty} \frac{\log x - x}{\log(1+e^x)}.$$

We begin by proving Taylor's formula in dimension $N = 1$.

Theorem 232 (Taylor's Formula) *Let $f \in C^{(m)}((a,b))$ and let $x_0 \in (a,b)$. Then for every $x \in (a,b)$,*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$
$$+ \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + R_m(x),$$

where the remainder $R_m(x, x_0)$ satisfies

$$\lim_{x\to x_0} \frac{R_m(x)}{(x - x_0)^m} = 0.$$

Lemma 233 *Let $g \in C^{(m)}((a,b))$ and let $x_0 \in (a,b)$. Then*

$$\lim_{x\to x_0} \frac{g(x)}{(x - x_0)^m} = 0 \qquad (26)$$

if and only if

$$g(x_0) = g'(x_0) = \cdots = g^{(m)}(x_0) = 0. \qquad (27)$$

Proof. Assume that (27) holds. By applying De l'Hôpital's theorem several times we get

$$\lim_{x\to x_0} \frac{g(x)}{(x - x_0)^m} = \lim_{x\to x_0} \frac{g'(x)}{m(x - x_0)^{m-1}} = \lim_{x\to x_0} \frac{g^{(2)}(x)}{m(m-1)(x - x_0)^{m-2}}$$
$$= \cdots = \lim_{x\to x_0} \frac{g^{(m-1)}(x)}{m!(x - x_0)} = \lim_{x\to x_0} \frac{g^{(m)}(x)}{m!1} = \frac{g^{(m)}(x_0)}{m!} = 0.$$

Conversely, assume (26). If $g^{(k)}(x_0) \neq 0$ for some $0 \le k < m$, then by what we just proved (with k in place of m)

$$\lim_{x\to x_0} \frac{g(x)}{(x - x_0)^k} = \frac{g^{(k)}(x_0)}{k!} \neq 0.$$

On the other hand,

$$\frac{g(x)}{(x - x_0)^k} = \frac{g(x)}{(x - x_0)^k} \frac{(x - x_0)^{m-k}}{(x - x_0)^{m-k}} = \frac{g(x)}{(x - x_0)^m}(x - x_0)^{m-k} \to 0$$

as $x \to x_0$, which is a contradiction. ■

We now turn to the proof of Theorem 232.

Proof of Theorem 232. Note that given a polynomial of degree m,

$$p(x) = a_0 + a_1 (x - x_0) + \cdots + a_m (x - x_0)^m = \sum_{i=0}^{m} a_i (x - x_0)^i,$$

we have that

$$p^{(k)}(x) = \sum_{i=k}^{m} i(i-1)\cdots(i-k+1)\, a_i (x - x_0)^{i-k},$$

so that

$$p^{(k)}(x_0) = k! a_k.$$

We apply the lemma to the function

$$g(x) := f(x) - p(x)$$

to conclude that

$$\lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^m} = 0$$

if and only if for all $k = 0, \ldots, m$,

$$0 = g^{(k)}(x_0) = f^{(k)}(x_0) - p^{(k)}(x_0) = f^{(k)}(x_0) - k! a_k,$$

that is

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Thus

$$g(x) = R_m(x) = f(x) - \left[f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \right].$$

■

Exercise 234 *Let $f : [a,b] \to \mathbb{R}$ be m-times differentiable in $[a,b]$ for some $m \in \mathbb{N}$ and let $x_0 \in (a,b)$. Prove that for every $x \in [a,b]$,*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + R_m(x),$$

where the remainder $R_m(x)$ satisfies

$$|R_m(x)| \le |x - x_0|^m \omega_{f^{(m)}}(|x - x_0|).$$

Wednesday, April 1, 2015

We are now ready to prove Taylor's formula in the general case.

Proof. Step 1: Since $\mathbf{x}_0 \in U$ and U is open, there exists $B(\mathbf{x}_0, r) \subseteq U$. Fix $\mathbf{x} \in B(\mathbf{x}_0, r)$, $\mathbf{x} \neq \mathbf{x}_0$ and let $\mathbf{v} := \frac{\mathbf{x}-\mathbf{x}_0}{\|\mathbf{x}-\mathbf{x}_0\|}$ and consider the function $g(t) := f(\mathbf{x}_0 + t\mathbf{v})$ defined for $t \in [0, r]$. By Theorem 200, we have that

$$\frac{dg}{dt}(t) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0 + t\mathbf{v})\, h_i = (\mathbf{v} \cdot \nabla) f(\mathbf{x}_0 + t\mathbf{v})$$

with for all $t \in [0, r]$. By repeated applications of Theorem 200, we get that

$$\frac{d^{(n)}g}{dt^n}(t) = (\mathbf{v} \cdot \nabla)^n f(\mathbf{x}_0 + t\mathbf{v})$$

for all $n = 1, \ldots, m$, where $(\mathbf{v} \cdot \nabla)^n$ means that we apply the operator

$$\mathbf{v} \cdot \nabla = h_1 \frac{\partial}{\partial x_1} + \cdots + h_N \frac{\partial}{\partial x_N}$$

n times to f. By the multinomial theorem, and the fact that for functions in C^m partial derivatives commute,

$$\begin{aligned}(\mathbf{v} \cdot \nabla)^n &= \left(v_1 \frac{\partial}{\partial x_1} + \cdots + v_N \frac{\partial}{\partial x_N}\right)^n \\ &= \sum_{\boldsymbol{\alpha} \text{ multi-index, } |\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{v}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}},\end{aligned}$$

and so

$$\frac{d^{(n)}g}{dt^n}(t) = \sum_{\boldsymbol{\alpha} \text{ multi-index, } |\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{v}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0 + t\mathbf{v}).$$

Using Taylor's formula for g, we get

$$g(t) = g(0) + \sum_{n=1}^{m} \frac{1}{n!} \frac{d^{(n)}g}{dt^n}(0)(t-0)^n + R_m(t),$$

where by Exercise 234,

$$|R_m(t)| \leq |t|^m \omega_{g^{(m)}}(t). \tag{28}$$

Substituting, we obtain

$$\begin{aligned}f(\mathbf{x}_0 + t\mathbf{v}) &= f(\mathbf{x}_0) + \sum_{n=1}^{m} \frac{t^n}{n!} \sum_{\boldsymbol{\alpha} \text{ multi-index, } |\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{v}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0) + R_m(t) \\ &= f(\mathbf{x}_0) + \sum_{\boldsymbol{\alpha} \text{ multi-index, } 1 \leq |\boldsymbol{\alpha}| \leq m} \frac{t^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \mathbf{v}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0) + R_m(t).\end{aligned}$$

Take $t=\|\mathbf{x}-\mathbf{x}_0\|$. Then

$$\mathbf{x}_0+t\mathbf{v}=\mathbf{x}_0+\|\mathbf{x}-\mathbf{x}_0\|\frac{\mathbf{x}-\mathbf{x}_0}{\|\mathbf{x}-\mathbf{x}_0\|}=\mathbf{x}$$

and

$$t^{|\boldsymbol{\alpha}|}\mathbf{v}^{\boldsymbol{\alpha}}=\|\mathbf{x}-\mathbf{x}_0\|^{|\boldsymbol{\alpha}|}\frac{(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}}{\|\mathbf{x}-\mathbf{x}_0\|^{|\boldsymbol{\alpha}|}}=(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}$$

and so

$$f(\mathbf{x})=f(\mathbf{x}_0)+\sum_{\boldsymbol{\alpha}\text{ multi-index, }1\le|\boldsymbol{\alpha}|\le m}\frac{(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)+R_m(\|\mathbf{x}-\mathbf{x}_0\|).$$

Moreover, by (28),

$$\frac{R_m(\|\mathbf{x}-\mathbf{x}_0\|)}{\|\mathbf{x}-\mathbf{x}_0\|^m}\le\omega_{g^{(m)}}(\|\mathbf{x}-\mathbf{x}_0\|).$$

Since $\|\mathbf{v}\|=1$, we have

$$\begin{aligned}\omega_{g^{(m)}}(\|\mathbf{x}-\mathbf{x}_0\|)&=\sup\left\{\left|\frac{d^{(n)}g}{dt^n}(t)-\frac{d^{(n)}g}{dt^n}(0)\right|:\ 0<t\le\|\mathbf{x}-\mathbf{x}_0\|\right\}\\&=\sup\left\{\left|\sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\mathbf{v}^{\boldsymbol{\alpha}}\left(\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+t\mathbf{v})-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right)\right|:\ 0<t\le\|\mathbf{x}-\mathbf{x}_0\|\right\}\\&\le\sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\sup\left\{\left|\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+t\mathbf{v})-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right|:\ 0<t\le\|\mathbf{x}-\mathbf{x}_0\|\right\}\\&\le\sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\sup\left\{\left|\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{y})-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right|:\ 0<\|\mathbf{y}\|\le\|\mathbf{x}-\mathbf{x}_0\|\right\}\\&=\sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\omega_{\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}}(\|\mathbf{x}-\mathbf{x}_0\|)\to 0\end{aligned}$$

as $\mathbf{x}\to\mathbf{x}_0$ by Exercise 219 and the fact that $\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}$ is continuous. This shows that $\lim_{\mathbf{x}\to\mathbf{x}_0}\frac{R_m(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|^m}=0$. ■

Exercise 235 *Calculate the limit*

$$\lim_{(x,y)\to(0,0)}\frac{\log\left(1+\sin^2(xy)\right)-x^2y^2}{(x^2+y^2)^4}.$$

14 Local Minima and Maxima

We recall that

Definition 236 *Let $f:E\to\mathbb{R}$, where $E\subseteq\mathbb{R}^N$, and let $\mathbf{x}_0\in E$. We say that* *(i) f attains a* local minimum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(ii) f attains a global minimum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$,*

(iii) f attains a local maximum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(iv) f attains a global maximum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$.*

Remark 237 *In view of Theorem 196, when looking for local minima and maxima, we have to search among the following:*

- *Interior points at which f is differentiable and $\nabla f(\mathbf{x}) = \mathbf{0}$, these are called* critical points. *Note that if $\nabla f(\mathbf{x}_0) = \mathbf{0}$, the function f may not attain a local minimum or maximum at $\mathbf{x}_0$. Indeed, consider the function $f(x) = x^3$. Then $f'(0) = 0$, but f is strictly increasing, and so f does not attain a local minimum or maximum at 0.*
- *Interior points at which f is not differentiable. The function $f(x) = |x|$ attains a global minimum at $x = 0$, but f is not differentiable at $x = 0$.*
- *Boundary points.*

To find sufficient conditions for a critical point to be a point of local minimum or local maximu, we study the second order derivatives of f.

Definition 238 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}^N$, and let $\mathbf{x}_0 \in E$. The Hessian matrix of f at $\mathbf{x}_0$ is the $N \times N$ matrix*

$$\begin{aligned} H_f(\mathbf{x}_0) &:= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_1}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_N}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_N^2}(\mathbf{x}_0) \end{pmatrix} \\ &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) \right)_{i,j=1}^N, \end{aligned}$$

whenever it is defined.

Remark 239 *If the hypotheses of Schwartz's theorem are satisfied for all $i, j = 1, \dots, N$, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0),$$

which means that the Hessian matrix $H_f(\mathbf{x}_0)$ is symmetric.

Given an $N \times N$ matrix H, the *characteristic polynomial* of H is the polynomial

$$p(t) := \det(tI_N - H), \quad t \in \mathbb{R}.$$

Theorem 240 *Let H be an $N \times N$ matrix. If H is symmetric, then all roots of the characteristic polynomial are real.*

Friday, April 3, 2015

Theorem 241 *Given a polynomial of the form*

$$p(t) = t^N + a_{N-1}t^{N-1} + a_{N-2}t^{N-2} + \cdots + a_1 t + a_0, \quad t \in \mathbb{R},$$

where the coefficients a_i are real for every $i = 0, \ldots, N-1$, assume that all roots of p are real. Then

(i) all roots of p are positive if and only if the coefficients alternate sign, that is, $a_{N-1} < 0$, $a_{N-2} > 0$, $a_{N-3} < 0$, etc.

(ii) all roots of p are negative if and only $a_i > 0$ for every $i = 0, \ldots, N-1$.

Remark 242 *Given an $N \times N$ matrix*

$$H = \begin{pmatrix} h_{1,1} & \cdots & h_{1,N} \\ \vdots & & \\ h_{N,1} & & h_{N,N} \end{pmatrix}$$

and a vector $\mathbf{x} \in \mathbb{R}^N$, we define $H\mathbf{x}$ as the vector of $\mathbb{R}^N$ of components

$$(h_{1,1}x_1 + \cdots + h_{1,N}x_N, \ldots, h_{N,1}x_1 + \cdots + h_{N,N}x_N).$$

We also define

$$(H\mathbf{x}, \mathbf{x}) := (H\mathbf{x}) \cdot \mathbf{x} = \sum_{j=1}^{N} \sum_{i=1}^{N} h_{i,j} x_i x_j.$$

If H is symmetric, then its eigenvalues $\lambda_1, \ldots, \lambda_N$ are real. Moreover, we can find corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_1$ which forms an orthonormal basis. Since

$$H\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

for every $i = 1, \ldots, N$, we get

$$(H\boldsymbol{v}_i, \boldsymbol{v}_i) = (\lambda_i \boldsymbol{v}_i) \cdot \boldsymbol{v}_i = \lambda_i \|\boldsymbol{v}_i\|^2 = \lambda_i 1. \tag{29}$$

Since $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_1\}$ forms a basis in $\mathbb{R}^N$, we can write

$$\mathbf{x} = c_1 \boldsymbol{v}_1 + \cdots + c_N \boldsymbol{v}_N.$$

Then

$$\begin{aligned}(H\mathbf{x}, \mathbf{x}) = (H\mathbf{x}) \cdot \mathbf{x} &= \left(H \sum_{j=1}^{N} c_j \boldsymbol{v}_j \right) \cdot \sum_{i=1}^{N} c_i \boldsymbol{v}_i \\ &= \left(\sum_{j=1}^{N} c_j H \boldsymbol{v}_j \right) \cdot \sum_{i=1}^{N} c_i \boldsymbol{v}_i = \sum_{j=1}^{N} \sum_{i=1}^{N} \lambda_j c_i c_j \boldsymbol{v}_i \cdot \boldsymbol{v}_j \\ &= \sum_{j=1}^{N} \lambda_j c_j^2,\end{aligned}$$

where we used the fact that $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 1$ if $i = j$ and 0 otherwise. In particular, if all the eigenvalues are positive, then letting $m := \min\{\lambda_1, \ldots, \lambda_N\}$, we have that

$$(H\mathbf{x}, \mathbf{x}) = \sum_{j=1}^{N} \lambda_j c_j^2 \geq m \sum_{j=1}^{N} c_j^2 = m \|\boldsymbol{x}\|^2 .$$

Remark 243 *If instead we want to use multiplication of matrices, we should identify $\mathbf{x}$ with the column vector or $N \times 1$ matrix*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Then $H\mathbf{x}$ could be considered as the product of matrices

$$\begin{pmatrix} h_{1,1} & \cdots & h_{1,N} \\ \vdots & & \\ h_{N,1} & & h_{N,N} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} h_{1,1}x_1 + \cdots + h_{1,N}x_N \\ \vdots \\ h_{N,1}x_1 + \cdots + h_{N,N}x_N \end{pmatrix}.$$

In turn we could define $(H\mathbf{x}, \mathbf{x})$ as

$$\begin{pmatrix} x_1 & \cdots & x_N \end{pmatrix} \begin{pmatrix} h_{1,1} & \cdots & h_{1,N} \\ \vdots & & \\ h_{N,1} & & h_{N,N} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

The next theorem gives necessary and sufficient conditions for a point to be of local minimum or maximum.

Theorem 244 *Let $U \subseteq \mathbb{R}^N$ be open, let $f : U \to \mathbb{R}$ be of class $C^2(U)$ and let $\mathbf{x}_0 \in U$ be a critical point of f.*

(i) If $H_f(\mathbf{x}_0)$ is positive definite, then f attains a local minimum at $\mathbf{x}_0$,

(ii) if f attains a local minimum at $\mathbf{x}_0$, then $H_f(\mathbf{x}_0)$ is positive semidefinite,

(iii) if $H_f(\mathbf{x}_0)$ is negative definite, then f attains a local maximum at $\mathbf{x}_0$,

(iv) if f attains a local maximum at $\mathbf{x}_0$, then $H_f(\mathbf{x}_0)$ is negative semidefinite.

Proof. (i) Assume that $H_f(\mathbf{x}_0)$ is positive definite. Then by Remark 242,

$$\sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0) v_i v_j \geq m \|\mathbf{v}\|^2 \tag{30}$$

for all $\mathbf{v} \in \mathbb{R}^N$ and for some $m > 0$.

We now apply Taylor's formula of order two to obtain

$$\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \sum_{|\boldsymbol{\alpha}|=2} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} (\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + R_2(\mathbf{x}) \\
&= f(\mathbf{x}_0) + 0 + \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i} (\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0)_i (\mathbf{x} - \mathbf{x}_0)_j + R_2(\mathbf{x}),
\end{aligned}$$

where we have used the fact that $\mathbf{x}_0$ is a critical point and where

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0.$$

Using the definition of limit with $\varepsilon = \frac{m}{2}$, we can find $\delta > 0$ such that

$$\left| \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right| \leq \frac{m}{2}$$

for all $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. Using this property and (30), we get

$$\begin{aligned}
f(\mathbf{x}) &\geq f(\mathbf{x}_0) + m \|\mathbf{x} - \mathbf{x}_0\|^2 + R_2(\mathbf{x}) = f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \left(m + \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right) \\
&\geq f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \left(m - \frac{m}{2} \right) = f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \frac{m}{2} > f(\mathbf{x}_0)
\end{aligned}$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. This shows that f attains a (strict) local minimum at $\mathbf{x}_0$.

(ii) Assume that if f attains a local minimum at $\mathbf{x}_0$. Then there exists $B(\mathbf{x}_0, r) \subseteq U$ such that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0)$$

for all $\mathbf{x} \in B(\mathbf{x}_0, r)$. Assume by contradiction that $H_f(\mathbf{x}_0)$ is not positive semidefinite. This means that there exists an eigenvalue $\lambda_i < 0$. Let $\boldsymbol{v}$ be an eigenvector of norm 1 for λ_i. Then

$$H_f(\mathbf{x}_0) \boldsymbol{v} = \lambda_i \boldsymbol{v}$$

Consider the function

$$g(t) := f(\mathbf{x}_0 + t\mathbf{v}_i), \quad t \in (-r, r).$$

Since $g(t) = f(\mathbf{x}_0 + t\mathbf{v}_i) \geq f(\mathbf{x}_0) = g(0)$ for all $t \in (-r, r)$, g attains a local minimum at $t = 0$. By Theorem 200, we have that for $t \in (-r, r)$,

$$\begin{aligned}
g'(t) &= \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} (\mathbf{x}_0 + t\mathbf{v}) v_i, \\
g''(t) &= \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i} (\mathbf{x}_0 + t\mathbf{v}) v_i v_j.
\end{aligned}$$

Then as in (29),

$$g''(0) = \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i} (\mathbf{x}_0) v_i v_j = \lambda_i < 0.$$

It follows by Taylor's formula for g that

$$\begin{aligned} g(t) &= g(0) + g'(0) + \frac{1}{2} g''(0) t^2 + o(t^2) \\ &= g(0) + 0 + \frac{\lambda_i}{2} t^2 + o(t^2) \\ &= g(0) + t^2 \left(\frac{\lambda_i}{2} + \frac{o(t^2)}{t^2} \right). \end{aligned}$$

As in part (i), taking $\varepsilon = -\frac{\lambda_i}{4} > 0$, we have that

$$\begin{aligned} g(t) &= g(0) + t^2 \left(\frac{\lambda_i}{2} + \frac{o(t^2)}{t^2} \right) \\ &\le g(0) + t^2 \left(\frac{\lambda_i}{2} - \frac{\lambda_i}{4} \right) \\ &= g(0) + t^2 \frac{\lambda_i}{4} < g(0), \end{aligned}$$

since $\lambda_i < 0$. This contradics the fact that g has a local minimum at $t = 0$. ■

Remark 245 *Note that in view of the previous theorem, if at a critical point $\mathbf{x}_0$ the characteristic polynomial of $H_f(\mathbf{x}_0)$ has one positive root and one negative root, then f does not admit a local minimum or a local maximum at $\mathbf{x}_0$.*

Monday, April 6, 2015

15 Implicit and Inverse Function

Definition 246 *Given an open set $U \subseteq \mathbb{R}^N$ and a function $\mathbf{f} : U \to \mathbb{R}^M$, we say that $\mathbf{f}$ is of class C^m for some nonnegative integer $m \in \mathbb{N}_0$, if all its components f_i, $i = 1, \dots, M$, are of class C^m. The space of all functions $\mathbf{f} : U \to \mathbb{R}^M$ of class C^m is denoted $C^m(U; \mathbb{R}^M)$. We set $C^\infty(U; \mathbb{R}^M) := \bigcap_{m=0}^{\infty} C^m(U; \mathbb{R}^M)$.*

Given a function f of two variables $(x, y) \in \mathbb{R}^2$, consider the equation

$$f(x, y) = 0.$$

We want to solve for y, that is, we are interested in finding a function $y = g(x)$ such that

$$f(x, g(x)) = 0.$$

We will see under which conditions we can do this. The result is going to be local.

In what follows given $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^M$ and $\mathbf{f}(\mathbf{x}, \mathbf{y})$, we write

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_1}{\partial x_N}(\mathbf{x}, \mathbf{y}) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_M}{\partial x_N}(\mathbf{x}, \mathbf{y}) \end{pmatrix}$$

and

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_1}{\partial y_M}(\mathbf{x}, \mathbf{y}) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial y_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_M}{\partial y_M}(\mathbf{x}, \mathbf{y}) \end{pmatrix}.$$

Theorem 247 (Implicit Function) *Let $U \subseteq \mathbb{R}^N \times \mathbb{R}^M$ be open, let $\mathbf{f} : U \to \mathbb{R}^M$, and let $(\mathbf{a}, \mathbf{b}) \in U$. Assume that $\mathbf{f} \in C^m(U)$ for some $m \in \mathbb{N}$, that*

$$\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad \textit{and} \quad \det \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b}) \neq 0.$$

Then there exist $B_N(\mathbf{a}, r_0) \subset \mathbb{R}^N$ and $B_M(\mathbf{b}, r_1) \subset \mathbb{R}^M$, with $B_N(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1) \subseteq U$, and a unique function

$$\mathbf{g} : B_N(\mathbf{a}, r_0) \to B_M(\mathbf{b}, r_1)$$

of class C^m such that $\mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ for all $\boldsymbol{x} \in B_N(\mathbf{a}, r_0)$ and $\mathbf{g}(\mathbf{a}) = \mathbf{b}$.

Proof. We present a proof in the case $N = M = 1$.
Step 1: Existence of g. Since $\frac{\partial f}{\partial y}(a, b) \neq 0$, without loss of generality, we can assume that $\frac{\partial f}{\partial y}(a, b) > 0$ (the case $\frac{\partial f}{\partial y}(a, b) < 0$ is similar). Using the fact that $\frac{\partial f}{\partial y}$ is continuous at (a, b), we can find $r > 0$ such that

$$R := [a - r, a + r] \times [b - r, b + r] \subseteq U$$

and

$$\frac{\partial f}{\partial y}(x, y) > 0 \quad \text{for all } (x, y) \in R.$$

Consider the function $h(y) := f(a, y)$, $y \in [b - r, b + r]$. Since

$$h'(y) = \frac{\partial f}{\partial y}(a, y) > 0 \quad \text{for all } y \in [b - r, b + r],$$

we have that h is strictly increasing. Using the fact that $h(b) = f(a, b) = 0$, it follows that

$$0 > h(b - r) = f(a, b - r), \quad 0 < h(b + r) = f(a, b + r).$$

Consider the function $k_1(x) := f(x, b-r)$, $x \in [a-r, a+r]$. Since $k_1(a) < 0$ and k_1 is continuous at a, there exists $0 < \delta_1 < r$ such that

$$0 > k_1(x) = f(x, b-r) \quad \text{for all } x \in (a-\delta_1, a+\delta_1).$$

Similarly, consider the function $k_2(x) := f(x, b+r)$, $x \in [a-r, a+r]$. Since $k_2(a) > 0$ and k_2 is continuous at a, there exists $0 < \delta_2 < r$ such that

$$0 < k_2(x) = f(x, b+r) \quad \text{for all } x \in (a-\delta_2, a+\delta_2).$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then for all $x \in (a-\delta, a+\delta)$,

$$f(x, b-r) < 0, \quad f(x, b+r) > 0.$$

Fix $x \in (a-\delta, a+\delta)$ and consider the function $k(y) := f(x, y)$, $y \in [b-r, b+r]$. Since

$$k'(y) = \frac{\partial f}{\partial y}(x, y) > 0 \quad \text{for all } y \in [b-r, b+r],$$

we have that k is strictly increasing. Using the fact that $k(b-r) = f(x, b-r) < 0$ and $k(b+r) = f(x, b+r) > 0$, it follows that there exists a unique $y \in (b-r, b+r)$ (depending on x) such that $0 = k(y) = f(x, y)$.

Thus, we have shown that for every $x \in (a-\delta, a+\delta)$ there exists a unique $y \in (b-r, b+r)$ depending on x such that $f(x, y) = 0$. We define $g(x) := y$.

Step 2: Continuity of g. Fix $x_0 \in (a-\delta, a+\delta)$. Note that $b-r < g(x_0) < b+r$. Let $\varepsilon > 0$ be so small that

$$b-r < g(x_0) - \varepsilon < g(x_0) < g(x_0) + \varepsilon < b+r.$$

Consider the function $j(y) := f(x_0, y)$, $y \in [b-r, b+r]$. Since

$$j'(y) = \frac{\partial f}{\partial y}(x_0, y) > 0 \quad \text{for all } y \in [b-r, b+r],$$

we have that j is strictly increasing. Using the fact that $j(g(x_0)) = f(x_0, g(x_0)) = 0$, it follows that

$$f(x_0, g(x_0) - \varepsilon) < 0, \quad f(x_0, g(x_0) + \varepsilon) > 0.$$

Consider the function $j_1(x) := f(x, g(x_0) - \varepsilon)$, $x \in (a-\delta, a+\delta)$. Since $j_1(x_0) < 0$ and j_1 is continuous at x_0, there exists $0 < \eta_1 < \delta$ such that

$$0 > j_1(x) = f(x, g(x_0) - \varepsilon) \quad \text{for all } x \in (x_0 - \eta_1, x_0 + \eta_1).$$

Similarly, consider the function $j_2(x) := f(x, g(x_0) + \varepsilon)$, $x \in (a-\delta, a+\delta)$. Since $j_2(x_0) > 0$ and j_2 is continuous at x_0, there exists $0 < \eta_2 < \delta$ such that

$$0 < j_2(x) = f(x, g(x_0) + \varepsilon) \quad \text{for all } x \in (x_0 - \eta_2, x_0 + \eta_2).$$

Let $\eta := \min\{\eta_1, \eta_2\}$. Then for all $x \in (a-\eta, a+\eta)$,

$$f(x, g(x_0) - \varepsilon) < 0, \quad f(x, g(x_0) + \varepsilon) > 0.$$

But $f(x, g(x)) = 0$ and $y \in [b-r, b+r] \mapsto f(x, y)$ is strictly increasing. It follows that

$$g(x_0) - \varepsilon < g(x) < g(x_0) + \varepsilon$$

and so g is continuous at x_0. ■

Wednesday, April 8, 2015

Proof. Step 3: Differentiability of g. Fix $x_0 \in (a-\delta, a+\delta)$. Consider the segment joining $(x, g(x))$ and $(x_0, g(x_0))$,

$$S = \{t(x, g(x)) + (1-t)(x_0, g(x_0)) : t \in [0,1]\}.$$

By the mean value theorem there exists $\theta \in (0,1)$ such that

$$0 = f(x, g(x)) - f(x_0, g(x_0)) = \frac{\partial f}{\partial x}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))(x - x_0)$$
$$+ \frac{\partial f}{\partial y}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))(g(x) - g(x_0)).$$

Hence,

$$\frac{g(x) - g(x_0)}{x - x_0} = -\frac{\frac{\partial f}{\partial x}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))}{\frac{\partial f}{\partial y}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))}$$

letting $x \to x_0$ and using the continuity of g and of $\frac{\partial f}{\partial x}$ and of $\frac{\partial f}{\partial y}$, we get

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = -\frac{\frac{\partial f}{\partial x}(x_0, g(x_0))}{\frac{\partial f}{\partial y}(x_0, g(x_0))}$$

and so

$$g'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0, g(x_0))}{\frac{\partial f}{\partial y}(x_0, g(x_0))}.$$

Since the right-hand side is continuous, it follows that g' is continuous. Thus g is of class C^1. ■

The next examples show that when $\det \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, then anything can happen.

Example 248 *In all these examples $N = M = 1$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$.*

(i) Consider the function

$$f(x, y) := (y - x)^2.$$

Then $f(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$ and $g(x) = x$ satisfies $f(x, g(x)) = 0$.

(ii) Consider the function

$$f(x, y) := x^2 + y^2.$$

Then $f(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$ but there is no function g defined near $x = 0$ such that $f(x, g(x)) = 0$.

(iii) Consider the function

$$f(x,y) := (xy-1)\left(x^2+y^2\right).$$

Then $f(0,0)=0$, $\frac{\partial f}{\partial y}(0,0)=0$ *but*

$$g(x) = \begin{cases} 0 & \text{if } x=0, \\ \frac{1}{x} & \text{if } x \neq 0, \end{cases}$$

which is discontinuous.

Next we prove the inverse function theorem.

Theorem 249 (Inverse Function) *Let* $U \subseteq \mathbb{R}^N$ *be open, let* $\mathbf{f} : U \to \mathbb{R}^N$*, and let* $\mathbf{a} \in U$*. Assume that* $\mathbf{f} \in C^m(U)$ *for some* $m \in \mathbb{N}$ *and that*

$$\det J_{\mathbf{f}}(\mathbf{a}) \neq 0.$$

Then there exists $B(\mathbf{a}, r_0) \subseteq U$ *such that* $\mathbf{f}(B(\mathbf{a}, r_0))$ *is open, the function*

$$\mathbf{f} : B(\mathbf{a}, r_0) \to \mathbf{f}(B(\mathbf{a}, r_0))$$

is invertible and $\mathbf{f}^{-1} \in C^m(\mathbf{f}(B(\mathbf{a}, r_0)))$*. Moreover,*

$$J_{\mathbf{f}^{-1}}(\mathbf{y}) = \left(J_{\mathbf{f}}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right)^{-1}.$$

Proof. We apply the implicit function theorem to the function $h : U \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$h(\mathbf{x}, \mathbf{y}) := \mathbf{f}(\mathbf{x}) - \mathbf{y}.$$

Let $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then $h(\mathbf{a}, \mathbf{b}) = 0$ and

$$\det \frac{\partial h}{\partial \mathbf{x}}(\mathbf{a}, \mathbf{b}) = \det J_{\mathbf{f}}(\mathbf{a}) \neq 0.$$

Hence, by the implicit function theorem there exists $B(\mathbf{a}, r_0) \subset \mathbb{R}^N$ and $B(\mathbf{b}, r_1) \subset \mathbb{R}^N$ such that $B(\mathbf{a}, r_0) \times B(\mathbf{b}, r_1) \subseteq U \times \mathbb{R}^N$ and a function $\mathbf{g} : B(\mathbf{b}, r_1) \to B(\mathbf{a}, r_0)$ of class C^m such that $h(\mathbf{g}(\mathbf{y}), \mathbf{y}) = 0$ for all $\mathbf{y} \in B(\mathbf{b}, r_1)$, that is,

$$\mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}$$

for all $\mathbf{y} \in B(\mathbf{b}, r_1)$. This implies that $\mathbf{g} = \mathbf{f}^{-1}$. Moreover,

$$\frac{\partial \mathbf{g}}{\partial y_k}(\mathbf{y}) = -\left(\frac{\partial h}{\partial \mathbf{x}}(\mathbf{g}(\mathbf{y}), \mathbf{y})\right)^{-1} \frac{\partial h}{\partial y_k}(\mathbf{g}(\mathbf{y}), \mathbf{y}),$$

that is,

$$\frac{\partial \mathbf{f}^{-1}}{\partial y_k}(\mathbf{y}) = \left(J_{\mathbf{f}}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right)^{-1} e_k.$$

■

The next exercise shows that differentiability is not enough for the inverse function theorem.

Exercise 250 *Consider the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$f_1(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \end{cases}$$
$$f_2(x,y) = y.$$

Prove that $\mathbf{f} = (f_1, f_2)$ *is differentiable in* $(0,0)$ *and* $J_{\mathbf{f}}(0,0) = 1$. *Prove that* $\mathbf{f}$ *is not one-to-one in any neighborhood of* $(0,0)$.

The next exercise shows that the existence of a local inverse at every point does not imply the existence of a global inverse.

Exercise 251 *Consider the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x,y) = (e^x \cos y, e^x \sin y).$$

Prove that $\det J_{\mathbf{f}}(x,y) \neq 0$ *for all* $(x,y) \in \mathbb{R}^2$ *but that* $\mathbf{f}$ *is not injective.*

Friday, April 10, 2015

16 Lagrange Multipliers

In Section 14 (see Theorem 244) we have seen how to find points of local minima and maxima of a function $f : E \to \mathbb{R}$ in the interior E° of E. Now we are ready to find points of local minima and maxima of a function $f : E \to \mathbb{R}$ on the boundary ∂E of E. We assume that the boundary of E has a special form, that is, it is given by a set of the form

$$\left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{g}(\mathbf{x}) = \mathbf{0}\right\}.$$

Definition 252 *Let* $f : E \to \mathbb{R}$, *where* $E \subseteq \mathbb{R}^N$, *let* $F \subseteq E$ *and let* $\mathbf{x}_0 \in F$. *We say that*

(i) f *attains a* constrained local minimum *at* $\mathbf{x}_0$ *if there exists* $r > 0$ *such that* $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ *for all* $\mathbf{x} \in F \cap B(\mathbf{x}_0, r)$,

(ii) f *attains a* constrained local maximum *at* $\mathbf{x}_0$ *if there exists* $r > 0$ *such that* $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ *for all* $\mathbf{x} \in F \cap B(\mathbf{x}_0, r)$.

The set F is called the *constraint*.

Theorem 253 (Lagrange Multipliers) *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $f : U \to \mathbb{R}$ *be a function of class* C^1 *and let* $\mathbf{g} : U \to \mathbb{R}^M$ *be a class of function* C^1, *where* $M < N$, *and let*

$$F := \{\mathbf{x} \in U : \mathbf{g}(\mathbf{x}) = \mathbf{0}\}.$$

Let $\mathbf{x}_0 \in F$ *and assume that* f *attains a constrained local minimum (or maximum) at* $\mathbf{x}_0$. *If* $J_{\mathbf{g}}(\mathbf{x}_0)$ *has maximum rank* M, *then there exist* $\lambda_1, \ldots, \lambda_M \in \mathbb{R}$ *such that*

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_M \nabla g_M(\mathbf{x}_0).$$

Proof. Assume that f attains a constrained local minimum at $\mathbf{x}_0$ (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in U \cap B(\mathbf{x}_0, r)$ such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. By taking $r > 0$ smaller, and since U is open, we can assume that $B(\mathbf{x}_0, r) \subseteq U$ so that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) \text{ for all } \mathbf{x} \in B(\mathbf{x}_0, r) \text{ with } \mathbf{g}(\mathbf{x}) = \mathbf{0}. \tag{31}$$

Since $J_{\mathbf{g}}(\mathbf{x}_0)$ has maximum rank M, there exists a $M \times M$ submatrix which has determinant different from zero. By relabeling the coordinates, if necessary, we may assume that $\mathbf{x} = (\mathbf{z}, \mathbf{y}) \in \mathbb{R}^{N-M} \times \mathbb{R}^M$, $\mathbf{x}_0 = (\mathbf{a}, \mathbf{b})$ and

$$\det \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b}) \neq 0.$$

By the implicit function theorem applied to the function $\mathbf{g} : B(\mathbf{x}_0, r) \to \mathbb{R}^M$ there exist $r_0 > 0$, $r_1 > 0$, and a function $\mathbf{h} : B_{N-M}(\mathbf{a}, r_0) \to B_M(\mathbf{b}, r_1)$ of class C^1 such that $B_{N-M}(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1) \subset B(\mathbf{x}_0, r)$, $\mathbf{h}(\mathbf{a}) = \mathbf{b}$, and

$$\mathbf{g}(\mathbf{z}, \mathbf{h}(\mathbf{z})) = \mathbf{0} \quad \text{for all } \mathbf{z} \in B_{N-M}(\mathbf{a}, r_0).$$

Consider the function $\mathbf{k} : B_{N-M}(\mathbf{a}, r_0) \to B_{N-M}(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1)$ defined by

$$\mathbf{k}(\mathbf{z}) := (\mathbf{z}, \mathbf{h}(\mathbf{z})).$$

Then

$$J_{\mathbf{k}}(\mathbf{a}) = \begin{pmatrix} I_{N-M} \\ \nabla h_1(\mathbf{a}) \\ \vdots \\ \nabla h_M(\mathbf{a}) \end{pmatrix},$$

which has rank $N - M$. Moreover, since $\mathbf{g}(\mathbf{k}(\mathbf{z})) = \mathbf{0}$ for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$, by Theorem 200,

$$\begin{aligned} \mathbf{0} &= J_{\mathbf{g}}(\mathbf{x}_0) J_{\mathbf{k}}(\mathbf{a}) \\ &= \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_1}{\partial x_N}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial g_M}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} \frac{\partial k_1}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_1}{\partial z_{N-M}}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial k_N}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_{N-M}}(\mathbf{a}) \end{pmatrix}. \end{aligned}$$

Considering the transpose of this expression, we get

$$\begin{aligned} \mathbf{0} &= (J_{\mathbf{k}}(\mathbf{a}))^T (J_{\mathbf{g}}(\mathbf{x}_0))^T \\ &= \begin{pmatrix} \frac{\partial k_1}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_1}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial k_1}{\partial z_{N-M}}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_{N-M}}(\mathbf{a}) \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_1}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial g_1}{\partial x_N}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix}, \end{aligned}$$

which implies that the vectors $\nabla g_i(\mathbf{x}_0)$, $i = 1, \ldots, M$, belong to the kernel of the linear transformation $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^M$ defined by

$$\mathbf{T}(\mathbf{x}) := (J_{\mathbf{k}}(\mathbf{a}))^T \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^N.$$

Hence,

$$V := \operatorname{span}\{\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_M(\mathbf{x}_0)\} \subseteq \ker \mathbf{T}.$$

But $\dim V = \operatorname{rank} J_{\mathbf{g}}(\mathbf{x}_0) = M = N - \operatorname{rank}(J_{\mathbf{k}}(\mathbf{a}))^T = \dim\ker \mathbf{T}$. Hence,

$$V = \ker \mathbf{T}.$$

Consider the function

$$p(\mathbf{z}) := f(\mathbf{k}(\mathbf{z})), \quad \mathbf{z} \in B_{N-M}(\mathbf{a}, r_0).$$

Since $\mathbf{g}(\mathbf{k}(\mathbf{z})) = \mathbf{0}$ for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$, it follows from (31) that

$$p(\mathbf{z}) = f(\mathbf{k}(\mathbf{z})) \geq f(\mathbf{x}_0) = f(\mathbf{k}(\mathbf{a})) = p(\mathbf{a})$$

for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$. Hence, the function p attains a local minimum at $\mathbf{a}$. It follows from Theorems 196 and 200 that

$$\mathbf{0} = J_p(\mathbf{a}) = \nabla f(\mathbf{x}_0) J_{\mathbf{k}}(\mathbf{a}).$$

Considering the transpose of this expression, we get

$$(J_{\mathbf{k}}(\mathbf{a}))^T (\nabla f(\mathbf{x}_0))^T = \mathbf{0},$$

which implies that the vector $\nabla f(\mathbf{x}_0)$, $i = 1, \ldots, M$ belong to the kernel of T, which coincides with V. It follows from the definition of V that there exist λ_1, $\ldots$, $\lambda_M \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_m \nabla g_M(\mathbf{x}_0).$$

■

Monday, April 13, 2015

Example 254 *Among all rectangles inscribed in the the unit circle in $\mathbb{R}^2$ find the one with maximal area. By rotation we can consider rectangles with sides parallel to the axes. Let (x, y) be the vertex on the first quadrant, so that $x^2 + y^2 = 1$, $x > 0$, $y > 0$. Then we want to find the maximum of the function*

$$f(x, y) = 4xy$$

in the open set $U = \{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y > 0\}$ subject to the constraint $x^2 + y^2 = 1$. Let's take $g(x, y) = x^2 + y^2 - 1$. We are looking for a solution of the following system

$$\nabla f = \lambda \nabla g,$$

subject to $g = 0$, that is

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) - \lambda \frac{\partial g}{\partial x}(x,y) = 0, \\ \frac{\partial f}{\partial y}(x,y) - \lambda \frac{\partial g}{\partial y}(x,y) = 0, \\ g(x,y) = 0, \end{cases}$$

that is,

$$\begin{cases} 4y - \lambda 2x = 0, \\ 4x - \lambda 2y = 0, \\ x^2 + y^2 - 1 = 0. \end{cases}$$

We get

$$\begin{cases} 2y = \lambda x, \\ 2x = \lambda y, \\ x^2 + y^2 - 1 = 0. \end{cases}$$

so multiplying the first two equations we get $4xy = \lambda^2 xy$, and since $x > 0$, $y > 0$, we have that $4 = \lambda^2$, so $\lambda = \pm 2$. If $\lambda = 2$ we get $x = y$, so $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, while if $\lambda = -2$ we get $x = -y$, which is not allowed since we are in U.

But how do we know $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is a solution? The set U is open.

Consider the set $K = \{(x,y) : x \geq 0, y \geq 0, x^2 + y^2 = 1\}$. This set is closed and bounded, and so K is compact. By the Weierstrass theorem the function g has a lobal minimum and a global maximum in K. If either $x = 0$ or $y = 0$, we get $g = 0$, which gives a global minimum. Thus the global maximum must be in U, and so it must be at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Could we solve this problem without Lagrange multipliers? Use polar coordinates to write $(x,y) = (\cos\theta, \sin\theta)$. Then we are looking for the maximum of the function

$$h(\theta) = 4\cos\theta\sin\theta = 2\sin(2\theta)$$

in $(0, \frac{\pi}{2})$.

Example 255 *Given a point $\boldsymbol{x}_0 \in \mathbb{R}^N$, find*

$$\operatorname{dist}(\boldsymbol{x}_0, S_{N-1}),$$

where $S_{N-1} := \partial B(\mathbf{0}, 1)$ is the unit sphere in $\mathbb{R}^N$. Note that

$$\begin{aligned} \operatorname{dist}(\boldsymbol{x}_0, S_{N-1}) &= \inf\{\|\boldsymbol{x}_0 - \boldsymbol{x}\| : \boldsymbol{x} \in S_{N-1}\} \\ &= \inf\{\|\boldsymbol{x}_0 - \boldsymbol{x}\| : \|\boldsymbol{x}\| = 1\}. \end{aligned}$$

To simplify our life, we can square everything, so we are looking for the minimum of the function

$$f(\boldsymbol{x}) = \|\boldsymbol{x}_0 - \boldsymbol{x}\|^2$$

subject to the constraint $\|\boldsymbol{x}\|^2 = 1$. Take $g(\boldsymbol{x}) := \|\boldsymbol{x}\|^2 - 1$. Since S_{N-1} is closed and bounded and f is countinuous, by the Weierstrass theorem f has a global *minimum and a global maximum in S_{N-1}. Thus, we can apply the theorem on Lagrange multipliers. We are looking for a solution of the following system*

$$\nabla f = \lambda \nabla g,$$

subject to $g = 0$, that is

$$\begin{cases} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) - \lambda \frac{\partial g}{\partial x_i}(\boldsymbol{x}) = 0, \quad i = 1, \ldots, N, \\ g(\boldsymbol{x}) = 0. \end{cases}$$

We have

$$\begin{cases} 2(x_i - x_{0,i}) - 2\lambda x_i = 0, \quad i = 1, \ldots, N, \\ \|\boldsymbol{x}\|^2 = 1, \end{cases}$$

that is,

$$\begin{cases} (1-\lambda) x_i = x_{0,i}, \quad i = 1, \ldots, N, \\ \|\boldsymbol{x}\|^2 = 1, \end{cases} \quad \Leftrightarrow \quad \begin{cases} (1-\lambda)\boldsymbol{x} = \boldsymbol{x}_0, \\ \|\boldsymbol{x}\|^2 = 1. \end{cases}$$

If $\boldsymbol{x}_0 = \mathbf{0}$, then $\lambda = 1$, and so every point on the sphere is at maximum distance from $\mathbf{0}$. If $\boldsymbol{x}_0 \neq \mathbf{0}$, then $\lambda \neq 1$, and so $\boldsymbol{x} = \frac{1}{1-\lambda}\boldsymbol{x}_0$. Plugging this into $\|\boldsymbol{x}\|^2 = 1$, we get

$$1 = \frac{1}{(1-\lambda)^2}\|\boldsymbol{x}_0\|^2 \quad \Leftrightarrow \quad (1-\lambda)^2 = \|\boldsymbol{x}_0\|^2$$

which gives $\lambda = 1 \pm \|\boldsymbol{x}_0\|$, and in turn

$$\boldsymbol{x} = \pm \frac{1}{\|\boldsymbol{x}_0\|}\boldsymbol{x}_0.$$

The closest point to $\boldsymbol{x}_0$ will be $\frac{1}{\|\boldsymbol{x}_0\|}\boldsymbol{x}_0$ and the furthest $-\frac{1}{\|\boldsymbol{x}_0\|}\boldsymbol{x}_0$, as we expected.

Example 256 (Eigenvalues of a symmetric matrix) *Consider an $N \times N$ matrix*

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,N} \\ \vdots & & \vdots \\ a_{N,1} & \cdots & a_{N,N} \end{pmatrix}$$

and let

$$f(\boldsymbol{x}) := (A\mathbf{x}, \mathbf{x}) = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{j,k} x_j x_k.$$

Let's find the points of minimum and maximum of f over the unit sphere $S_{N-1} := \partial B(\mathbf{0}, 1)$. Since S_{N-1} is closed and bounded and f is countinuous, by the Weierstrass theorem f has a global minimum and a global maximum in S_{N-1}. Let

$$f(\mathbf{v}_1) = \max\{f(\mathbf{x}) : \|\mathbf{x}\| = 1\}.$$

Take $g(\boldsymbol{x}) := \|\boldsymbol{x}\|^2 - 1$. Thus, we can apply the theorem on Lagrange multipliers. We are looking for a solution of the following system

$$\nabla f = \lambda \nabla g,$$

subject to $g = 0$, that is

$$\begin{cases} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) - \lambda \frac{\partial g}{\partial x_i}(\boldsymbol{x}) = 0, \quad i = 1, \dots, N, \\ g(\boldsymbol{x}) = 0. \end{cases}$$

Synce A is symmetric, we have

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) &= \sum_{j=1}^{N} \sum_{k=1}^{N} a_{j,k} \delta_{i,j} x_k + \sum_{j=1}^{N} \sum_{k=1}^{N} a_{j,k} x_j \delta_{i,k} \\ &= \sum_{k=1}^{N} a_{i,k} x_k + \sum_{j=1}^{N} a_{j,i} x_j = 2 \sum_{k=1}^{N} a_{i,k} x_k \end{aligned}$$

and so we get

$$\begin{cases} 2 \sum_{k=1}^{N} a_{i,k} x_k - \lambda 2 x_i = 0, \quad i = 1, \dots, N, \\ \|\boldsymbol{x}\|^2 - 1 = 0, \end{cases}$$

that is

$$\begin{cases} A\boldsymbol{x} = \lambda \boldsymbol{x}, \\ \|\boldsymbol{x}\|^2 - 1 = 0. \end{cases}$$

Since $\mathbf{v}_1$ is a point of constrained maximum, by the Lagrange multipliers theorem there exist $\mu_1 \in \mathbb{R}$ such that

$$A\boldsymbol{v}_1 = \mu_1 \boldsymbol{v}_1,$$

which shows that A has one real eigenvalue with eigenvector $\boldsymbol{v}_1$. Moreover, since $\|\boldsymbol{v}_1\|^2 = 1$, we have

$$f(\mathbf{v}_1) = (A\boldsymbol{v}_1, \mathbf{v}_1) = \mu_1 \boldsymbol{v}_1 \cdot \mathbf{v}_1 = \mu_1,$$

and so

$$\mu_1 = f(\mathbf{v}_1) = \max \{(A\mathbf{x}, \mathbf{x}) : \|\mathbf{x}\| = 1\}.$$

Wednesday, April 15, 2015

Example 257 (Eigenvalues of a symmetric matrix, continued) *Next assume by induction that n eigenvalues and n orthonormal eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ have been been found, with $n < N-1$. Let's find the minimum and maximum of f over the set*

$$K_n := \{\boldsymbol{x} \in S_{N-1} : \ \boldsymbol{x} \cdot \boldsymbol{v}_1 = \dots = \boldsymbol{x} \cdot \boldsymbol{v}_n = 0\}.$$

The set K_n is closed and bounded, and so again by the Weierstrass theorem f has a global minimum and a global maximum in K_n. Let $\mathbf{v}_{n+1} \in K_n$ be such that

$$f(\mathbf{v}_{n+1}) = \max \{f(\mathbf{x}) : K_n\}.$$

Take $g_{n+1}(\boldsymbol{x}) := \|\boldsymbol{x}\|^2 - 1$ *and* $g_i(\boldsymbol{x}) := 2\boldsymbol{x} \cdot \boldsymbol{v}_i$, $i = 1, \ldots, n$. *Then we can apply the theorem on Lagrange multipliers. We are looking for a solution of the following system*

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_{n+1} \nabla g_{n+1},$$

subject to $g_1 = 0, \ldots,$ *and* $g_2 = 0$ *We will get*

$$\begin{cases} A\boldsymbol{x} = \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n + \lambda_{n+1} \boldsymbol{x} \\ \boldsymbol{x} \cdot \boldsymbol{v}_1 = \cdots = \boldsymbol{x} \cdot \boldsymbol{v}_n = 0, \\ \|\boldsymbol{x}\|^2 - 1 = 0, \end{cases}$$

Since $\mathbf{v}_{n+1}$ *is a point of constrained maximum, by the Lagrange multipliers theorem there exist* $\lambda_1, \lambda_{n+1} \in \mathbb{R}$ *such that*

$$A\boldsymbol{v}_{n+1} = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_1 + \cdots + \lambda_{n+1} \boldsymbol{v}_{n+1}.$$

Taking the inner product with $\mathbf{v}_i$ *gives*

$$\begin{aligned} 0 + \cdots + \lambda_i 1 + \cdots + 0 &= \lambda_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_i + \cdots + \lambda_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i + \cdots + \lambda_{n+1} \boldsymbol{v}_{n+1} \cdot \boldsymbol{v}_i \\ &= (A\mathbf{v}_{n+1}) \cdot \mathbf{v}_i \\ &= \mathbf{v}_{n+1} \cdot (A\mathbf{v}_i) = \mathbf{v}_{n+1} \cdot (\mu_i \mathbf{v}_i) = \mu_i (\mathbf{v}_{n+1} \cdot \mathbf{v}_i) = 0, \end{aligned}$$

and so $\lambda_i = 0$ *and* $A\boldsymbol{v}_{n+1} = \lambda_{n+1} \boldsymbol{v}_{n+1}$.*Thus* $\boldsymbol{v}_{n+1}$ *is a second eigenvector. Moreover, since* $\|\boldsymbol{v}_{n+1}\|^2 = 1$, *we have*

$$f(\mathbf{v}_{n+1}) = (A\boldsymbol{v}_{n+1}, \mathbf{v}_{n+1}) = \mu_{n+1} \boldsymbol{v}_{n+1} \cdot \mathbf{v}_{n+1} = \mu_{n+1},$$

and so

$$\mu_{n+1} = f(\mathbf{v}_{n+1}) = \max \{(A\mathbf{x}, \mathbf{x}) : \|\mathbf{x}\| = 1, \ \boldsymbol{x} \cdot \boldsymbol{v}_1 = \cdots = 0\}.$$

Inductively, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *are orthonormal eigenvectors, with* $1 \leq n < N$, *let find the minimum and maximum of* f *over the set*

$$K_n := \{\boldsymbol{x} \in S_{N-1} : \ \boldsymbol{x} \cdot \boldsymbol{v}_1 = \cdots = \boldsymbol{x} \cdot \boldsymbol{v}_n = 0\}.$$

In this way we can find $N - 1$ *eigenvectors. To find the last one, we consider a unit vector* $\boldsymbol{v}_N$ *which is orthogonal to all other vectors* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$. *Reasoning as before, taking the inner product with* $\mathbf{v}_i$ *gives*

$$A\mathbf{v}_N \cdot \mathbf{v}_i = \mathbf{v}_N \cdot (A\mathbf{v}_i) = \mathbf{v}_N \cdot (\mu_i \mathbf{v}_i) = \mu_i (\mathbf{v}_N \cdot \mathbf{v}_i) = 0,$$

which shows that $A\mathbf{v}_N$ *is orthogonal to* $\mathbf{v}_i$ *for all* $i = 1, \ldots,, N - 1$, *Hence,* $A\mathbf{v}_N$ *is parallel to* $\mathbf{v}_N$ *and so* $A\mathbf{v}_N = \lambda_N \mathbf{v}_N$ *for some* $\lambda_N \in \mathbb{R}$.

17 Integration

Given N bounded intervals $I_1, \ldots, I_N \subset \mathbb{R}$, a *rectangle* in $\mathbb{R}^N$ is a set of the form

$$R := I_1 \times \cdots \times I_N.$$

The elementary measure of a rectangle is given by

$$\operatorname{meas} R := \operatorname{length} I_1 \cdot \cdots \cdot \operatorname{length} I_N,$$

where if I_n has endpoints $a_n \leq b_n$, then we set $\operatorname{length} I_n := b_n - a_n$. To highlight the dependence on N, we will use the notation meas_N.

Remark 258 *Note that the intersection of two rectangles is still a rectangle. We will use this fact a lot in what follows.*

Given a rectangle R, by a *partition* $\mathcal{P}$ of R we mean a finite set of rectangles $R_1, \ldots, R_n$ such that $R_i \cap R_j = \emptyset$ if $i \neq j$ and

$$R = \bigcup_{i=1}^{n} R_i.$$

Exercise 259 *Let $R \subset \mathbb{R}^N$ be a rectangle and let $\mathcal{P} = \{R_1, \ldots, R_n\}$ be a partition of R. Prove that*

$$\operatorname{meas} R = \sum_{i=1}^{n} \operatorname{meas} R_i.$$

Hint: Use induction on N.

Given any bounded set $E \subset \mathbb{R}^N$, if we want to measure E a natural idea is to approximate E with rectangles. If we approximate E from the outside, we have two possible choices

$$\operatorname{meas}_1 E := \inf \left\{ \sum_{i=1}^{n} \operatorname{meas} R_i : \ R_i \text{ rectangles}, \ \bigcup_{i=1}^{n} \operatorname{meas} R_i \supseteq E, \ n \in \mathbb{N} \right\}$$

and

$$\operatorname{meas}_2 E := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{meas} R_i : \ R_i \text{ rectangles}, \ \bigcup_{i=1}^{\infty} \operatorname{meas} R_i \supseteq E \right\}. \tag{32}$$

The next example shows that meas_1 and meas_2 are different in general.

Example 260 *Let $E := [0,1] \cap \mathbb{Q}$. The set E is countable, so we can write E as*

$$E = \{r_n : \ n \in \mathbb{Q}\}.$$

If $I_1, \ldots, I_m$ are intervals that cover E then these intervals cover $[0,1]$ (exercise). Hence,

$$\sum_{i=1}^{m} \operatorname{length} I_i \geq 1$$

and so $\operatorname{meas}_1 E \geq 1$*. Taking* $I = [0,1]$ *shows that* $\operatorname{meas}_1 E = 1$*. On the other hand, taking* $I_n = \{r_n\}$*, then* $E = \bigcup_{i=1}^{\infty} I_i$ *and so*

$$\operatorname{meas}_2 E \leq \sum_{i=1}^{\infty} \operatorname{length} I_i = 0,$$

which shows that $\operatorname{meas}_2 E = 0$*.*

We will see that meas_1 is used to define the Peano–Jordan measure and Riemann's integration, while meas_2 is used to define the Lebesgue measure and Lebesgue's integration.

Remark 261 *Note that in* meas_2 *we could use infinite sums provided we only consider open rectangles.*

Friday, April 17, 2015

Carnival

Monday, April 20, 2015

Definition 262 *A set* $E \subset \mathbb{R}^N$ *is called a* pluri-rectangle *if it can be written as a finite union of rectangles.*

Exercise 263 *Prove that a pluri-rectangle can be written as a finite union of disjoint rectangles.*

Given a bounded set $E \subset \mathbb{R}^N$, the *Peano–Jordan inner measure* of E is given by

$$\operatorname{meas}_i E := \sup \{\operatorname{meas} P : P \text{ pluri-rectangle}, E \supseteq P\},$$

while the *Peano–Jordan outer measure* of E is given by

$$\operatorname{meas}_o E := \inf \{\operatorname{meas} P : P \text{ pluri-rectangle}, E \subseteq P\}.$$

We say that E is *Peano–Jordan measurable* if $\operatorname{meas}_i E = \operatorname{meas}_0 E$ and we call the common value the *Peano–Jordan measure*. We write $\operatorname{meas} E := \operatorname{meas}_i E = \operatorname{meas}_0 E$.

Remark 264 *Note that a rectangle is Peano–Jordan measurable and its Peano–Jordan measure coincides with its elementary measure (see Remark 275).*

Remark 265 *If* $E, F \subset \mathbb{R}^N$ *are two Peano–Jordan measurable sets, with* $E \subseteq F$*, then*

$$\operatorname{meas} E \leq \operatorname{meas} F.$$

Exercise 266 *Prove that if* $E \subset \mathbb{R}^N$ *and* $F \subset \mathbb{R}^N$ *are Peano–Jordan measurable and* R *is a rectangle containing* E*, then* $E \cup F$*,* $E \cap F$*,* $R \setminus E$ *are Peano–Jordan measurable.*

Exercise 267 *Prove that if $E_1, \ldots, E_n \subset \mathbb{R}^N$ are Peano–Jordan measurable and pairwise disjoint, then*

$$\operatorname{meas}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \operatorname{meas} E_i.$$

Exercise 268 *Prove that if $E_1, \ldots, E_n \subset \mathbb{R}^N$ are Peano–Jordan measurable and $E \subseteq \bigcup_{i=1}^{n} E_i$ is Peano–Jordan measurable, then*

$$\operatorname{meas} E \leq \sum_{i=1}^{n} \operatorname{meas} E_i.$$

Exercise 269 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{x} \in E$, let $\mathbf{y} \in \mathbb{R}^N \setminus E$. Prove that the segment S joining $\mathbf{x}$ and $\mathbf{y}$ intersects ∂E.*

Theorem 270 *A bounded set $E \subset \mathbb{R}^N$ is Peano–Jordan measurable if and only if its boundary is Peano–Jordan measurable and it has Peano–Jordan measure zero.*

Proof. We begin by observing that if P is a pluri-rectangle, then $\operatorname{meas} \partial P = 0$ (why?), and so

$$\operatorname{meas} P = \operatorname{meas} \overline{P} = \operatorname{meas} P^{\circ}.$$

Step 1: Assume that $E \subset \mathbb{R}^N$ is Peano–Jordan measurable and let R be a rectangle containing E. Since

$$\operatorname{meas}_i E = \operatorname{meas}_0 E,$$

for every $\varepsilon > 0$ there exist a pluri-rectangle P_1 contained in E and and a pluri-rectangle P_2 containing E such that

$$0 \leq \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \varepsilon.$$

Hence,

$$\begin{aligned}\operatorname{meas}\left(\overline{P_2} \setminus P_1^{\circ}\right) &= \operatorname{meas} \overline{P_2} - \operatorname{meas} P_1^{\circ} \\ &= \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \varepsilon.\end{aligned}$$

Note that $\overline{P_2} \setminus P_1^{\circ}$ is still a pluri-rectangle (exercise) and since

$$\overline{E} \subseteq \overline{P_2}, \quad P_1^{\circ} \subseteq E^{\circ},$$

we have that

$$\partial E = \overline{E} \setminus E^{\circ} \subseteq \overline{P_2} \setminus P_1^{\circ}.$$

Hence,

$$\begin{aligned}0 &\leq \sup\{\operatorname{meas} P : P \text{ pluri-rectangle}, \partial E \supseteq P\} \\ &\leq \inf\{\operatorname{meas} P : P \text{ pluri-rectangle}, \partial E \subseteq P\} \leq \operatorname{meas}\left(\overline{P_2} \setminus P_1^{\circ}\right) \leq \varepsilon,\end{aligned}$$

which, by letting $\varepsilon \to 0^+$, implies that

$$\begin{aligned} 0 &= \sup \{\operatorname{meas} P : P \text{ pluri-rectangle}, \partial E \supseteq P\} \\ &= \inf \{\operatorname{meas} P : P \text{ pluri-rectangle}, \partial E \subseteq P\} = 0. \end{aligned}$$

It follows that ∂E is Peano–Jordan measurable with measure zero.
Step 2: Assume that $\partial E \subset \mathbb{R}^N$ is Peano–Jordan measurable with measure zero. Since E is bounded, so is $\overline{E}$ and so there exists a rectangle R containing $\overline{E}$. Since $\operatorname{meas} \partial E = 0$ by the previous theorem there exists a a pluri-rectangle P containing ∂E such that

$$\operatorname{meas} P \leq \varepsilon.$$

The set $R \setminus P$ is a pluri-rectangle and thus we can write it as disjoint unions of rectangles,

$$R \setminus P = \bigcup_{i=1}^{n} R_i.$$

Let P_1 be the pluri-rectangle given by the union of all the rectangles R_i that are contained in E, so that $P_1 \subseteq E$. Let $P_2 := P \cup P_1$. We claim that

$$E \subseteq P_2.$$

Fix $\mathbf{x} \in E$. If $\mathbf{x}$ does not belong to P_2, then in particular it cannot belong to P and so it belongs to $R \setminus P$. Hence, there exists R_i such that $\mathbf{x} \in R_i$. But then R_i must be contained in E. Indeed, if not, then there exists $\mathbf{y} \in R_i \cap (R \setminus E)$. It follows by Exercise 269 that the segment S joining $\mathbf{x}$ and $\mathbf{y}$ must contained a point on the boundary of ∂E, which is a contradiction since the segment S is contained in R_i and R_i does not intersect $P \supset \partial E$. This proves the claim.

Since the claim holds, for every $\varepsilon > 0$ we have found a pluri-rectangle P_1 contained in E and and a pluri-rectangle P_2 is containing E such that

$$0 \leq \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \varepsilon.$$

It follows by the previous theorem that E is Peano–Jordan measurable. ∎

Wednesday, April 22, 2015

Given a rectangle R and two partitions $\mathcal{P}$ and $\mathcal{Q}$ of R, we say that $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, if each rectangle of $\mathcal{Q}$ is contained in some rectangle of $\mathcal{P}$.

Next we define the integral of a bounded function. Consider first the case in which f is nonnegative. Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to [0, \infty)$ be a bounded function. The integral of f should measure the set

$$S_f := \{(\mathbf{x}, y) \in R \times [0, \infty) : 0 \leq y \leq f(\mathbf{x})\}.$$

Thus it makes sense to define the upper Riemann integral of f as the outer Peano–Jordan measure of S_f in $\mathbb{R}^{N+1}$, that is,

$$\overline{\int_R} f(\mathbf{x}) \, d\mathbf{x} = \operatorname{meas}_{o,N+1}(S_f). \tag{33}$$

Similarly, we could define the lower Riemann integral of f as the inner Peano–Jordan measure of S_f in $\mathbb{R}^{N+1}$, that is,

$$\underline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \mathrm{meas}_{i,N+1}(S_f). \tag{34}$$

Now consider a pluri-rectangle $Q \subset \mathbb{R}^{N+1}$ which contains S_f. Then we can write

$$Q = \bigcup_{i=1}^{m} Q_i,$$

where Q_i are disjoint rectangles. By considering a refinement of the Q_i, without loss of generality, we can assume that $Q_i = R_i \times I_i$, where $R_i \subset \mathbb{R}^N$, I_i is an interval, and the rectangles R_i are pairwise disjoint. Also, without loss of generality we can put together all the rectangles $Q_j = R_j \times I_j$ which have the same base R_i. Thus we can assume that the rectangles Q_i are of the form $Q_i = R_i \times [l_i, M_i]$, where $l_i \leq 0$. Note that since $S_f \subset R \times [0, \infty)$, necessarily, $M_i \geq \sup_{R_i \cap R} f$.

Define $R_i' := R_i \cap R$, $Q_i' := R_i' \times [0, \sup_{R_i'} f]$, and

$$Q' := \bigcup_{i=1}^{m} Q_i',$$

Since $S_f \subset R \times [0, \infty)$, we have that Q' is still a pluri-rectangle which contains S_f, and so

$$\overline{\int_R} f(\mathbf{x})\ d\mathbf{x} \leq \sum_{i=1}^{m} \sup_{R_i'} f \,\mathrm{meas}_N R_i' = \mathrm{meas}_{N+1}(Q') \leq \mathrm{meas}_{N+1}(Q).$$

Thus, we have shown that

$$\overline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \inf \left\{ \sum_{i=1}^{m} \sup_{R_i} f \,\mathrm{meas}_N R_i : \ R_i \text{ are a partition of } R,\ m \in \mathbb{N} \right\}. \tag{35}$$

In many books the right-hand side is used as a definition for the upper Riemann integral. Note that it makes sense also for functions which takes negatives values. We will use this as a definition.

Definition 271 *Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the* upper Riemann integral $\overline{\int_R} f(\mathbf{x})\ d\mathbf{x}$ *of f is defined as in (35).*

Another way to look at the upper Riemann integral is to use step functions.

Definition 272 *Let $R \subset \mathbb{R}^N$ be a rectangle. A function $s : R \to \mathbb{R}$ is called a* step function *if there exists a partition $R_1, \ldots, R_n$ of R such that s is constant on each rectangle R_i of the partition, that is, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $s(\boldsymbol{x}) = c_i$ for all $\boldsymbol{x} \in R_i$, $i = 1, \ldots, n$.*

The Riemann integral of a step function is defined by

$$\int_R s(\mathbf{x})\,d\mathbf{x} := \sum_{i=1}^{m} c_i \operatorname{meas}_N R_i.$$

It follows from the discussion above that the upper integral of a bounded function f can be written as

$$\overline{\int_R} f(\mathbf{x})\,d\mathbf{x} = \inf\left\{\int_R s(\mathbf{x})\,d\mathbf{x} : \ s \text{ step function, } f \le s\right\}. \tag{36}$$

Now let's look at the lower integral. Consider a pluri-rectangle $Q \subset \mathbb{R}^{N+1}$ which is contained in S_f. Then we can write

$$Q = \bigcup_{i=1}^{m} Q_i,$$

where Q_i are disjoint rectangles. By considering a refinement of the Q_i, without loss of generality, we can assume that $Q_i = R_i \times I_i$, where $R_i \subset \mathbb{R}^N$, I_i is an interval, and the rectangles R_i are pairwise disjoint. Also, without loss of generality we can put together all the rectangles $Q_j = R_j \times I_j$ which have the same base R_i. Thus we can assume that the rectangles Q_i are of the form $Q_i = R_i \times [l_i, m_i]$. Since $Q \subset S_f \subset R \times [0, \infty)$, necessarily, $R_i \subseteq R$, $l_i \ge 0$, $m_i \le \inf_{R_i} f$.

Define $Q_i' := R_i \times [0, \inf_{R_i} f]$, and

$$Q' := \bigcup_{i=1}^{m} Q_i',$$

Then $S_f \supseteq Q' \supseteq Q$, and so

$$\underline{\int_R} f(\mathbf{x})\,d\mathbf{x} \ge \sum_{i=1}^{m} \inf_{R_i'} f \operatorname{meas}_N R_i' = \operatorname{meas}_{N+1}(Q') \ge \operatorname{meas}_{N+1}(Q).$$

Thus, we have shown that

$$\underline{\int_R} f(\mathbf{x})\,d\mathbf{x} = \sup\left\{\sum_{i=1}^{m} \inf_{R_i} f \operatorname{meas}_N R_i : \ R_i \text{ are a partition of } R,\ m \in \mathbb{N}\right\}. \tag{37}$$

In many books the right-hand side is used as a definition for the lower Riemann integral. Note that it makes sense also for functions which takes negatives values.

Definition 273 *Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the* lower Riemann integral $\underline{\int_R} f(\mathbf{x})\,d\mathbf{x}$ *of f is defined as in (37).*

Observe that

$$\underline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \sup\left\{\int_R s(\mathbf{x})\ d\mathbf{x} : \ s \text{ step function, } f \geq s\right\}.$$

Definition 274 *Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, we say that f is* Riemann integrable *over R if the lower and upper integral coincide. We call the common value the* Riemann integral *of f over R and we denote it by $\int_R f(\mathbf{x})\ d\mathbf{x}$. Thus, for a Riemann integrable function,*

$$\int_R f(\mathbf{x})\ d\mathbf{x} := \underline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \overline{\int_R} f(\mathbf{x})\ d\mathbf{x}.$$

Remark 275 *Note that the function $f = 1$ is Riemann integrable over a rectangle R and*

$$\int_R 1\, dx = \operatorname{meas} R.$$

We can also define the Riemann integral over bounded sets E. Given a bounded set $E \subset \mathbb{R}^N$, let R be a rectangle containing E. We say that function $f : E \to \mathbb{R}$ is *Riemann integrable over E* if the function

$$g(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ 0 & \text{if } \mathbf{x} \in R \setminus E \end{cases}$$

is Riemann integrable over R and we define the *Riemann integral of f over E* to be

$$\int_E f(\mathbf{x})\ d\mathbf{x} := \int_R g(\mathbf{x})\ d\mathbf{x}.$$

Exercise 276 *Prove the previous definition does not depend on the choice of the particular rectangle R containing E.*

Note that in the discussion above we have proved the following important theorems.

Theorem 277 *Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to [0, \infty)$ be a bounded function. Then*

$$\overline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \operatorname{meas}_{o,N+1}(S_f), \quad \underline{\int_R} f(\mathbf{x})\ d\mathbf{x} = \operatorname{meas}_{i,N+1}(S_f).$$

In particular, f is Riemann integrable over R if and only if the set S_f is Peano–Jordan measurable in $\mathbb{R}^{N+1}$ and in this case

$$\operatorname{meas}_{N+1} S_f = \int_R f(\mathbf{x})\ d\mathbf{x}.$$

Remark 278 *With a similar proof one can show that if $R \subset \mathbb{R}^N$ is a rectangle and $f : R \to \mathbb{R}$ is a bounded function, with $f(\mathbf{x}) \geq c$ for all $\mathbf{x} \in R$. Then f is Riemann integrable over R if and only if the set*

$$T_f := \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y \leq f(\mathbf{x})\}$$

is Peano–Jordan measurable in $\mathbb{R}^{N+1}$ and in this case

$$\operatorname{meas}_{N+1} T_f = \int_R (f(\mathbf{x}) - c)\, d\mathbf{x}.$$

Theorem 279 *Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to [0, \infty)$ be a bounded function. Then*

$$\overline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \inf \left\{ \int_R s(\mathbf{x})\, d\mathbf{x} : \ s \text{ step function, } f \leq s \right\}$$

and

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \sup \left\{ \int_R s(\mathbf{x})\, d\mathbf{x} : \ s \text{ step function, } f \geq s \right\}.$$

The following theorem characterizes Riemann integrable functions.

Theorem 280 *Given a rectangle R, a bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the set of its discontinuity points has Lebesgue measure zero (see* (32)*).*

Proof. We will skip this proof. ■

In view of the previous theorem, we have the following.

Corollary 281 *Given a rectangle R, a bounded continuous function $f : R \to \mathbb{R}$ is Riemann integrable.*

Remark 282 *If R is a closed rectangle and $f : R \to \mathbb{R}$ is continuous, then we do not need to assume that f is bounded. Indeed, since R is closed and bounded, it is sequentially compact, and so by the Weierstrass theorem f is bounded.*

Corollary 283 *Given $f : [a, b] \to \mathbb{R}$,*

(i) if f is continuous, then f is Riemann integrable,

(ii) if f is monotone, then f is Riemann integrable.

Proof. If f is continuous, then by the Weierstrass theorem it is bounded, and thus by the previous theorem it is Riemann integrable.

If f is monotone, then it is bounded from below by $\min\{f(a), f(b)\}$ and from above by $\max\{f(a), f(b)\}$. Moreover, by Theorem 138 its set of discontinuity points is at most countable. Since a countable set has Lebesgue measure zero, it follows from the previous theorem that f is Riemann integrable. ■

Example 284 *Some examples of functions that are Riemann integrable and others that are not.*

(i) *The function*

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is Riemann integrable over $[0,1]$*, since it is bounded and discontinuous only at* $x = 0$.

(ii) *The function*

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}$$

is Riemann integrable over $[0,1]$*, since it is bounded and its set of discontinuity points is*

$$E = \left\{ \frac{1}{n} \right\}_n \cup \{0\},$$

which is countable.

(iii) *The function*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise}, \end{cases}$$

is bounded but not Riemann integrable over $[0,1]$*, since it is bounded and its set of discontinuity points is* $[0,1]$.

Exercise 285 *Consider the function* $g : [0,1] \to [0,1]$ *defined by*

$$g(x) := \begin{cases} 0 & \text{if } x \text{ is irrational}, \\ \frac{1}{p} & \text{if } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ relatively prime, } 0 < p < q, \\ 1 & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(i) *Prove that* g *is discontinuous at every rational point of* $[0,1]$.

(ii) *Prove that* g *is continuous at every irrational point of* $[0,1]$.

(iii) *Prove that* g *is Riemann integrable.*

Friday, April 24, 2015

Corrections Homework #9.

Monday, April 27, 2015

Exercise 286 (Important) *Let* $R \subset \mathbb{R}^N$ *be a rectangle, let* $f : R \to \mathbb{R}$ *be a bounded function, let* $\mathcal{P} = \{R_1, \ldots, R_n\}$ *be a partition of* R *and let* $\mathcal{P} =$

$\{R_1, \dots, R_n\}$ *be a partition of* R *and let* $\mathcal{Q} = \{S_1, \dots, S_m\}$ *be a refinement of* $\mathcal{P}$*. Prove that*

$$\sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\mathbf{x} \in R_i} f(\mathbf{x}) \le \sum_{j=1}^{m} \operatorname{meas} S_j \inf_{\mathbf{x} \in S_j} f(\mathbf{x}),$$
$$\sum_{j=1}^{m} \operatorname{meas} S_j \sup_{\mathbf{x} \in S_j} f(\mathbf{x}) \le \sum_{i=1}^{n} \operatorname{meas} R_i \sup_{\mathbf{x} \in R_i} f(\mathbf{x}).$$

Exercise 287 *Given a rectangle* R*, consider a bounded function* $f : R \to \mathbb{R}$*. Prove that*

$$\operatorname{meas} R \inf_{\mathbf{x} \in R} f(\mathbf{x}) \le \underline{\int_R} f(\mathbf{x})\, d\mathbf{x} \le \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} \le \operatorname{meas} R \sup_{\mathbf{x} \in R} f(\mathbf{x}).$$

Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to \mathbb{R}$ be a bounded function. To simplify the notation, given a partition $\mathcal{P} = \{R_1, \dots, R_n\}$ of R, we define the *lower* and *upper sums* of f for the partition $\mathcal{P}$ respectively by

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\mathbf{x} \in R_i} f(\mathbf{x}),$$
$$U(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \sup_{\mathbf{x} \in R_i} f(\mathbf{x}).$$

Thus we can rewrite the lower and upper Riemann integrals as

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \sup \{L(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\},$$
$$\overline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \inf \{U(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\}.$$

The next theorem is very important in exercises. It allows to calculate triple, double, etc.. integrals by integrating one variable at a time.

Theorem 288 (Repeated Integration) *Let* $S \subset \mathbb{R}^N$ *and* $T \subset \mathbb{R}^M$ *be rectangles, let* $f : S \times T \to \mathbb{R}$ *be Riemann integrable and assume that for every* $\mathbf{x} \in S$*, the function* $\mathbf{y} \in T \mapsto f(\mathbf{x}, \mathbf{y})$ *is Riemann integrable. Then the function* $\mathbf{x} \in S \mapsto \int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ *is Riemann integrable and*

$$\int_{S \times T} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \int_S \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}. \tag{38}$$

Similarly, if for every $\mathbf{y} \in T$*, the function* $\mathbf{x} \in S \mapsto f(\mathbf{x}, \mathbf{y})$ *is Riemann integrable, then the function* $\mathbf{y} \in T \mapsto \int_S f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}$ *is Riemann integrable and*

$$\int_{S \times T} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \int_T \left(\int_S f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \right) d\mathbf{y}.$$

Proof. Let $R := S \times T$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of R. Construct a refinement (exercise) $\mathcal{P}' = \{R_1, \ldots, R_n\}$ of $\mathcal{P}$ and $\mathcal{Q}$ with the property that each rectangle R_k can be written as $R_k = S_i \times T_j$, where $\mathcal{P}_N = \{S_1, \ldots, S_m\}$ and $\mathcal{P}_M = \{T_1, \ldots, T_\ell\}$ are partitions of S and T, respectively. Using the fact that

$$\operatorname{meas}_{N+M} R_k = \operatorname{meas}_{N+M} (S_i \times T_j) = \operatorname{meas}_N S_i \operatorname{meas}_M T_j,$$

and Exercise 286 we have

$$\begin{aligned}
L(f, \mathcal{P}) \le L(f, \mathcal{P}') &= \sum_{i=1}^{m} \sum_{j=1}^{\ell} \operatorname{meas}_N S_i \operatorname{meas}_M T_j \inf_{\mathbf{x} \in S_i} \inf_{\mathbf{y} \in T_j} f(\mathbf{x}, \mathbf{y}) \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \inf_{\mathbf{x} \in S_i} \sum_{j=1}^{\ell} \operatorname{meas}_M T_j \inf_{\mathbf{y} \in T_j} f(\mathbf{x}, \mathbf{y}) \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \inf_{\mathbf{x} \in S_i} \int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \le \underline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le \sum_{i=1}^{m} \operatorname{meas}_N S_i \sup_{\mathbf{x} \in S_i} \int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \sup_{\mathbf{x} \in S_i} \sum_{j=1}^{\ell} \operatorname{meas}_M T_j \sup_{\mathbf{y} \in T_j} f(\mathbf{x}, \mathbf{y}) \\
&\le \sum_{i=1}^{m} \sum_{j=1}^{\ell} \operatorname{meas}_N S_i \operatorname{meas}_M T_j \sup_{\mathbf{x} \in S_i} \sup_{\mathbf{y} \in T_j} f(\mathbf{x}, \mathbf{y}) \\
&= U(f, \mathcal{P}') \le U(f, \mathcal{Q}),
\end{aligned}$$

which shows that

$$L(f, \mathcal{P}) \le \underline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le \overline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le U(f, \mathcal{Q}).$$

Taking the supremum over all partitions $\mathcal{P}$ of R, we get

$$\begin{aligned}
\underline{\int_{S \times T}} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) &\le \underline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le U(f, \mathcal{Q}).
\end{aligned}$$

Taking the infimum over all partitions $\mathcal{Q}$ of R, we get

$$\begin{aligned}
\underline{\int_{S \times T}} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) &\le \underline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le \overline{\int_{S \times T}} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Since f is Riemann integrable, it follows that

$$\int_{S\times T} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) = \underline{\int_S}\left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} = \overline{\int_S}\left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x},$$

which implies that the function $\mathbf{x}\in S\mapsto \int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}$ is Riemann integrable and that (38). ■

Remark 289 *Note that if in Theorem 288 we remove the hypothesis that for every $\mathbf{x}\in S$, the function $\mathbf{y}\in T\mapsto f(\mathbf{x},\mathbf{y})$ is Riemann integrable, we can still prove that the functions $\mathbf{x}\in S\mapsto \underline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}$ and $\mathbf{x}\in S\mapsto \overline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}$ are Riemann integrable with*

$$\begin{aligned}\int_{S\times T} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) &= \int_S\left(\underline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x}\\ &= \int_S\left(\overline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x}.\end{aligned}$$

The proof is similar to the one of Theorem 288 and we leave it as an exercise. In turn,

$$\int_S\left(\overline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} = 0.$$

Since $\overline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\geq 0$, it follows from Exercise **??** *that there exists a set $E\subseteq S$ of Lebesgue measure zero such that*

$$\overline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x},\mathbf{y})\, d\mathbf{y} = 0$$

for all $\mathbf{x}\in S\setminus E$. Thus, for every $\mathbf{x}\in S\setminus E$, the function $\mathbf{y}\in T\mapsto f(\mathbf{x},\mathbf{y})$ is Riemann integrable.

The following example shows that without assuming that f is Riemann integrable, the previous theorem fails.

Example 290 *Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be defined by*

$$f(x,y) := \begin{cases} 1 & \text{if there exist } p\geq 2 \text{ prime and } m,n\in\mathbb{N}\\ & \text{such that } (x,y)=\left(\frac{m}{p},\frac{n}{p}\right),\\ 0 & \text{otherwise.}\end{cases}$$

Let E be the set of all $(x,y)\in[0,1]\times[0,1]$ for which there exist $p\geq 2$ prime and $m,n\in\mathbb{N}$ such that $(x,y)=\left(\frac{m}{p},\frac{n}{p}\right)$. Using the density of the rationals and of the irrationals, it can be shown that both E and $([0,1]\times[0,1])\setminus E$ are dense in $[0,1]\times[0,1]$. Hence, the set of discontinuity points of f is $[0,1]\times[0,1]$. Thus,

f is not Riemann integrable. On the other hand, if we fix $x \in [0,1]$ and we consider the function $f(x,\cdot)$, then we have the following two cases. If $x = \frac{m}{p}$ for some $p \geq 2$ prime and some $m \in \mathbb{N}$, then

$$f(x,y) = \begin{cases} 1 & \text{if } y \in \left\{\frac{1}{p}, \ldots, \frac{p-1}{p}, \frac{p}{p}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x,\cdot)$ is only discontinuous at a finite number of points and so it is Riemann integrable in $[0,1]$ with

$$\int_0^1 f(x,y)\, dy = 0.$$

In the second case, x cannot be written in the form $\frac{m}{p}$ for some $p \geq 2$ prime and some $m \in \mathbb{N}$. In this case $f(x,y) = 0$ for all $y \in [0,1]$ and so again $\int_0^1 f(x,y)\, dy = 0$, which shows that

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dy = \int_0^1 0\, dy = 0$$

and similarly,

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx\right) dx = \int_0^1 0\, dx = 0.$$

So the iterated integrals exist and are equal, but the integral $\int_{[0,1]\times[0,1]} f(x,y)\, d(x,y)$ does not exist.

The following example shows that the fact that $f : S \times T \to \mathbb{R}$ be Riemann integrable does not imply that for every $\mathbf{x} \in S$, the function $\mathbf{y} \in T \mapsto f(\mathbf{x},\mathbf{y})$ is Riemann integrable.

Example 291 *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by*

$$f(x,y) := \begin{cases} 1 & \text{if } y \in [0,1] \cap \mathbb{Q} \text{ and } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The function f is discontinuous only on the segment $\left\{\frac{1}{2}\right\} \times [0,1]$, which has Lebesgue measure zero. Hence, f is Riemann integrable in $[0,1] \times [0,1]$. On the other hand, if we fix $x = \frac{1}{2}$ and we consider the function $g(y) = f\left(\frac{1}{2}, y\right)$, $y \in [0,1]$, we have that g is discontinuous at every $y \in [0,1]$, and so g is not Riemann integrable in $[0,1]$.

Exercise 292 *Calculate $\int_{[0,1]\times[0,1]} f(x,y)\, d(x,y)$ in two different ways, where*

$$f(x,y) := x \sin(x+y).$$

Wednesday, April 29, 2015

Definition 293 *Given a bounded set $E \subset \mathbb{R}^N$, let R be a rectangle containing E. We say that function $f : E \to \mathbb{R}$ is* Riemann integrable over E *if the function*

$$g(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ 0 & \text{if } \mathbf{x} \in R \setminus E \end{cases}$$

is Riemann integrable over R and we define the Riemann integral of f over E *to be*

$$\int_E f(\mathbf{x})\, d\mathbf{x} := \int_R g(\mathbf{x})\, d\mathbf{x}.$$

Exercise 294 *Prove the previous definition does not depend on the choice of the particular rectangle R containing E.*

Exercise 295 *Given two sets $E, F \subseteq \mathbb{R}^N$, prove that*

$$\partial(E \setminus F) \subseteq \partial E \cup \partial F$$

Corollary 296 *Let $R \subset \mathbb{R}^N$ be a rectangle, let $\alpha : R \to \mathbb{R}$ and $\beta : R \to \mathbb{R}$ be two Riemann integrable functions, with $\alpha(\mathbf{x}) \leq \beta(\mathbf{x})$ for all $\mathbf{x} \in R$, let*

$$E := \{(\mathbf{x}, y) \in R \times \mathbb{R} : \alpha(\mathbf{x}) \leq y \leq \beta(\mathbf{x})\},$$

and let $f : E \to \mathbb{R}$ be a bounded continuous function. Then f is Riemann integrable over E and

$$\int_E f(\mathbf{x}, y)\, d(\mathbf{x}, y) = \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x}, y)\, dy \right) d\mathbf{x}.$$

Proof. Consider a rectangle $R \times [a, b]$ containing E and let

$$g(\mathbf{x}, y) := \begin{cases} f(\mathbf{x}, y) & \text{if } (\mathbf{x}, y) \in E, \\ 0 & \text{if } (\mathbf{x}, y) \in (R \times [a, b]) \setminus E. \end{cases}$$

We need to show that g is Riemann integrable over $R \times [a, b]$. Hence, we need to look at the set of discontinuity points of g. Since g is continuous in E, we have that the discontinuity points of g are on the boundary of E. Thus, we need to show that ∂E has Lebesgue measure zero in $\mathbb{R}^{N+1}$. We will show more, namely that it has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$. Let $c \in \mathbb{R}$ be such that $\alpha(\mathbf{x}) \geq c$ for all $\mathbf{x} \in R$. Then

$$\begin{aligned} E &= \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y \leq \beta(\mathbf{x})\} \setminus \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y < \alpha(\mathbf{x})\} \\ &= T_\beta \setminus T_\alpha. \end{aligned}$$

By the previous exercise, $\partial E \subseteq \partial T_\alpha \cup \partial T_\beta$ and since ∂T_α and ∂T_β have Peano–Jordan measure zero in $\mathbb{R}^{N+1}$ in view of the previous corollary, it follows that ∂E has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$. This shows that g is Riemann integrable over $R \times [a, b]$.

For every $\mathbf{x} \in R$, the function

$$y \in [a,b] \mapsto g(\mathbf{x},y) = \begin{cases} 0 & \text{if } y > \beta(\mathbf{x}), \\ f(\mathbf{x},y) & \text{if } \alpha(\mathbf{x}) \le y \le \beta(\mathbf{x}), \\ 0 & \text{if } y < \alpha(\mathbf{x}), \end{cases}$$

is Riemann integrable since it is discontinuous at most at the two points $y = \alpha(\mathbf{x})$ and $y = \beta(\mathbf{x})$. Hence, by Theorem 288, the function $\mathbf{x} \in R \mapsto \int_{[a,b]} g(\mathbf{x},y)\, dy$ is Riemann integrable and

$$\int_{R\times[a,b]} g(\mathbf{x},y)\, d(\mathbf{x},y) = \int_R \left(\int_{[a,b]} g(\mathbf{x},y)\, dy \right) d\mathbf{x}$$
$$= \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x},y)\, dy \right) d\mathbf{x}.$$

This completes the proof. ■

Remark 297 *If α and β are continuous, then the set of discontinuity points of g is given by the union of the graphs of α and β, but if α and β are discontinuous, then the set of discontinuity points of g is larger (why?).*

Next we discuss some properties of Riemann integration.

Proposition 298 *Given a rectangle R, let $f, g : R \to \mathbb{R}$ be Riemann integrable.*

(i) If $\lambda \in \mathbb{R}$, then λf is Riemann integrable and

$$\int_R \lambda f(\mathbf{x})\, d\mathbf{x} = \lambda \int_R f(\mathbf{x})\, d\mathbf{x}. \tag{39}$$

(ii) The functions $f + g$ and fg are Riemann integrable and

$$\int_R (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} = \int_R f(\mathbf{x})\, d\mathbf{x} + \int_R g(\mathbf{x})\, d\mathbf{x}. \tag{40}$$

(iii) If $f \le g$, then

$$\int_R f(\mathbf{x})\, d\mathbf{x} \le \int_R g(\mathbf{x})\, d\mathbf{x}.$$

(iv) The function $|f|$ is Riemann integrable and

$$\left| \int_R f(\mathbf{x})\, d\mathbf{x} \right| \le \int_R |f(\mathbf{x})|\, d\mathbf{x}.$$

Proof. We only prove (ii). Since the set of discontinuities points of $f + g$ and fg is contained in the union of the sets of discontinuities points of f and g, using the fact that the finite union of sets of Lebesgue measure zero still has measure zero, it follows that the functions $f+g$ and fg are Riemann integrable.

To prove (40), let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of R and construct a refinement $\mathcal{S}$ of both $\mathcal{P}$ and $\mathcal{Q}$. Using the fact that

$$\inf_E f + \inf_E g \le \inf_E (f+g),$$

and Exercise 286 we have

$$\begin{aligned} L(f,\mathcal{P}) + L(g,\mathcal{Q}) &\le L(f,\mathcal{S}) + L(g,\mathcal{S}) \\ &\le L(f+g,\mathcal{S}) \le \underline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x}. \end{aligned}$$

Taking the supremum over all partitions $\mathcal{P}$ of R, we get

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} + L(g,\mathcal{Q}) \le \underline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x}.$$

Taking the supremum over all partitions $\mathcal{Q}$ of R, we get

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} + \underline{\int_R} g(\mathbf{x})\, d\mathbf{x} \le \underline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x}. \tag{41}$$

Similarly, let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of R and construct a refinement $\mathcal{S}$ of both $\mathcal{P}$ and $\mathcal{Q}$. Using the fact that

$$\sup_E (f+g) \le \sup_E f + \sup_E g,$$

and Exercise 286 we have

$$\begin{aligned} \overline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} &\le U(f+g,\mathcal{S}) \le U(f,\mathcal{S}) + U(g,\mathcal{S}) \\ &\le U(f,\mathcal{P}) + U(g,\mathcal{Q}). \end{aligned}$$

Taking the infimum over all partitions $\mathcal{P}$ of R, we get

$$\overline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} \le \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} + U(g,\mathcal{Q}).$$

Taking the infimum over all partitions $\mathcal{Q}$ of R, we get

$$\overline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} \le \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} + \overline{\int_R} g(\mathbf{x})\, d\mathbf{x}. \tag{42}$$

Combining (41) and (42) and Exercise 287 we obtain

$$\begin{aligned} \underline{\int_R} f(\mathbf{x})\, d\mathbf{x} + \underline{\int_R} g(\mathbf{x})\, d\mathbf{x} &\le \underline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} \\ &\le \overline{\int_R} (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} \le \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} + \overline{\int_R} g(\mathbf{x})\, d\mathbf{x}. \end{aligned}$$

Since f and g are Riemann integrable, it follows that the left and right hand side of the previous inequalities are the same and so (40) holds. ∎

Exercise 299 *Give an example of a bounded function $f : R \to \mathbb{R}$ such that $|f|$ is Riemann integrable over R, but f is not.*

Friday, May 1, 2015

Exercise 300 *Consider a bounded function $f : [a, b] \to \mathbb{R}$. Prove that for every constant $C \in \mathbb{R}$,*

$$\underline{\int_a^b} (f(x) + C)\, dx = \underline{\int_a^b} f(x)\, dx + C(b-a),$$
$$\overline{\int_a^b} (f(x) + C)\, dx = \overline{\int_a^b} f(x)\, dx + C(b-a).$$

Proposition 301 *Consider a bounded function $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then*

$$\underline{\int_a^b} f(x)\, dx = \underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx, \tag{43}$$
$$\overline{\int_a^b} f(x)\, dx = \overline{\int_a^c} f(x)\, dx + \overline{\int_c^b} f(x)\, dx. \tag{44}$$

Proof. To highlight the dependence of the interval I where the lower and upper sums are taken, we write $L(f, P, I)$ and $U(f, P, I)$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Consider the new partition $P' = P \cup \{c\}$ (note that if c is already in P, then $P' = P$). Let $P_1 := P' \cap [a, c]$ and $P_2 := P' \cap [c, b]$. Then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. Hence,

$$\begin{aligned}\underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx &\geq L(f, P_1, [a, c]) + L(f, P_2, [c, b]) \\ &= L(f, P', [a, b]) \geq L(f, P, [a, b]),\end{aligned}$$

where in the last inequality we have used Exercise 286. Taking the supremum over all partitions P of $[a, b]$, we get

$$\underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx \geq \underline{\int_a^b} f(x)\, dx = \sup_{P \text{ partition of } [a,b]} L(f, P, [a, b]). \tag{45}$$

To prove the opposite inequality, fix $\varepsilon > 0$. Using the definition of supremum, we may find a partition P_1^ε of $[a, c]$ and a partition P_2^ε of $[c, b]$ such that

$$L(f, P_1^\varepsilon, [a, c]) \geq \underline{\int_a^c} f(x)\, dx - \varepsilon,$$
$$L(f, P_2^\varepsilon, [c, b]) \geq \underline{\int_c^b} f(x)\, dx - \varepsilon.$$

Then $P^{\varepsilon} := P_1^{\varepsilon} \cup P_2^{\varepsilon}$ is a partition of $[a,b]$ and so P_2 is a partition of $[c,b]$. Hence,

$$\underline{\int_a^b} f(x)\,dx \geq L\left(f, P^{\varepsilon}, [a,b]\right) = L\left(f, P_1^{\varepsilon}, [a,c]\right) + L\left(f, P_2^{\varepsilon}, [c,b]\right)$$
$$\geq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx - 2\varepsilon.$$

Letting $\varepsilon \to 0^+$ and using also (45), we obtain (43). The proof of (44) is similar and we omit it. ■

Theorem 302 (Fundamental Theorem of Calculus, I) *Consider a bounded function $f : [a,b] \to \mathbb{R}$ and let*

$$F(x) := \begin{cases} \underline{\int_a^x} f(y)\,dy & \text{if } a < x \leq b, \\ 0 & \text{if } x = a, \end{cases} \qquad G(x) := \begin{cases} \overline{\int_a^x} f(y)\,dy & \text{if } a < x \leq b, \\ 0 & \text{if } x = a. \end{cases}$$

Then F and G are differentiable at every point $x_0 \in [a,b]$ at which f is continuous, with

$$F'(x_0) = G'(x_0) = f(x_0).$$

Proof. Assume that f is continuous at $x_0 \in [a,b]$. We consider the case $x_0 \in (a,b)$ (the cases $x_0 = a$ and $x_0 = b$) are simpler. We want to prove that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

For $h \neq 0$ consider the different quotient

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \begin{cases} \frac{1}{x - x_0} \underline{\int_{x_0}^x} \left(f(y) - f(x_0)\right) dy & \text{if } x > x_0, \\ -\frac{1}{x - x_0} \underline{\int_x^{x_0}} \left(f(y) - f(x_0)\right) dy & \text{if } x < x_0, \end{cases}$$

where we have used (43) and Exercise 300. Since f is continuous at x_0, given $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|f(x) - f(x_0)| \leq \varepsilon$$

for all $x \in [a,b]$ with $|x - x_0| \leq \delta$. Take $|x - x_0| \leq \delta$. Then for $x > x_0$ (the case $x < x_0$ is similar), by Exercise 287, we have

$$-\varepsilon \leq \frac{x - x_0}{x - x_0} \inf_{y \in [x_0, x]} \left(f(y) - f(x_0)\right) \leq \frac{1}{x - x_0} \underline{\int_{x_0}^x} \left(f(y) - f(x_0)\right) dy$$
$$\leq \frac{x - x_0}{x - x_0} \sup_{y \in [x_0, x]} \left(f(y) - f(x_0)\right) \leq \varepsilon,$$

which shows that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \varepsilon.$$

This concludes the proof. ■

Theorem 303 (Fundamental Theorem of Calculus, II) *Let* $F : [a,b] \to \mathbb{R}$ *be a differentiable function. Assume that* F' *is Riemann integrable over* $[a,b]$*. Then*

$$F(b) - F(a) = \int_a^b F'(x)\, dx. \tag{46}$$