21-269 Vector Analysis Final.

2017-05-09

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 7 questions and 70 points.*
- *You may use any result proved in class or any regular homework problem* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *Difficulty wise, Q1 ≈ Q2 ≈ Q3 ≈ Q4 ⩽ Q5 < Q6 ⩽ Q7. Good luck* ◡̈.

$\boxed{10}$ 1. Does there exist a strictly increasing sequence of prime numbers (p_n) such that $\lim_{n\to\infty} \sin(p_n)$ exists? Prove or disprove your answer.

$\boxed{10}$ 2. Let $C \subseteq \mathbb{R}^n$ be a nonempty set. Define a function $d : \mathbb{R}^n \to [0, \infty)$ by $d(x) = \inf\{|x - c| \bigm| c \in C\}$. Show that for any $x, y \in \mathbb{R}^n$ we have $|d(x) - d(y)| \leqslant |x - y|$. Hence, or otherwise, show that d is continuous.

$\boxed{10}$ 3. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $S \subseteq \mathbb{R}^3$ and $a \in \mathbb{R}^3$ by

$$f(x,y,z) = \begin{pmatrix} x^2 - 2xy + z^2 \\ x^2y + 3xz \end{pmatrix}, \quad S = \{(x,y,z) \in \mathbb{R}^3 \bigm| f(x,y,z) = (1,0)\}, \quad \text{and} \quad a = (1,0,0) \in \mathbb{R}^3 .$$

Suppose there exists an open set $U \subseteq \mathbb{R}^3$ such that $a \in U$ and the set $S \cap U$ is a manifold. Find a basis for the tangent space of this manifold at the point a.

$\boxed{10}$ 4. Compute $\displaystyle\int_{x=0}^{1} \int_{y=x}^{1} \sqrt{1+y^2}\, dy\, dx$.

$\boxed{10}$ 5. Attempt **ONLY ONE** part of this question. Part (b) is easier *and is only worth 50% of the points of the first part.* If you attempt part (b), then you will only receive credit for part (b) and part (a) *will not* be graded.

Both parts are special cases of a theorems we proved in class. Please provide a complete proof here, and not simply quote the result from class. You may, of course, use all other lemmas and results from class / homework that are independent of what you are asked to prove.

(a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that both $\partial_1 f$ and $\partial_2 f$ exist and are continuous. Show that f is differentiable.

(b) Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be differentiable, and $a, b \in \mathbb{R}^d$. Show that there exists $\xi \in \mathbb{R}^d$ such that

$$f(b) - f(a) = (b - a) \cdot \nabla f(\xi) .$$

$\boxed{10}$ 6. Let $f, h\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions, and suppose the set $C = \{x \in \mathbb{R}^2 \,|\, f(x) = 0\}$ is a C^1 curve (aka a 1-dimensional manifold). Let $a = (a_1, a_2) \in C$, and suppose $\partial_2 f(a) \neq 0$. If h attains a constrained minimum at a, subject to the constraint $f = 0$, then show that

$$\partial_1 f(a) \partial_2 h(a) - \partial_2 f(a) \partial_1 h(a) = 0 .$$

NOTE: This is a special case of the Lagrange multiplier theorem which proved in class. Please provide a complete proof here, and not just deduce it from the general theorem we proved. You may, however, use any result from class/homework that is independent of the above result.

$\boxed{10}$ 7. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function, and e_1, e_2, e_3 denote the standard basis vectors in $\mathbb{R}^3$. For any $R > 0$ define $\gamma_R\colon [0, 2\pi] \to \mathbb{R}^3$ by $\gamma_R(t) = R\sin(t)e_1 + R\cos(t)e_3$, and let Γ_R be the curve traced out by γ_R. (Note Γ_R is a circle in the x_1-x_3 plane with center 0 and radius R, traversed counter clockwise with respect to an observer located at e_2.) Compute

$$\lim_{R\to 0} \frac{1}{R^2} \oint_{\Gamma_R} F \cdot dl .$$

Your final answer may involve F and derivatives of F, but not any integrals or limits.

NOTE: You need not provide any proof or justification for your answer. However, if you have no justification, then you will get no partial credit whatsoever if your answer has even the slightest mistake in it. If you do provide a justification, it need not be completely rigorous, and will be used to assign partial credit in case your answer isn't completely correct.