# 21-272 Introduction to PDE: Midterm 2.

Fri 11/06/2015

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 40 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

10 1. Find the full Fourier series of the function $f(x) = x$ on the interval $(-\pi, \pi)$.

10 2. For a general function $f : [-\pi, \pi] \to \mathbb{R}$, find $\int_{-\pi}^{\pi} f(x)^2\, dx$ in terms of the coefficients of its full Fourier series. Use this to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$. [This was HW5, Q2(a).]

10 3. Let $D = \{x \in \mathbb{R}^2 \mid |x| \leqslant a\}$ be a disk of radius $a$. Let $u$ be a function such that $\Delta u = 0$ on the interior of $D$, and $u(x) \geqslant 0$ for all $x \in D$. If $0 < r < a$ and $|x| \leqslant r$, show that

$$u(x) \leqslant \Big(\frac{a+r}{a-r}\Big) u(0).$$

[Hint: Use the Poisson formula; this is the Harnack inequality and was on HW6 Q5]

10 4. Suppose $u$ solves the heat equation $\partial_t u - \kappa \partial_x^2 u = 0$ for $x \in (0, \pi)$ and $t > 0$ with Dirichlet boundary conditions $u(0, t) = u(\pi, t) = 0$ and initial data $u(x, 0) = f(x)$. You may assume $\kappa > 0$ and $\int_0^{\pi} |f(x)|^2\, dx < \infty$. True or false: Does there exist $\alpha > 0$ such that

$$\lim_{t \to \infty} e^{+\alpha t} \int_0^{\pi} u(x, t)^2\, dx = 0?$$

Prove it, or find a counter example. [Hint: Separate variables, and use results about the Fourier series. We've seen a similar result on the homework before, however, the convergence required in this question is stronger.]