21-272 Introduction to PDE: Midterm 1.

Fri 10/02/2015

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 40 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

$\boxed{10}$ 1. Let $D \subset \mathbb{R}^3$ be the region occupied by a fluid body (e.g. a lake), $u(x,t)$ be the instantaneous velocity of the fluid at point $x \in D$ and time t, and $\rho(x,t)$ be the concentration of some pollutant at time t and position $x \in \mathbb{R}^3$. Fick's law says that the rate of flow of the pollutant is proportional to the concentration gradient. Use this to derive a PDE for ρ. [This was on your homework.]

$\boxed{8}$ 2. (a) Use separation of variables and find the general (series form) solution of the equation $\partial_t^2 u + \partial_x^2 u = 0$ for $x \in [0, L]$, $t \geqslant 0$ with boundary conditions $u(0,t) = u(L,t) = 0$ for all t.

$\boxed{2}$ (b) If we restrict our attention to the rectangle $[0, L] \times [0, T]$, is it possible to additionally impose the boundary conditions $u(x, 0) = f(x)$ and $u(x, T) = g(x)$ for two given functions f and g? [Assume f and g are differentiable and $f(0) = f(L) = 0 = g(0) = g(L)$. No justification required for this part. Just guess. A correct answer is worth full credit, a blank answer is worth half credit, and an incorrect answer is worth no credit.]

$\boxed{10}$ 3. Suppose $\partial_t u + 6u\partial_x u + \partial_x^3 u = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = f(x)$. Find $\alpha \in \mathbb{R}$ so that the quantity

$$\int_{-\infty}^{\infty} [u(x,t)^3 + \alpha(\partial_x u(x,t))^2] \, dx$$

is constant in time. You may assume assume u and all its derivatives vanish rapidly at $x = \pm\infty$. [This is HW3 Q3(c), with an added hint.]

$\boxed{10}$ 4. Given a domain $\Omega \subseteq \mathbb{R}^3$ and a function $f : \partial\Omega :\to \mathbb{R}$, define $\mathcal{S} = \{v : \Omega \to R \mid v = 0 \text{ on } \partial\Omega\}$. Define

$$\mathcal{E}(v) = \frac{\int_\Omega |\nabla v|^2 \, dV}{\int_\Omega v^2 \, dV}.$$

If there exists a C^2 function $u \in \mathcal{S}$ which minimises $\mathcal{E}$ (i.e. $\mathcal{E}(u) \leqslant \mathcal{E}(v)$ for all $v \in \mathcal{S}$), then show that $-\Delta u = \lambda u$ where $\lambda = \mathcal{E}(u)$. [Hint: Let $w \in \mathcal{S}$ and let $g(\varepsilon) = \mathcal{E}(u + \varepsilon w)$. Show that $g'(0) = 0$, and use this to find the PDE for u. Half credit for doing it assuming $\Omega = [0, L] \subseteq \mathbb{R}$.]