Math 272: Homework.

Assignment 1: Assigned Fri 09/04. Due Fri 09/11

1. (a) Show by example that the divergence theorem *does not* hold if the region of integration is not bounded.

   (b) Suppose now that the vector function $f : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

   $$|f(x)| \leqslant \frac{1}{1 + |x|^3}$$

   for all $x \in \mathbb{R}^3$. Show that $\int_{\mathbb{R}^3} \nabla \cdot f \, dV = 0$.

2. Let $D \subset \mathbb{R}^3$ be the region occupied by a fluid body (e.g. a lake), $u(x,t)$ be the instantaneous velocity of the fluid at point $x \in D$ and time t, and $\rho(x,t)$ be the concentration of some pollutant at time t and position $x \in \mathbb{R}^3$. Fick's law says that the rate of flow of the pollutant is proportional to the concentration gradient. Use this to derive a PDE for ρ. [This is called the advection diffusion equation.]

3. *Strauss 1.3.2:* A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.

4. For this problem, identify $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* if for every $z \in \mathbb{C}$, $\lim_{\zeta \to 0} \frac{f(z+\zeta)-f(z)}{\zeta}$ exists. Note: The limit is taken as $\zeta \in \mathbb{C}$ approaches 0.

   If f is holomorphic, let $f' = \lim_{\zeta \to 0} \frac{f(z+\zeta)-f(z)}{\zeta}$ denote the "complex" derivative of f. Amazingly, if f is holomorphic, then f' is also holomorphic! (In contrast, if f is only differentiable, f' need not even be continuous, let alone differentiable.) Assuming f and f' are holomorphic, show that $u \stackrel{\text{def}}{=} \operatorname{Re} f$ is harmonic (i.e. show $\Delta u = 0$). [Hint: First find a relation between $\partial_x f$ and $\partial_y f$.]

5. Consider the equation $\partial_t u - (0.1)\partial_x^2 u = 0$ on the interval $x \in (-1, 1)$ with boundary conditions $u(-1,t) = 0 = u(1,t)$ and initial data $u(x, 0) = 1 - |x|$.

   (a) Guess what the solution will look like as time evolves.

   (b) Numerically solve this equation using MATLAB (or Python), and obtain solutions plots confirming (or disproving) your guess above. [To do this in MATLAB, look up the function `pdepe`; the first example in the online documentation can quickly be adapted in this setting.]

Assignment 2: Assigned Fri 09/11. Due Fri 09/18

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function so that $f(1) = 0$, $f(x+4) = f(x)$ and $f(1-x) = -f(1+x)$ for all $x \in \mathbb{R}$.

   (a) Let $u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct))$. Show that u satisfies the wave equation $\partial_t^2 u - c^2 \partial_x^2 u = 0$ on the interval $[-1, 1]$, with initial data $u(x, 0) = f(x)$, $\partial_t u(x, 0) = 0$, and with boundary conditions $u(-1,t) = u(1,t) = 0$ for all $t > 0$.

   (b) Find a formula for the solution of the wave equation $\partial_t^2 u - c^2 \partial_x^2 u = 0$ on the interval $[-1, 1]$ with initial data $u(x, 0) = 1 - |x|$, $\partial_t u(x, 0) = 0$ for $x \in [-1, 1]$, and with boundary conditions $u(-1,t) = u(1,t) = 0$. Draw plots, and contrast your answer with the solution to the heat equation from the previous homework.

Consider an oddly shaped wire loop $\Gamma \subseteq \mathbb{R}^3$. It turns out that amongst all surfaces in $\mathbb{R}^3$ with boundary Γ, the one that *minimises* the strain is the graph of a harmonic function! The next problem outlines a proof of this.

2. Given a domain $\Omega \subseteq \mathbb{R}^2$ and a function $f : \partial\Omega :\to \mathbb{R}$, define $\mathcal{S} = \{v : \Omega \to R \mid v = f \text{ on } \partial\Omega\}$. Define

   $$\mathcal{E}(v) = \int_\Omega |\nabla u|^2 \, dA.$$

   If there exists a C^2 function $u \in \mathcal{S}$ which minimises $\mathcal{E}$ (i.e. $\mathcal{E}(u) \leqslant \mathcal{E}(v)$ for all $v \in \mathcal{S}$), then show that $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$. [Think of the graph of f as the above wire loop Γ, and the graph of v to be any surface with boundary Γ, and $\mathcal{E}(v)$ as the "strain" in the surface. HINT: Let $w : \Omega \to \mathbb{R}$ be any function such that $w = 0$ on $\partial\Omega$. Let $g(\varepsilon) = \mathcal{E}(u + \varepsilon w)$. Show that $g'(0) = 0$, and use this to show that $\int_\Omega w \Delta u = 0$ for all such functions w.]

3. If we replaced $\mathcal{E}$ above with $\mathcal{E}'$ defined by

   $$\mathcal{E}'(v) = \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 - gu\Big) \, dA,$$

   for a given function g, then show that the minimising function u now satisfies $-\Delta u = g$ in Ω with $u = f$ on $\partial\Omega$.

Instead of minimising the strain as above, if we minimise the surface area, we get the equation satisfied by a soap film. This is the next problem.

4. If $\mathcal{E}$ in the previous problem is replaced with

   $$\mathcal{A}(v) = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dA,$$

   find a PDE and boundary conditions satisfied by the minimiser u. [We know from calculus that $\mathcal{A}(v)$ is the surface area of the graph of v.]

Assignment 3: Assigned Fri 09/18. Due Fri 09/25

1. *(Strauss 1.4.6)* Two homogeneous rods have the same cross section, specific heat c, and density ρ, but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux $k\partial_x u$ at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.

   (a) Find the equilibrium temperature distribution in the composite rod.

   (b) Sketch it as a function of x in case $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but

2. Let D be a region in $\mathbb{R}^3$, and $c, r > 0$ be constants, and a, f be functions depending only on the spatial variables x_1, x_2 and x_3. Show that solutions to the PDE
$$\partial_t^2 u - c^2 \Delta u + a^2 u + r\partial_t u = f$$
with Dirichlet boundary conditions $u = 0$ on ∂D, and initial data
$$u(x, 0) = \varphi(x), \qquad \partial_t u(x, 0) = \psi(x)$$
are unique. That is, if u_1 and u_2 are two solutions to the above PDE, with the same boundary conditions and initial data, show that they are equal. [Hint: Suppose u_1 and u_2 are two solutions. Set $v = u_1 - u_2$. Now try and cook up some 'energy' which will help you show v is 0. Note, the energy you cook up (if you do it right) won't be conserved! It will however decrease with time.]

3. Solitary waves (or solitons) are waves that travel great distances without changing shape. Tsunami's are one example. Scientific study began with Scott Russell in 1834, who followed such a wave in a channel on horseback, and was fascinated by it's rapid pace and unchanging shape. In 1895, Kortweg and De Vries showed that the evolution of the profile is governed by the equation
$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0.$$
For this question, suppose u is a solution to the above equation for $x \in \mathbb{R}$, $t > 0$. Suppose further that u and all derivatives (including higher order derivatives) of u decay to 0 as $x \to \pm\infty$.

   (a) Let $p = \int_{-\infty}^{\infty} u(x, t)\, dx$. Show that p is constant in time. [Physically, p is the momentum of the wave.]

   (b) Let $E = \int_{-\infty}^{\infty} u(x, t)^2\, dx$. Show that E is constant in time. [Physically, E is the energy of the wave.]

   (c) It turns out that the KdV equation has *infinitely many* conserved quantities. The energy and momentum above are the *only two* which have any physical meaning. Can you find a non-trivial conserved quantity that's not a linear combination of p and E?

4. The *forward time, centered space* difference scheme for the heat equation is:
$$\frac{U_j^{m+1} - U_j^m}{\Delta t} - \kappa\Big(\frac{U_{j+1}^m + U_{j-1}^m - 2U_j^m}{(\Delta x)^2}\Big) = 0, \quad \text{for } 1 \leqslant j \leqslant N-1 \tag{1}$$
$$U_j^{m+1} = U_j^m = 0 \quad \text{for } j \in 0, N. \tag{2}$$

   (a) Explicitly find a matrix A so that $U^{m+1} = AU^m$, and use a computer to numerically solve this equation.

   (b) Let $s = \kappa\Delta t/(\Delta x)^2$. Verify numerically for various initial data that when $s < 1/2$ the solution U^m decays (exponentially) to 0.

   (c) Verify numerically that if $s > 1/2$ then the numerically computed U^m is "unphysical": Explicitly, exhibit non-negative initial data for which the solution becomes negative. Also find initial data for which the solution grows without bound.

   (d) Suppose now A is the "infinite" matrix corresponding to (1) for all $j \in \mathbb{Z}$. Show that for all $k \in \mathbb{N}$, $1 - 2s(1 - \cos(k\Delta x))$ is an eigenvalue of A. Consequently, if $s > 1/2$ show that the matrix A has an eigenvalue λ with $|\lambda| > 1$.

Assignment 4: Assigned Fri 10/02. Due Fri 10/09

In low density (or high energy) plasmas energy transport is modelled by the *Kompaneets* equation: $\partial_t n + \partial_x F = 0$ where $F = F(x, n) = (2x - x^2)n - n^2 - x^2 \partial_x n$. (Note n is a function of x, so $\partial_x F = \partial_1 F + \partial_2 F \partial_x n$.) Here $x > 0$ is proportional to the magnitude of the wave vector (and hence the energy) of a photon, and $n(x, t)$ represents the number density of photons with energy x. Assume for this question that $F \to 0$ as $x \to 0$ and $x \to \infty$.

1. (a) Let $N(t) \stackrel{\text{def}}{=} \int_0^\infty n(x, t)\, dx$ be the total photon number. Show that N is constant in time.
   (b) Let $f = n/x^2$. Show $x^2 \partial_t f = \partial_x [x^4(\partial_x f + f + f^2)]$. [This is how the equation is usually stated in Physics.]
   (c) Let $h(x_1, x_2) = x_2 \ln x_2 - (1+x_2)\ln(1+x_2) + x_1 x_2$, and define the *quantum entropy* H by $H(t) = \int_0^\infty x^2 h(x, f)\, dx$. Show that
   $$\partial_t H = -\int_0^\infty x^4 f(1+f)[\partial_x \partial_2 h(x, f)]^2\, dx \leqslant 0.$$
   (d) Show that all nonnegative stationary solutions are of the form $f_\mu(x) = 1/(e^{x+\mu} - 1)$ for $\mu \geqslant 0$. [Hint: For stationary solutions $\partial_x \partial_2 h(x, f) = 0$.]
2. *(Strauss 4.1.4)* Consider waves in a resistant medium that satisfy the problem
   $$\begin{gathered} \partial_t^2 u = c^2 \partial_x^2 u + r \partial_t u \quad \text{for } 0 < x < l, \\ u = 0 \quad \text{at both ends}, \\ u(x, 0) = \phi(x), \quad \partial_t u(x, 0) = \psi(x), \end{gathered}$$
   where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.
3. *(Strauss 4.2.2)* Consider the equation $\partial_t^2 u = c^2 \partial_x^2 u$ for $0 < x < L$, with the boundary conditions $u_x(0, t) = 0$, $u(L, t) = 0$.
   (a) Show that the eigenfunctions are $\cos((n + 1/2)\pi x/L)$.
   (b) Write the series expansion for a solution $u(x, t)$.

This problem outlines a short proof of a few basic properties of the Discrete Fourier Transform.

4. Let $N \in \mathbb{N}$ and define $e_n \in \mathbb{C}^N$ to be the vector $e_n = \sum_{k=0}^{N-1} \exp(2\pi i n k/N)\hat{e}_k$, where $\hat{e}_k$ is the k^{th} elementary basis vector.
   (a) Show that $\{e_n \mid 0 \leqslant n < N\}$ forms an orthogonal basis of $\mathbb{C}^N$.
   (b) For $x \in \mathbb{C}^N$, and define the discrete Fourier transform of x (denoted by X) to be the element of $\mathbb{C}^N$ with coordinates $X_n = \sum_{k=0}^{N-1} x_k \exp(-2\pi i n k/N)$. (Note $X_n = \langle x, e_n \rangle$.) Show that $x_k = \frac{1}{N} \sum_{n=0}^{N-1} X_n \exp(+2\pi i n k/N)$.
   (c) Find a real symmetric matrix T for which e_n's are the eigenvectors. [Hint: Try the discrete version of the (periodic) second derivative.]

Assignment 5: Assigned Fri 10/09. Due Fri 10/16

1. (a) Determine whether the following Fourier series converge pointwise, L^2 or uniformly. When it converges, determine the limit. [It might help to draw pictures to confirm your guess. When applicable, use the Fourier series convergence theorems from class instead of proving everything from scratch.]
      i. Fourier sine series of $\cos x$ on $[0, \pi]$.
      ii. Fourier cosine series of $\sin x$ on $[0, \pi]$.
      iii. Full Fourier series of x on $[-\pi, \pi]$.
      iv. Fourier cosine series of $f(x) = 1$ for $x < \pi/2$ and $f(x) = 0$ otherwise on $[0, \pi]$.
   (b) Compute $1 - 1/3 + 1/5 - 1/7 \cdots$. [Hint: Compute the Fourier coefficients of the last part above.]
2. (a) Show that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. [Hint: Find an L^2 function for which $|\hat{f}(n)|^2 = \frac{1}{n^2}$, and use Parseval's identity.]
   (b) Compute $\sum_{n=1}^\infty \frac{1}{n^4}$.
3. Let f be a piecewise differentiable $2L$-periodic (complex valued) function with $\int_{-L}^{L} |f|^2 < \infty$ and $\int_{-L}^{L} |f'|^2 < \infty$.
   (a) Show that $|\hat{f}(n)| \leqslant C/n$ for some constant C that is independent of n.
   (b) *(Unrelated)* If $\int_{-L}^{L} f = 0$, show that $\|f\|^2 \leqslant \frac{\pi^2}{L^2} \|f'\|^2$.
4. Let f be a continuous function on $[0, L]$, and A_n be the Fourier Sine coefficients of f.
   (a) Suppose u satisfies the heat equation on $[0, L]$ with Dirichlet boundary conditions $u(0, t) = u(L, t) = 0$ and initial data $u(x, 0) = f(x)$. Show that for any $t > 0$ the function u is infinitely differentiable as a function of x.
   (b) Explain why your proof above won't work for the wave equation.
5. Suppose u satisfies the heat equation $\partial_t u - \partial_x^2 u = 0$ for $x \in (0, L)$ and $t > 0$ with Neumann boundary conditions $\partial_x u(0, t) = \partial_x u(L, t) = 0$ and initial data $u(x, 0) = f(x)$. You may assume $\|f\| < \infty$.

   Consider now the limit of the functions $u(\cdot, t)$ as $t \to \infty$. Namely, for any fixed $t > 0$, view the slice of u at time t as a function of x. Then consider the limit of these functions as $t \to \infty$. Does this limit exist in the pointwise, uniform or L^2 sense? Prove it. Also compute the limit.

Assignment 6: Assigned Fri 10/16. Due Fri 10/23

1. Let $D \subset \mathbb{R}^2$ be the *exterior* of the disk of radius a and center 0 (i.e. $D = \{(x,y) \mid x^2+y^2 \geqslant 1\}$). Find the solution of the PDE $\Delta u = 0$ in D, with boundary conditions

$$u(a,\theta) = f(a,\theta) \quad \& \quad \lim_{r\to\infty} u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(a,\phi)\,d\phi.$$

   Express your answer as an integral involving f (without an infinite sum). [Hint: Option 1: Switch to polar coordinates, and separate variables, and suffer. Option 2: Let $v(r,\theta) = u(1/r,\theta)$.]

2. Let $\Omega = B(0,a)$, and $f : \partial\Omega \to \mathbb{R}$ be differentiable, and $g(a,\theta) = -\partial_\theta f(a,\theta)$.
   (a) Suppose $\Delta v = 0$ in Ω and $\partial_r v = g$ on $\partial\Omega$. Find $Q = Q_a(r,\theta)$ so that

$$v(r,\theta) = \int_{-\pi}^{\pi} g(\phi) Q_a(r,\theta-\phi)\,d\phi.$$

   (b) Express Q, and the above formula in Cartesian coordinates.
   (c) Suppose u solves $\Delta u = 0$ in Ω with $u = f$ on $\partial\Omega$. Find a relation between ∇u and ∇v. [Hint: Find a relationship between the Poisson kernel P and Q. The answer should be something familiar.]

3. *(Separation of variables on an annulus)* Given $a, b \in \mathbb{R}$ such that $0 < a < b$, let the annulus A be defined by $A = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$. Given two periodic functions f, g, find a function u such that $\Delta u = 0$ in A, with boundary conditions $u(a,\theta) = f(\theta)$ and $u(b,\theta) = g(\theta)$. You may leave your answer as an infinite series involving the appropriate Fourier coefficients of f and g.

4. (a) Let $R > 0$ and $D = \{x \in \mathbb{R}^2 \mid |x - x_0| < a\}$. If $\Delta u = 0$ in D, then show

$$u(x_0) = \frac{1}{\text{area}\, D}\int_D u.$$

   (b) *(Liouville's theorem)* Suppose $\Delta u = 0$ in *all of* $\mathbb{R}^2$, and u is bounded, then show that u must be constant.

5. *(Harnack inequality)* Let $D = \{x \in \mathbb{R}^2 \mid |x| \leqslant a\}$ be a disk of radius a. Let u be a function such that $\Delta u = 0$ on the interior of D, and $u(x) \geqslant 0$ for all $x \in D$. If $|x| \leqslant r < a$, then show that

$$u(0)\frac{a-r}{a+r} \leqslant u(x) \leqslant u(0)\frac{a+r}{a-r}.$$

   [This is called the *Harnack Inequality*, and is a rather striking result. In words it says that if u is a positive harmonic function on a disk of radius a, then the oscillation on any smaller disk can be uniformly controlled! The hint is to use the Poisson formula.]

Assignment 7: Assigned Fri 10/23. Due Fri 10/30

1. Show that the Poisson kernel is an approximate identity. Explicitly show:
   (a) $P_a(r,\theta) \geqslant 0$ and $\int_{-\pi}^{\pi} P_a(r,\theta)\,d\theta = 1$.
   (b) For any $\delta > 0$, $\lim_{r\to a^-} [\int_{-\pi}^{-\delta} P_a(r,\theta)\,d\theta + \int_{\varepsilon}^{\pi} P_a(r,\theta)\,d\theta] = 0$.

2. Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, and $u \in C^2(\Omega) \cap C(\bar\Omega)$. Let $b : \Omega \to \mathbb{R}^d$ be a bounded function.
   (a) Suppose $-\Delta u + b \cdot \nabla u < 0$, show that u can not attain a maximum in the interior of Ω.
   (b) *(Weak maximum principle)* Suppose $-\Delta u + b \cdot \nabla u \leqslant 0$, show that

$$\max_{x\in\Omega} u(x) \leqslant \max_{x\in\partial\Omega} u(x).$$

   Hint: Find a function φ so that $-\Delta\varphi + b\cdot\nabla\varphi < 0$, and set $u = v + \varepsilon\varphi$.

3. Suppose $u \in C^2(\Omega) \cap C(\bar\Omega)$ and $-\Delta u \leqslant 0$ in Ω.
   (a) *(Hopf lemma)* If u attains it's maximum at a point $a \in \partial\Omega$, show that $\frac{\partial u}{\partial \hat n}(a) > 0$ unless u is constant. [Hint: Use weak maximum principle and either the Harnack inequality or the separation of variables on the annulus.]
   (b) *(Strong maximum principle)* If u attains an interior maximum show that u is constant.

4. Let $D = [0,L] \times [0,L]$, and b be some (bounded) vector function. Suppose u is a solution of the PDE $-\Delta u + b \cdot \nabla u = b_1$, with $u = 0$ on ∂D. (Note: b_1 is the first component of the vector b.) Find a constant c which *only depends on* L such that $u(x) \leqslant c$ for all $x \in D$.

Assignment 8: Assigned Fri 11/06. Due Fri 11/13

1. Find the Greens function for the upper half plane $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Use this to find a function u such that $\Delta u = 0$ in Ω and $u(x, 0) = \operatorname{sign}(x)$.
2. Find the Greens function for the half sphere $\Omega = \{x \in \mathbb{R}^3 \mid |x| < a \;\&\; x_3 > 0\}$.
3. Suppose $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$. Show that

$$u(x) = \int_\Omega G(x, y) f(y)\, dV(y) + \int_{\partial\Omega} G(x, y) g(y)\, dS(y),$$

   where G is the Greens function on Ω.
4. The *Neumann function* $\mathcal{N}$ of a domain Ω is a function $\mathcal{N} : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ such that:
   (i) $\mathcal{N}(x, x_0)$ is C^2 as a function of x, provided $x \neq x_0$.
   (ii) $\Delta_x \mathcal{N}(x, x_0) = 0$ (here Δ_x denotes the Laplacian in x, treating x_0 as fixed).
   (iii) $H(x) \stackrel{\text{def}}{=} \mathcal{N}(x, x_0) - N(x - x_0)$ extends to a harmonic function on all of Ω (including $x = x_0$). (Here N is the Newton potential).
   (iv) $\partial_{\hat{n}} \mathcal{N}(x, x_0) = 1/\operatorname{area}(\partial\Omega)$ for $x \in \partial\Omega$.

   Suppose $\Delta u = f$ in Ω and $\partial_{\hat{n}} u = g$ on $\partial\Omega$ for some functions f, g such that $\int_{\partial\Omega} g\, dS = 0$. Find a formula expressing u in terms of f, g and $\mathcal{N}$, and prove it.
5. Find the Neumann function for the two dimensional ball of radius a. Use this to find the analogue of the Poisson formula with Neumann boundary condition.

Assignment 9: Assigned Fri 11/13. Due Fri 11/20

1. Show that $G(\cdot, t)$ is an approximate identity as $t \to 0$.
2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a (nice) function, and u solve $\partial_t u - \kappa\Delta u = 0$ for $x \in \mathbb{R}^d$ and $t > 0$ with initial data $u(x, 0) = f(x)$. Find a formula expressing u in terms of f (as some convolution). [Assume $d = 2$ or $d = 3$ if you are uncomfortable with d-dimensional integrals.]
3. (a) Suppose $\partial_t u - \kappa\partial_x^2 u = 0$ for $x > 0$, $t > 0$, with initial data $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = 0$ and decay at infinity. Find a formula expressing u in terms of f (as some convolution). [Hint: Can you extend f to a function defined on all of $\mathbb{R}$ so that the solution to the heat equation (on all of $\mathbb{R}$) and initial data f will automatically satisfy the required boundary condition at $x = 0$?]
   (b) Do part (a) assuming $\partial_t u - \kappa\partial_x^2 u = g$ instead. Here $g : [0, \infty) \times [0, \infty)$ is a given function.
   (c) Do part (a) with boundary conditions $u(0, t) = h(t)$ instead. [Hint: Find a function w so that $v = u - w$ satisfies the equation (and boundary conditions) for which you already have a formula.]
   (d) Do part (a) with Neumann boundary conditions $\partial_x u(0, t) = 0$ instead.
4. Let f be a function, and suppose $u(x, t) = \int_{-\infty}^{\infty} f(y) G(x - y, t)\, dy$.
   (a) If f is bounded and $\lim_{x\to\pm\infty} f(x) = 0$, show that for every $t > 0$ we also have $\lim_{x\to\pm\infty} u(x, t) = 0$.
   (b) If instead $\int_{-\infty}^{\infty} |f| < \infty$, must we still have $\lim_{x\to\pm\infty} u(x, t) = 0$? Prove it.
5. Compute $G * N$, where $G : \mathbb{R}^3 \to \mathbb{R}$ is defined by $G(x) = \frac{1}{(2\pi)^{3/2}} \exp(-|x|^2/2)$ and $N(x) = -1/(4\pi|x|)$ is the three dimensional Newton potential. [There is a clean and easy way to do this.]

Assignment 10: Assigned Fri 11/27. Due Fri 12/04

1. Suppose $\partial_t u - \partial_x^2 u = 0$ for $x \in (0, L)$ and $t > 0$ with initial data $u(x, 0) = 1$ and Dirichlet boundary conditions $u(0, t) = u(L, t) = 0$ for all $t > 0$. True or false: For every $x \in (0, L)$ the function $t \mapsto u(x, t)$ is a strictly decreasing function of t. Prove or disprove your answer.
2. The *reaction diffusion equation* is

$$\partial_t u - \partial_x^2 u = u(1 - u). \tag{3}$$

Physically, this is used to describe evolution of temperature in an exothermic reaction (e.g. burning fuel), or model the spread of a species.

   (a) Let R be the space-time rectangle $(0, L) \times (0, T)$, and $\partial_P R$ denote the parabolic boundary. If u_1, u_1 are two bounded solutions to (3) such that $u_1, u_2 \in C(\bar{R})$, and $u_1 \leqslant u_2$ on $\partial_P R$, then show that $u_1 \leqslant u_2$ on R.

   (b) If u solves (3) in R and $0 \leqslant u \leqslant 1$ on $\partial_P R$, prove $0 \leqslant u \leqslant 1$ on all of R.

3. Let u be a solution to $\partial_t^2 u - c^2 \partial_x^2 u = 0$ for $x \in \mathbb{R}$ and $t > 0$ with initial data $u(x, 0) = \varphi(x)$ and $\partial_t u(x, 0) = \psi(x)$. Find φ and ψ so that $\partial_t u(x, 10) = 0$, $u(x, 10) = 1$ for $x \in (0, 1)$ and $u(x, 10) = 0$ for $x \notin (0, 1)$.
4. (a) Find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2 \partial_x^2 u = 0$, for $x > 0, t > 0$, with Dirichlet boundary conditions $u(0, t) = 0$ and initial data $u(x, 0) = \varphi(x)$, and $\partial_t u(x, 0) = \psi(x)$.

   (b) For the previous subpart, sketch the domain of dependence of a point (x, t). [Do two cases: $x < ct$ and $x \geqslant ct$. Your pictures will be different!]

   (c) Find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2 \partial_x^2 u = g(x, t)$, for $x > 0, t > 0$, with Neumann boundary conditions $\partial_x u(0, t) = 0$ and initial data $u(x, 0) = \varphi(x)$, and $\partial_t u(x, 0) = \psi(x)$. Sketch the domain of dependence of a point (x, t).

Assignment 11: Assigned Fri 12/04. Due Fri 12/11

1. Let u solve $\partial_t^2 u - c^2 \Delta u = 0$ for $x \in \mathbb{R}^2$ and $t \geqslant 0$, with initial data $u(x, 0) = \varphi(x)$ and $\partial_t u(x, 0) = \psi(x)$. Show that

$$u(x,t) = \frac{1}{2\pi c} \int_{B(x,ct)} \frac{\psi(y)}{\left(c^2 t^2 - |y - x|^2\right)^{1/2}} \, dA(y) + \partial_t \Big[\frac{1}{2\pi c} \int_{B(x,ct)} \frac{\varphi(y)}{\left(c^2 t^2 - |y - x|^2\right)^{1/2}} \, dA(y) \Big]$$

[Hint: For $x \in \mathbb{R}^2$, $z \in \mathbb{R}$, treat $(x, z) \in \mathbb{R}^3$. Let $v(x, z, t) = u(x, t)$, and observe that v satisfies the three dimensional wave equation, for which you know the Kirchoff formula. Simplify that to obtain the above.

It's also important to observe the difference between the solution formulae in 2D and 3D. The formula in 3D involve integrals over the *surface* of the ball of radius ct. In 2D, however, the formula involves integrals over the *solid* disk of radius ct.]

Suppose a signal is emitted at point $x_0 \in \mathbb{R}^3$ at time 0. In 3D, at time t, the signal will only be seen (or heard) by an observer that is exactly a distance ct away from x_0. You can easily make this precise by choosing φ to be an approximation to the δ function at the origin and using the 3D Kirchoff formula.

In two dimensions, however, the signal will only be seen (or heard) by an observer that is *at most* a distance of ct away from x_0. In fact, an observer at y_0 will hear nothing before time $|x_0 - y_0|/c$, hear the loudest signal at time exactly $|x_0 - y_0|/c$, and at later times hear a softer whose amplitude eventually decays to 0. This is the content of the next two problems.

2. Let u solve $\partial_t^2 u - c^2 \Delta u = 0$ for $x \in \mathbb{R}^2$ and $t \geqslant 0$, with initial data $u(x, 0) = \varphi(x)$ and $\partial_t u(x, 0) = \psi(x)$. Suppose φ and φ are bounded functions that vanish outside $B(0, R)$. Show that for any $x \in \mathbb{R}^2$ there exists a constant $C > 0$ such that $u(x, t) \leqslant C/t$. In particular, $u(x, t) \to 0$ as $t \to \infty$.
3. Suppose in the previous question u was a solution of the *three dimensional* wave equation instead. Show that $u(x, t) = 0$ for all sufficiently large t.