21-272 Introduction to PDE: Midterm 2.

Mon 12/14

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You may bring in a formula sheet that fits on two sides of one US Letter sized piece of paper, provided it is handwritten. No print outs.*
- *You have 3 hours. The exam has a total of 8 questions and 90 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

$\boxed{10}$ 1. Let $f, g : (0, \pi) \to \mathbb{R}$ be two bounded continuous functions whose Fourier sine series are given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

respectively. Find

$$\int_0^\pi f(x)\, g(x)\, dx$$

in terms of the coefficients A_n's and B_n's. [No proof is required. But even a slightly incorrect formula will early you 0 credit unless you provide some justification.]

$\boxed{10}$ 2. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded region, and suppose u_1 and u_2 are both solutions of the PDE $-\Delta u = 1$ in Ω with boundary conditions

$$u + \frac{\partial u}{\partial \hat{n}} = 0 \quad \text{on } \partial\Omega.$$

Must $u_1 = u_2$ in Ω? Prove it, or find a counter example.

$\boxed{15}$ 3. Let Ω be the semi-circular disk expressed in polar coordinates by

$$\Omega = \{(r, \theta) \bigm| 0 < r < 1 \text{ and } 0 < \theta < \pi\}.$$

Suppose $\Delta u = 0$ in Ω with boundary conditions

$$u(r, 0) = 0, \quad \partial_\theta u(r, \pi) = 0, \quad \text{and} \quad u(1, \theta) = 1.$$

Use separation of variables to express u as an infinite series. (You should explicitly find all coefficients in the series, and not leave them as integrals.) [Recall in polar coordinates we have $\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u$.]

$\boxed{10}$ 4. Suppose $\partial_t u - \frac{1}{2}\partial_x^2 u = 0$ for $x \in \mathbb{R}$, $t > 0$, with initial data $u(x, 0) = f(x)$. Suppose $\int_{-\infty}^{\infty} |f| = 1$. Does there exist $C \in \mathbb{R}$ and $\alpha > 0$ such that

$$u(x, t) \leqslant \frac{C}{t^\alpha},$$

for all $x \in \mathbb{R}$ and $t > 0$? If yes, find α and C and prove it. If no, find a counter example.

$\boxed{10}$ 5. Find the Greens function for the upper half plane $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Use this to find a function u such that $\Delta u = 0$ in Ω and $u(x, 0) = \operatorname{sign}(x)$. [This was HW8 Q1]

$\boxed{10}$ 6. Let $R = (0, L) \times (0, T)$ and $\partial_P R = \{(x, t) \mid x = 0, \text{ or } x = L, \text{ or } t = 0\}$. Let $c : R \to \mathbb{R}$ be a continuous, bounded function. Suppose u satisfies

$$\partial_t u - \partial_x^2 u + c(x) u \geqslant 0$$

in R with $u \geqslant 0$ on $\partial_p R$, must $u \geqslant 0$ on all of R? Prove it, or find a counter example. [This was an intermediate step required in the proof of HW10 Q1]

$\boxed{10}$ 7. Let u solve $\partial_t^2 u - c^2 \Delta u = 0$ for $x \in \mathbb{R}^2$ and $t \geqslant 0$, with initial data $u(x, 0) = \varphi(x)$ and $\partial_t u(x, 0) = \psi(x)$. Find a formula for u in terms of φ and ψ. You may use without proof the Kirchoff formula in three dimensions. [This was HW11 Q1.]

8. Let $B = B(0,1) \subseteq \mathbb{R}^2$ be the unit disk, $B^* = B - \{(0,0)\}$ be the unit disk with the origin removed, and $\bar{B} = B \cup \partial B$.

$\boxed{3}$ (a) Suppose $u : B^* \to \mathbb{R}$ is a C^2 function such that $\Delta u = 0$ in B^* and $u = 0$ on ∂B. Must $u = 0$ in B^*? Prove it, or find a counter example. [Even though u is not defined on ∂B, by $u = 0$ on ∂B we mean for every $a \in \partial B$, $u(x) \to 0$ as $x \to a$.]

$\boxed{12}$ (b) Suppose $u : \bar{B} \to \mathbb{R}$ is continuous on all of $\bar{B}$, C^2 on B^* and satisfies $\Delta u = 0$ in B^* with $u = 0$ on ∂B. Must $u = 0$ in B^*? Prove it, or find a counter example. [Hint: Let $\varepsilon > 0$ and $v_\varepsilon = u - \varepsilon \ln|x|$.]