21-268 Multidimensional Calculus: Sample Questions on Integration.

Wed 09/23

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

Here are a few questions of the level you might find on the exam. Your actual exam WILL NOT be these same questions with the fonts and margins changed. To help you practice, I have approximate times (in multiples of 5 minutes) on each question and you should try and do each question in the time allotted.

1 Questions Similar to Results From Class / Homework

These questions are similar (or even identical) to material presented in class, or problems you did in the homework.

10 mins 1. Let $U \subseteq \mathbb{R}^2$ be the region enclosed by the curves $y = e^x$, $y = 0$, $x = 0$ and $x = 1$. Compute $\int_U (10 + x^2 - y^2)\, dA$.

10 mins 2. Compute $\int_0^1 \int_{x=0}^{x=\tan^{-1}(y)} 1\, dx\, dy$.

15 mins 3. Find the volume of the region bounded by $z = 9 - x^2 - y^2$, $x^2 + y^2 = 4$ and the x-y-plane.

10 mins 4. Let C be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$ (oriented counter-clockwise). Compute

$$\oint_C \sqrt{1 + x^3}\, dx + 2xy\, dy.$$

10 mins 5. Find the arc-length of the helix Γ defined by $\Gamma = \{(\cos t, \sin t, t) \mid 0 \leqslant t \leqslant 2\pi\}$.

15 mins 6. Let Σ be the portion of the unit sphere contained in an upward cone whose vertex is at the origin and has angle α. What is the surface area of Σ? [For extra practice, also find the volume of the region enclosed by Σ and the cone.]

5 mins 7. (a) Let $U \subseteq \mathbb{R}^2$ be a bounded region such that $0 \notin U$. Show that

$$\oint_{\partial U} \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dx = 0$$

You may assume ∂U is the finite union of piecewise C^1 curves.

5 mins (b) Let $\Gamma = \partial B(0, \varepsilon)$ oriented counter clockwise. Compute

$$\oint_{\Gamma} \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dx = 2\pi.$$

10 mins (c) Let Γ be a simple closed curve that encloses the origin, oriented counter clockwise. Show that

$$\oint_{\Gamma} \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dx = 2\pi.$$

[A related question was on your homework *before* you knew Greens theorem. You can of course follow the same strategy here, but a proof using Greens theorem is much simpler. Hint (which will not appear on an exam): Let $U' = U - B(0, \varepsilon)$, and use the previous parts.]

15 mins 8. Let $U \subseteq \mathbb{R}^2$ be a bounded domain whose boundary is the closed C^1 curve Γ. At any point $x \in \Gamma$, let $\hat{n} = \hat{n}(x) \in \mathbb{R}^2$ denote the outward pointing unit normal vector. If $v : U \to \mathbb{R}^2$ is C^1, show that

$$\int_U (\partial_1 v_1 + \partial_2 v_2)\, dA = \oint_{\partial U} v \cdot \hat{n}\, |d\ell|.$$

[HINT: Use Green's theorem. It is also instructive to compare this with the divergence theorem.]

10 9. If $\Sigma \subseteq \mathbb{R}^3$ is a closed surface and $v : \Sigma \to \mathbb{R}^2$ is C^1, must $\oint_\Sigma \nabla \times v\, dS = 0$? Prove it, or find a counter example.

2 Unfamiliar territory

These questions are probably not of the type you have seen before, but can be solved using the techniques you've learnt so far.

15 mins 10. Given a function f, find $\alpha, \beta, \gamma \in \mathbb{R}$ so that

$$\frac{d}{dt} \int_0^t f(s,t)\, ds = \alpha f(t,t) + \beta \int_0^t \partial_1 f(s,t)\, dt + \gamma \int_0^t \partial_2 f(s,t)\, dt.$$

3 Quid erat quod iterum?

These might be a bit harder...

20 mins 11. Let $U \subseteq \mathbb{R}^2$ be a bounded region, and suppose $h : U \to \mathbb{R}$ is a differentiable function such that the level sets $\Omega_c = \{x \in U \mid h(x) = c\}$ are all "concentric" connected closed curves. Show that

$$\int_U |\nabla h|\, dA = \int_m^M \operatorname{arc\,len}(\Omega_r)\, dr,$$

where $m = \min h$ and $M = \max h$. [HINT: Assume there exists a function θ so that $\nabla h \cdot \nabla \theta = 0$ and $\varphi = (h, \theta)$ is a coordinate change function. Now transform the integral on the left into (h, θ) coordinates.]

12. Let $U = \mathbb{R}^2 - \{(0,0)\}$.

10 mins (a) Find a C^1 function $v : U \to \mathbb{R}^2$ such that $\partial_1 v_2 - \partial_2 v_1 = 0$, but there does not exist a C^1 function $\varphi : U \to \mathbb{R}$ such that $v = \nabla \varphi$.

30 mins (b) If $v : U \to \mathbb{R}^2$ is a C^1 function such that $\nabla \times v = 0$ show that there exist $\varphi : U \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$v = \frac{\alpha}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \nabla \varphi$$

[HINT: Let $\Gamma \subseteq \mathbb{R}^2$ be the unit circle traversed counter clockwise, and $\alpha = \oint_\Gamma v \cdot d\ell$.]