21-268 Multidimensional Calculus: Sample Midterm 2 questions.

Wed 09/23

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

Here are a few questions of the level you might find on the exam. Your actual exam WILL NOT be these same questions with the fonts and margins changed. To help you practice, I have approximate times (in multiples of 5 minutes) on each question and you should try and do each question in the time allotted.

1 Questions Similar to Results From Class / Homework

These questions are similar (or even identical) to material presented in class, or problems you did in the homework.

10 mins

1. Let $f(x, y) = \frac{xy^3}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Compute $\partial_x \partial_y f(0, 0)$ and $\partial_y \partial_x f(0, 0)$.

10 mins

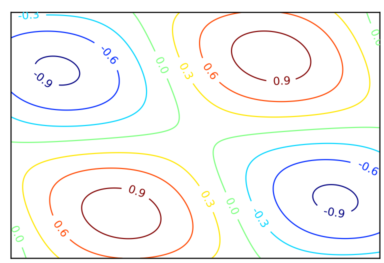
2. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is C^2 and define
$$g(r, \theta, \phi) = f(r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$
Compute $\partial_r \partial_\theta g$ in terms of r, θ, ϕ and partial derivatives of f.

10 mins

3. Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a differentiable function such that Hf is always positive semi-definite. Show that f is convex. [This, along with its converse, was on your homework.]

5 mins

4. The graph shown alongside shows various level sets of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. At which points do you think $\nabla f = 0$? At each of these points, guess whether f has a local minimum, local maximum or saddle.



10 mins

5. Find the maximum and minimum of the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$.

10 mins

6. Let $f(x, y, z) = x^2 + y^2 + z^2 + xz$. Find all points at which $\nabla f = 0$. Classify these as local maxima, minima, saddles (or neither).

5 mins

7. Consider the equations
$$xy + z^2w^2 = 1 \quad \text{and} \quad xz + y^2w^2 = 1.$$
Find $\partial_y w$.

10 mins

8. Given a differentiable function $f = f(x, y)$, we use the relations $x = r\cos\theta$ and $y = r\sin\theta$ to treat f as a function of r and θ. Compute $\partial_x f$ in terms of $\partial_r f$, $\partial_\theta f$, r and θ. [A version of this was on your homework.]

10 mins

9. Let $\varphi(x, y) = (u, v)$, where
$$u = e^{x^2-y^2}\cos(2xy), \quad v = e^{x^2-y^2}\sin(2xy),$$
Near the point $(u, v) = \varphi(1, 0) = (e, 0)$ can x and y be expressed as differentiable functions of u and v? If yes, let $(x, y) = \psi(u, v)$ and compute $\det(D\psi_{(e,0)})$.

$\boxed{\text{10 mins}}$ 10. Consider the system of equations

$$xyz\sin(xyz) + x + y + z = 0 \quad \text{and} \quad e^{xyz} + 2x + \sin(2y) + z = 1,$$

for which $x = y = z = 0$ is a solution. Near this point, find a combination of variables that can be solved for and expressed as differentiable functions of the remaining variables. Further, at the point $(0, 0, 0)$, compute the derivative of the function expressing your chosen variables as differentiable functions of the remaining.

$\boxed{\text{5 mins}}$ 11. Find the tangent space, tangent line and a normal vector to the curve $e^{x^2-y^2}\sin(2xy) = 1$ at the point $x = \sqrt{\pi}/2$, $y = \sqrt{\pi}/2$.

$\boxed{\text{5 mins}}$ 12. Find the tangent space, tangent plane and a normal vector to the surface $1 = e^{x^2-y^2}\sin(2xyz)$ at the point $x = \sqrt{\pi}/2$, $y = \sqrt{\pi}/2$, $z = 1$.

$\boxed{\text{10 mins}}$ 13. Find the tangent space, tangent line and two linearly independent normal vectors to the curve defined by the equations

$$1 = e^{x^2-y^2}\sin(2xyz) \quad \text{and} \quad \cos(x^2 - z^2) - \cos(y^2 - z^2) = z - 1$$

at the point $x = \sqrt{\pi}/2$, $y = \sqrt{\pi}/2$, $z = 1$.

$\boxed{\text{10 mins}}$ 14. Maximise y subject to the constraint $x^2 - xy + y^2 = 3$.

2 Unfamiliar territory

These questions are probably not of the type you have seen before, but can be solved using the techniques you've learnt so far.

$\boxed{\text{10 mins}}$ 15. Find the shortest distance between the plane $x + y + z = 0$ and the surface $z = x^2 + y^2 + 1$.

$\boxed{\text{10 mins}}$ 16. If $x_1, \ldots x_n > 0$, show that $\frac{1}{n}\sum_{i=1}^n x_i \geqslant (\prod_{i=1}^n x_i)^{1/n}$. [Hint: Minimise $\sum x_i$ subject to the constraint $\prod x_i = c$. If you find it a bit abstract, try it for $n = 3$ first.]

$\boxed{\text{10 mins}}$ 17. Let S be the surface $z = \frac{-1}{2x^2+3y^2}$. A ball is placed on this surface at the point $(x_0, y_0, \frac{-1}{2x_0^2+3y_0^2})$. In what direction will the ball roll? Where do you think the ball will eventually end up?

$\boxed{\text{10 mins}}$ 18. Let $S \subseteq \mathbb{R}^4$ be defined by the equations

$$x_1x_2x_3x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \quad \text{and} \quad x_4(x_1^4 + x_2^4 + x_3^4) + x_4^4 = 1.$$

At the point $(0, 0, 0, 1)$ find the tangent plane and tangent space of S. Find as many linearly independent normal vectors as possible at this point.

3 Quid erat quod iterum?

These might be a bit harder...

$\boxed{\text{10 mins}}$ 19. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2 and $\partial_x f(0,0) = \partial_y f(0,0) = 0$. Let $Hf_{(0,0)} = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. Let

$$g(t) = f\big(\operatorname{sign}(t)\sqrt{|t|}, \operatorname{sign}(t)\sqrt{|t|}\big)$$

Is f differentiable at 0? Prove or find a counter example.

$\boxed{\text{10 mins}}$ 20. Let $R = (0,1)^2 \subseteq \mathbb{R}^2$, and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable function and suppose $\operatorname{rank}(Df) = 2$ at all points in the domain of f. The set $S = \{f(x) \mid x \in R\}$ will describe a surface in $\mathbb{R}^3$. Find the tangent space of S in terms of Df. [Hint: Suppose there is a differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$ so that the surface S can be written implicitly as $\{g = 0\}$, and use the fact that the tangent space of S is exactly the kernel of Dg. (Note – on an actual exam, the hint will be much more cryptic or even entirely absent.)]