Math 268: Homework.

There is a firm 'no late homework' policy. I will often assign harder optional problems. I recommend doing (but not turning in) the optional problems. They often involve useful concepts that will come in handy as the semester progresses.

Assignment 1: Assigned Wed 09/02. Due Wed 09/09

1. Determine (without proving) whether the following sets are open and/or connected. Give a reasonably convincing reason.
   (a) $\{x \in \mathbb{R}^3 \mid 1 < |x| < 10\}$.
   (b) $\{x \in \mathbb{R}^3 \mid 1 \leqslant |x| < 10\}$.
   (c) $\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < |x_3|\}$.
   (d) $\{x \in \mathbb{R}^{268} \mid \sum_{i<268} x_i^2 < |x_{268}|\}$.
2. Decide whether $\{x \in \mathbb{R}^2 \mid x_1 x_2 > 0\}$ is open, and prove your answer.
3. (a) Let $f : \mathbb{R}^d \to \mathbb{R}$, and suppose $\lim_{x \to x_0} f(x) = \ell$. Given $v \in \mathbb{R}^d$, consider the one variable function g_v defined by $g_v(t) = f(x_0 + tv)$. True or false: For any $v \in \mathbb{R}^d$, $\lim_{t \to 0} g_v(t) = \ell$?
   (b) Prove your answer to the previous part (as rigorously as you are able).
   (c) Is the converse to the first part true? Namely, if for every $v \in \mathbb{R}^d$ we know $\lim_{t \to 0} g_v(t) = \ell$, then must $\lim_{x \to x_0} f(x) = \ell$? Justify.
4. (a) Does $\displaystyle\lim_{|x| \to 0} \frac{x_1 x_2}{|x|^2}$ exist? Justify. [Here $x \in \mathbb{R}^2$.]
   (b) Does $\displaystyle\lim_{|x| \to 0} \frac{x_1 x_2}{|x|}$ exist? Justify. [Here $x \in \mathbb{R}^2$.]
5. Let $U \subset \mathbb{R}^d$ be a domain, and $f, g : U \to \mathbb{R}^n$ be two functions.
   (a) Prove that if $\lim_{x \to x_0} f(x) = \ell$ and $\lim_{x \to x_0} g(x) = m$ then $\lim_{x \to x_0} f(x) + g(x) = l + m$.
   (b) If $\lim_{x \to x_0} f(x) = \ell$, then for every $i \in \{1, n\}$ must $\lim_{x \to x_0} f_i(x) = \ell_i$? Justify as best as you can. [Here f is the vector function $(f_1, f_2, \dots, f_n)$.]
   (c) Conversely, if for every $i \in \{1, n\}$ must $\lim_{x \to x_0} f_i(x) = \ell_i$, then must we have $\lim_{x \to x_0} f(x) = \ell$? Prove it rigorously.

The next few questions are a little more 'interesting'. Try it anyway, and feel free to exercise the "phone a friend" option.

6. Prove that any open set in $\mathbb{R}$ can be expressed as a countable union of open sets.
7. Prove that $\mathbb{R}$ is connected. [To prove it, you will need to know what the supremum of a set is. Look it up (or come ask me). This is a bit harder ....]

Assignment 2: Assigned Wed 09/09. Due Wed 09/16

1. For each of the following functions defined on $\mathbb{R}^2 - \{(0,0)\}$ determine if the function has a limit as $(x, y) \to (0, 0)$. Prove your answer.
   (a) $f(x, y) = \dfrac{x^3 y^4}{x^5 + y^6}$
   (b) $f(x, y) = \dfrac{x^2 y^3}{(x^4 + y^6)^{1/3}}$
2. Compute the following partial derivatives.
   (a) $\partial_x \tan^{-1}\left(\dfrac{y}{x}\right)$
   (b) $\partial_1 \ln|x|$
   (c) $\dfrac{\partial}{\partial x} x^y$
   (d) $\dfrac{\partial}{\partial y} x^y$
   (e) $\partial_t \displaystyle\int_0^t \exp(-|x|^2 - s^2)\, ds$
   (f) $\partial_1 \displaystyle\int_0^t \exp(-|x|^2 - s^2)\, ds$
3. Let $f(x, y) = x^2 y / (x^2 + y^2)$ for $(x, y) \neq 0$, and $f(0, 0) = 0$.
   (a) At what points in $\mathbb{R}^2$ do $\partial_x f$ and $\partial_y f$ exist? Justify.
   (b) At what points in $\mathbb{R}^2$ are $\partial_x f$ and $\partial_y f$ continuous? Justify.
   (c) Find all $v \in \mathbb{R}^2$ so that $D_v f(0, 0)$ exists. Justify.
4. Determine whether each of these statements are true or false. If true, prove it. If not, find a counter example. (Assume $d > 1$ below.)
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is necessarily continuous.
   (b) If $f : \mathbb{R}^d \to \mathbb{R}$ is such that $\partial_i f$ exists for some $i \in \{1, \dots, d\}$, then f is necessarily continuous.
   (c) If $f : \mathbb{R}^d \to \mathbb{R}$ is such that $\partial_i f$ exists for all $i \in \{1, \dots, d\}$, then f is necessarily continuous.
   (d) If $f : \mathbb{R}^d \to \mathbb{R}$ is such that $D_v f$ exists for all $D_v f$, then f is necessarily continuous.
5. Let $U \subseteq \mathbb{R}^d$ be a domain, and $f : U \to R$ be a function. Suppose for some $a \in U$, all the partial derivatives $\partial_1 f, \dots, \partial_d f$ exist *and are continuous* at a.
   (a) If for some $v \in \mathbb{R}^d$ the directional derivative $D_v f(a)$ exists, can you guess a formula expressing $D_v f(a)$ in terms of the partial derivatives of f and v.
   (b) For every $v \in \mathbb{R}^d$, must the directional derivative $D_v f(a)$ exist? Prove it, or find a counter example
   (c) Redo the previous part *without* the assumption that all partial derivatives of f are continuous.

Assignment 3: Assigned Wed 09/16. Due Wed 09/23

1. Compute the derivative (or Jacobian matrix) of the following functions.
   (a) $f(x, y) = x/y$
   (b) $f(x) = \ln|x|$, $x \in \mathbb{R}^2$.
   (c) $f(x, y) = \dfrac{xy}{1 - x - y}$
   (d) $z = (x - 2y)^5 e^{xy}$.
   (e) $f(x, y) = \tan^{-1} \frac{y}{x}$
   (f) $f(x) = 1/|x|$, $x \in \mathbb{R}^3$.
2. (a) Let $U = \mathbb{R}^2 - \{(x, y) \mid x \leqslant 0\}$. Define $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \tan^{-1}(y/x)$, and $\varphi : U \to V$ by $\varphi(x, y) = (r(x, y), \theta(x, y))$. Compute $D\varphi$ and $\det(D\varphi)$.
   (b) Let $V = \{(r, \theta) \in \mathbb{R}^2 \mid r > 0, |\theta| < \pi\}$. Let $x(r, \theta) = r \cos\theta$, and $y(r, \theta) = r \sin\theta$ and $\psi(r, \theta) = (x(r, \theta), y(r, \theta))$. Compute $D\psi$ and $\det(D\psi)$.
3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = |x|^2 \sin(1/|x|)$ when $x \neq 0$, and $f(0) = 0$.
   (a) Show that $\partial_1 f$ and $\partial_2 f$ are continuous in $\mathbb{R}^2$ except at 0.
   (b) Show that f is differentiable at all points in $\mathbb{R}^2$, including at $a = 0$.
4. Decide whether each of the following statements are true or false. If true, prove it. If false, provide a counter example.
   (a) Any differentiable function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous.
   (b) Any continuous function $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable.
5. (a) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable, and f' is increasing. Show that for all $x, y \in [a, b]$, $\theta \in [0, 1]$ we have $f(\theta x + (1 - \theta)y) \leqslant \theta f(x) + (1 - \theta)f(y)$.
   (b) Conversely, suppose for all $x, y \in [a, b]$, $\theta \in [0, 1]$ we have $f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1 - \theta)f(y)$. If f is differentiable, show that f' is increasing.

   A function that satisfies either of the above properties is called convex. (The second form is preferable since it doesn't assume differentiability.)

   (c) If $p, q > 1$ with $1/p + 1/q = 1$ and $x, y \in \mathbb{R}$ show that $xy \leqslant |x|^p/p + |y|^q/q$. [Hint: This is part (c) of a question. Google "Young's inequality" if you need more help.]
   (d) Does $\displaystyle\lim_{x \to 0} \frac{x_1 x_2}{(|x_1|^{4/3} + x_2^4)^{.99}}$ exist? How about $\displaystyle\lim_{x \to 0} \frac{x_1 x_2}{(|x_1|^{4/3} + x_2^4)^{1.01}}$ exist? Prove it.

Assignment 4: Assigned Wed 09/23. Due Never

1. For each of the functions y below express the derivative (with respect to x) as a product of two matrices, and evaluate the matrices at the given values of x below.
   (a) $y = \begin{pmatrix} u_1 u_2 - 3u_1 \\ u_2^2 + 2u_1 u_2 + 2u_1 - u_2 \end{pmatrix}$, $u = \begin{pmatrix} x_1 \cos(3x_2) \\ x_1 \sin(3x_2) \end{pmatrix}$ at $x = 0$.
   (b) $y = \begin{pmatrix} u_1^2 + u_2^2 - 3u_1 + u_3 \\ u_1^2 - u_2^2 + 2u_1 - 3u_3 \end{pmatrix}$, $u = \begin{pmatrix} x_1 x_2 x_3^2 \\ x_1 x_2^2 x_3^2 \\ x_1^2 x_2 x_3 \end{pmatrix}$, at $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
2. (a) Let $F(x, y) = xy$. Given two differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be defined by $\gamma(t) = (f(t), g(t))$. Observe $\frac{d}{dt}(fg) = D(F \circ \gamma)$. Compute this using the chain rule, and derive the product rule.
   (b) Derive the quotient rule from the chain rule using a method similar to the previous question.
3. Let $U = R^2 - \{(x, 0) \mid x \leqslant 0\}$, and $V = \{(r, \theta) \mid r > 0, \theta \in (-\pi, \pi)\}$. Given a differentiable function f defined on U, we treat it as a function of the coordinates x and y. Using the relation $x = r \cos\theta$ and $y = r \sin\theta$ for $(r, \theta) \in V$, we can now treat f as a function of r and θ.
   (a) Express $\partial_r f$ and $\partial_\theta f$ in terms of $\partial_x f$, $\partial_y f$, r and θ.
   (b) Let $u = x^2 + y^2$ and $v = y/x$. Explicitly express u, v in terms of r and θ and compute $\partial_r u$, $\partial_\theta u$, $\partial_r v$ and $\partial_\theta v$ directly. Verify that this agrees with the formulae in the previous part.
   (c) If further r, θ are functions functions of variables s and t, compute $\partial_s f$ in terms of $\partial_x f$, $\partial_y f$, r, θ, $\partial_s r$ and $\partial_s \theta$.
   (d) Express r, θ in terms of x and y.
   (e) Suppose now g is a differentiable function defined on V, which we treat as a function as a function of r and θ. Using the previous part, we can treat g as a function of x and y. Compute $\partial_x g$ and $\partial_y g$ in terms of $\partial_r g$, $\partial_\theta g$, x and y. Verify your formula is correct for the function $g(r, \theta) = r\theta$.
4. Suppose $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ are two C^1 functions. Let α, $\beta \in \mathbb{R}$ be two constants and A be a 3×3 (constant) matrix. Prove the following identities. [You don't need the limit definition, or ε-δ's for even one of these. You can do them all by using partial derivatives and your one variable differentiation rules.]
   (a) $D(au + bv) = aDu + bDv$
   (b) $D(Au) = A(Du)$.
   (c) $\nabla(u \cdot v) = (Du)^T v + (Dv)^T u$
   (d) $\partial_i(u \times v) = \partial_i u \times v + u \times \partial_i v$
5. Use the proof outline from class to give a full ε-δ proof of the chain rule.

Assignment 5: Assigned Wed 09/30. Due Wed 10/07

1. Do question 3 from homework 4.

In class we stated (but didn't prove) the fact that if mixed partials are equal they must be continuous. Here's a proof

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Suppose $h, k > 0$ and let $\delta = \delta(h,k)$ be defined by $\delta(h,k) = f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x,y)$.
   (a) Use the mean value theorem to show that there exists $\theta \in (0,h)$ so that
   $$\delta(h,k) = h(\partial_x f(x+\theta, y+k) - \partial_x f(x+\theta, y))$$
   (b) Conclude $\delta(h,k) = hk\partial_y\partial_x f(x+\theta, y+\eta)$ for some $\theta \in (0,h)$ and $\eta \in (0,k)$.
   (c) Show that $\lim_{(h,k)\to(0,0)} \frac{\delta(h,k)}{hk} = \partial_y\partial_x f(x,y)$, provided $\partial_y\partial_x f$ is continuous.
   (d) Similarly, show $\lim_{(h,k)\to(0,0)} \frac{\delta(h,k)}{hk} = \partial_x\partial_y f(x,y)$, if $\partial_x\partial_y f$ is continuous.
   (e) Conclude $\partial_x\partial_y f = \partial_y\partial_x f$ if $f \in C^2$.
3. If $f : \mathbb{R}^d \to \mathbb{R}$ is C^2 define $\Delta f = \sum_1^d \partial_i^2 f$ (this is called the Laplacian of f).
   (a) Suppose $u, v : \mathbb{R}^2 \to \mathbb{R}$ are C^2 functions such that $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Show that $|\nabla u|^2 = |\nabla v|^2$, $\nabla u \cdot \nabla v = 0$, and $\Delta u = \Delta v = 0$.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2, and let $g = \Delta f$. Compute $\Delta(f(u,v))$ in terms of g, u and v. [If you do it correctly, your answer will only involve g and $|\nabla u|^2$.]
4. We say a function $f : \mathbb{R}^d \to \mathbb{R}$ is *convex* if for every $x, y \in \mathbb{R}^d$ and $\theta \in [0,1]$ we have $f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1-\theta) f(y)$.

   Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^2. Show that f is convex if and only if its Hessian is positive semi-definite.

Optional problems, and details from class I left for you to check.

* Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.
   (a) Compute the eigenvalues of A in terms of a, b, c.
   (b) Show that A is positive semi-definite if and only if $a \geqslant 0$ and $ac - b^2 \geqslant 0$.
* Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. For any $a, b \in \mathbb{R}^d$ show that there exists ξ on the line segment joining a and b such that $f(b) - f(a) = (b-a)\cdot\nabla f(\xi)$.

Assignment 6: Assigned Wed 10/07. Due Wed 10/14

1. If $f : \mathbb{R}^d \to \mathbb{R}$ is C^2, $Df_a = 0$ and Hf_a is positive definite, then show that f has a local minimum at a. [We outlined a proof of this in class, but left a few details out. Polish them up and write a complete rigorous proof.]
2. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 \neq 0$. Find all critical points of $ax^2 + 2bxy + cy^2$. Find conditions on a, b, c that would classify this as a local minimum, maximum or saddle.
3. Find the critical points of each of these functions. For each critical point, determine whether it is a local maximum, local minimum, saddle or neither.
   (a) $\frac{x}{x^2+y^2}$
   (b) $[x^2 + (y+1)^2][x^2 + (y-1)^2]$
   (c) $\sin x \cosh y$
   (d) $x^2 - 2xy + y^2$
4. (a) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. Compute
   $$\lim_{h\to 0} \frac{1}{4h^2}(f(x+h, y+h) - f(x-h, y+h) + f(x-h, y-h) - f(x+h, y-h))$$
   and express your answer in terms of derivatives of f.
   (b) Do the same for
   $$\lim_{h\to 0} \frac{1}{h^2}(f(x+h, y) + f(x, y+h) + f(x-h, y) + f(x, y-h) - 4f(x,y)).$$
5. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^3 function, $a, h \in \mathbb{R}^d$ and define $g(t) = f(a+th)$.
   (a) Using the third order (one dimensional) Taylor expansion for $g(1) - g(0)$, show that there exists a point ξ on the line segment joining a and $a+h$ such that
   $$f(a+h) = f(a) + \sum \partial_i f(a) h_i + \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(a) h_i h_j + \frac{1}{6}\sum_{i,j,k=1}^d \partial_i\partial_j\partial_k f(\xi) h_i h_j h_k$$
   (b) Conclude there exists a function $R_3(h)$ such that
   $$f(a+h) = f(a) + \sum \partial_i f(a) h_i + \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(a) h_i h_j + \frac{1}{6}\sum_{i,j,k=1}^d \partial_i\partial_j\partial_k f(a) h_i h_j h_k + R_3(h)$$
   where $\lim_{h\to 0} R_3(h)/|h|^3 = 0$. [This can of course be continued inductively to obtain higher order Taylor expansions of f.]

Assignment 7: Assigned Wed 10/14. Due Wed 10/21

1. *(Spherical coordinates)* Let $V = \{(r, \theta, \phi) \mid r > 0, \theta \in (-\pi, \pi), \phi \in (0, \pi)\}$ and define $\varphi(r, \theta, \phi) = (x, y, z)$ where $x = r \sin\phi \cos\theta$, $y = r \sin\phi \sin\theta$ and $z = r\cos\phi$. (Geometrically, ϕ is the angle between (x, y, z) and the positive z-axis, and θ is the angle between the projection (x, y) and the positive x-axis.)
   (a) Let $U \subseteq \mathbb{R}^3$ be the set of points where either $x > 0$ or $y \neq 0$. Show $\det(D\varphi) \neq 0$ in V, and conclude (by the inverse function theorem) that r, θ, φ can be (locally) expressed as differentiable functions of x, y, z.
   (b) Explicitly express r, θ, ϕ as functions of x, y, z and show that $\varphi : V \to U$ is bijective (and hence a coordinate change function). Let $(r, \theta, \phi) = \psi(x, y, z)$ denote the inverse function. Compute $\det D\psi$ explicitly and verify $\det D\psi \neq 0$ in U.
   (c) If f is a differentiable function compute $\partial_x f$, $\partial_y f$ and $\partial_z f$ in terms of $\partial_r f$, $\partial_\theta f$, $\partial_\phi f$, r, θ and ϕ.
   (d) If g is a differentiable function compute $\partial_r g$, $\partial_\theta g$ and $\partial_\phi g$ in terms of $\partial_x g$, $\partial_y g$ and $\partial_z g$, x, y and z.
2. For each of the equations below near the given point, which variables can be solved for and expressed as differentiable functions of the remaining variables (according to the implicit function theorem). For each of these variables, compute all the partials at the given point. (That is, if you say w, x can be expressed as differentiable functions of y, z, compute $\partial_y w$, $\partial_z w$, $\partial_y x$ and $\partial_z x$ at the given point.)
   (a) $x^2 + y^2 - \cos(xy) = 0$ near $(1, 0)$.
   (b) $e^{xz} + y\sin(yz) + z = 0$ near $(0, 0, -1)$.
   (c) $\sin(xy) + \sin(yz) + \sin(xz) = 0$ and $e^{xyz} + x + y + z = 2$, near $(0, 0, 1)$.
3. Decide whether each of the following implicitly defined sets are curves or surfaces. At the given point, find a basis of the tangent space, and as many many linearly independent normal vectors as possible. Also find the tangent line (or tangent plane).
   (a) $x\sin(x) = y + xe^y$ at $(0, 0)$.
   (b) $\ln(xy) = y - x$ at $(1, 1)$
   (c) $x\sin y + y\sin z + z\sin x = 0$ at $(0, \pi, 2\pi)$.
   (d) $z^2 = x^2 + y^2 - 1$ and $2(x-1) + y - z = 0$ at $(1, 0, 0)$.
4. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be two differentiable functions, $c, d \in \mathbb{R}$. Consider the two surfaces $\Gamma = \{x \in \mathbb{R}^3 \mid f(x) = c\}$ and $\Delta = \{x \in \mathbb{R}^3 \mid g(x) = d\}$. Suppose $C = \Gamma \cap \Delta$ is a curve, $a \in C$ and that the vectors $\nabla f(a)$ and $\nabla g(a)$ are linearly independent. Find the tangent space of C at a in terms of ∇f and ∇g. Verify your formula by explicitly computing it when $f(x, y, z) = z^2 - x^2 - y^2 + 1$ and $g(x, y, z) = 2(x-1) + y - z$ at the point $(1, 0, 0)$.

Assignment 8: Assigned Wed 10/21. Due Wed 10/28

1. Parametrize the following curves.
   (a) $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, where $a, b > 0$ are constants.
   (b) $\cos x \cos y = 1/2$ for $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
2. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a differentiable function so that $\gamma(t)$ represents the position of a particle at time t. Let $v(t) = \gamma'(t)$ denotes the velocity, and $a(t) = v'(t)$ denote the acceleration. As we've seen in class, if γ parametrises a curve Γ, then $v(t)$ is tangent to Γ at the point $\gamma(t)$.
   (a) If $|v(t)| = 1$ for all t, show that $a(t)$ and $v(t)$ are perpendicular.

   If when moving while keeping the *magnitude* of the velocity constant, a particle experiences acceleration it is because the path taken by the particle is curved. The sharper the curve, the more the acceleration experienced (think about the force you feel when driving around a sharp curve).

   Definition: If a curve Γ is parametrized by the function γ, then we define the *curvature* at the point $\gamma(t)$ by $\kappa = \frac{1}{|\gamma'(t)|}|(\frac{\gamma'(t)}{|\gamma'(t)|})'| = \frac{1}{|v|}|(\frac{v}{|v|})'|$.

   If $|\gamma'(t)| = 1$ for all t, then the curvature is exactly the magnitude of the acceleration. (Note, one can show that the curvature κ only depends on the curve, and not the parametrization.)
   (b) Compute the curvature at any point on a circle of radius R.
   (c) Compute the curvature of the curve $y^2 = x^2 - 1$ at the point $(x, 1)$.
   (d) If $d = 3$, show $\kappa^2 = |v|^{-6}(|a|^2|v|^2 - (a \cdot v)^2)$, and hence conclude $\kappa = \frac{|a \times v|}{|v|^3}$.
3. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Sometimes when maximising a function in the region $\{g \leqslant c\}$, the maximum (or minimum) could be attained on the boundary $\{g = c\}$. In this case you can of course find all *interior* local maxima and minima by solving $\nabla f = 0$ and looking at Hf. For maxima and minima on the boundary, it is often convenient to use Lagrange multipliers.

4. Find the absolute maxima and minima of e^{-xy} on the set $x^2 + 4y^2 \leqslant 1$.
5. Let $S \subseteq \mathbb{R}^3$ be the surface $z^2 = x^2 + y^2 - 1$, and $(a, b, c) \in S$. There are exactly two lines that pass through (a, b, c) that are *completely contained* in the surface S. Find them. Express them as $\ell_i = \{(a, b, c) + t(u_i, v_i, w_i) \mid t \in \mathbb{R}\}$ for $i \in \{1, 2\}$ and two linearly independent vectors (u_1, v_1, w_1) and (u_2, v_2, w_2). [Hint: What is the intersection of S with it's tangent plane? Start with $(a, b, c) = (1, 0, 0)$.]
6. *(Optional challenge)* If Γ is any curve contained on the sphere $x^2 + y^2 + z^2 = 1$, then show that the curvature at any point on Γ is at least 1.

Assignment 9: Assigned Wed 10/28. Due Never

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and $\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\}$ be the graph of f. Show that the curvature of Γ is $|f''|/(1 + (f')^2)^{3/2}$.

   (b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be C^2, $c \in \mathbb{R}$ and Γ be the curve $\{(x, y) \mid g(x, y) = c\}$. Suppose further $\nabla g \neq 0$ on the curve Γ. Show that the curvature of Γ is

   $$\frac{|\partial_x^2 g(\partial_y g)^2 + \partial_y^2 g(\partial_x g)^2 - 2\partial_x\partial_y g \partial_x g \partial_y g|}{|\nabla g|^3}.$$

   HINT: Let $(a, b) \in \Gamma$, and suppose $\partial_y g(a, b) \neq 0$. Use the implicit function theorem to write Γ as the curve $y = f(x)$ near the point (a, b). Now express f'' in terms of derivatives of g by implicit differentiation, and use the previous part.

2. For each of the curves below, find the local maxima and minima of the curvature.
   (a) A circle of radius R.
   (b) A straight line.
   (c) The parabola $y = x^2$.
   (d) The ellipse $x^2/a^2 + y^2/b^2 = 1$, for $a, b > 0$.
   (e) The curve $\cos x \cos y = 1/2$ in the region $x, y \in (-\pi/2, \pi/2)$.

3. Maximise the volume of an open box given the surface area is $3a^2$. (That is, maximise xyz under the constraint $xy + 2(yz + zx) = 3a^2$.)

4. Maximise the volume of a cylinder given that the total surface area is $6\pi a^2$. [For fun, check if the proportions of your optimal cylinder agrees with your standard coke can; if not, write to Coco-cola with a proposal to save money and the environment...]

5. Let $p, q > 1$ be such that $1/p + 1/q = 1$. In the region $x, y > 0$ maximise xy subject to the constraint $x^p/p + y^q/q = C$. Use this to give a different proof of Young's inequality from homework 3.

6. *(Cauchy-Schwartz inequality)* If $x, y \in \mathbb{R}^n$ show $|x \cdot y| \leqslant |x||y|$. [HINT: Maximise $x \cdot y$ subject to the constraint $|x| = a$ and $|y| = b$.]

Assignment 10: Assigned Wed 11/04. Due Wed 11/11

1. Do question 1 from assignment 9.

2. Do question 4 from assignment 9.

3. In each of the cases below, find $\int_R f(x, y)\, dA$.
   (a) $f(x, y) = 16 - x^2 - y^2$, and R is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(2, 1)$.
   (b) $f(x, y) = y$, where R is bounded by the curves $x = y^2$ and $y = x - 2$.
   (c) $f(x, y) = x^2 + 2y$, where R is bounded by the curves $y = x$, $y = x^3$ in the region $x \geqslant 0$.

4. Compute the integrals integrals:
   (a) $\displaystyle\int_{x=-1}^{1} \int_{y=1}^{2} \sin(xy^2)\, dy\, dx.$
   (b) $\displaystyle\int_{y=0}^{1} \int_{x=-\cos^{-1} y}^{\cos^{-1} y} e^{\sin x}\, dx\, dy.$

   [Just for fun, try plugging these into a computer. Most computer algebra systems won't be able to evaluate these integrals symbolically.]

5. Compute both iterated integrals of the function

   $$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

   on the region $[0, 1] \times [0, 1]$. Are they equal? [HINT: First compute $\partial_x \partial_y \tan^{-1}(y/x)$.]

Assignment 11: Assigned Wed 11/11. Due Wed 11/18

1. Evaluate $\int_R f(x,y,z)\,dV$ in the following cases:
   (a) $f(x,y,z) = x^2y^2z$ where R is the cylinder $x^2+y^2<1$, $0<z<1$.
   (b) $f(x,y,z) = x^2+z^2$, where R is the pyramid with vertices $(\pm1,\pm1,0)$ and $(0,0,1)$.
2. (a) Let $D = B(0,1) \subseteq \mathbb{R}^2$ be the two dimensional disk of radius 1 and center $(0,0)$. For what $p \in \mathbb{R}$ is $\displaystyle\int_D \frac{1}{|x|^p}\,dA < \infty$. [Here $x = (x_1,x_2) \in \mathbb{R}^2$.]
   (b) Let $S = B(0,1) \subseteq \mathbb{R}^3$ be the three dimensional sphere with radius 1 and center $(0,0,0)$. For what $p \in \mathbb{R}$ is $\displaystyle\int_S \frac{1}{|x|^p}\,dV < \infty$. [Here $x = (x_1,x_2,x_3) \in \mathbb{R}^3$.]
3. Let $D \subseteq \mathbb{R}^2$ represent an irregular plate whose density is given by $\rho(x,y)$. Let $\ell \subseteq \mathbb{R}^2$ be a straight line, representing a knife edge upon which D is balanced. The magnitude of the *torque* experienced when D is balanced on ℓ is given by $T_\ell = \int_D \rho d\,dx\,dy$. Here $d = p\cdot\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to ℓ, and $p = p(x,y)$ is the vector from (x,y) to the closest point on ℓ. The plate D will balance on a knife edge along ℓ if (and only if) $T_\ell = 0$.
   (a) Given $a \in \mathbb{R}$, let $\ell = \{(a,t) \mid t \in \mathbb{R}\}$. Show that $T_\ell = 0$ if and only if $a = (\int_D x\rho\,dA)/\int_D \rho dA$.
   (b) Given $b \in \mathbb{R}$ let $\ell = \{(t,b) \mid t \in \mathbb{R}\}$. Show that $T_\ell = 0$ if and only if $b = (\int_D y\rho\,dA)/\int_D \rho dA$.

   The point (a,b) above for which $T_\ell = 0$ is called the *center of mass* of D.
   (c) Find the center of mass of the triangle with vertices $(0,0)$, (a,b), $(0,c)$ that has a uniform density. [Assume $0<a<c$ and $b>0$.]
   (d) If ℓ is *any line* (not necessarily parallel to the coordinate axis) passing through the center of mass of D must $T_\ell = 0$? Prove or disprove.
4. Let $U \subseteq \mathbb{R}^2$ be a (bounded) oddly shaped region containing the origin. Define $C \subseteq \mathbb{R}^3$ to be the cone with vertex $(0,0,h)$ and base U given by
$$C = \Big\{(x,y,z) \;\Big|\; 0<z<h,\ \Big(\frac{hx}{h-z}, \frac{hy}{h-z}\Big) \in U\Big\}$$
   Find $\operatorname{vol}(C)$ in terms of h and $\operatorname{area}(U)$. [Hint: Let $u = hx/(h-z)$, $v = hy/(h-z)$ and $w = z$, and transform $\int_C 1\,dx\,dy\,dz$ into u, v, w coordinates.]
5. Compute $\displaystyle\int_{\mathbb{R}^2} \frac{1}{y^2+4\sqrt{x^2+y^2}+4}\,\frac{dx\,dy}{\sqrt{x^2+y^2}}$. [Hint: Let $x = u^2-v^2$ and $y = 2uv$ and change coordinates.]
6. Compute $\int_0^\infty \frac{\sin x}{x}\,dx$. [Hint: Substitute $1/x = \int_0^\infty e^{-xy}\,dy$ above and switch the order of integration. This is one situation where the hypothesis of Fubini's theorem *won't* be satisfied; however, flipping the order of the integrals can still be justified using other methods. Extra kudos for justifying it!]

Assignment 12: Assigned Wed 11/18. Due Wed 11/25

1. A non-uniform wire is bent along the semi-circle $x^2+y^2=1$ with $y \geqslant 0$. The density of the wire is given by $\rho(x,y) = 2-y$. Find the total mass and the *center of mass* of the wire. Is the center of mass on the wire? [You've derived a formula for the center of mass of a plate previously. See if you can use your intuition from there to guess a formula for the center of mass in this context.]
2. Let $S \subseteq \mathbb{R}^3$ be a surface whose boundary is the closed curve Γ. Amperes law says that the *total current* passing through the surface S is given by $\frac{1}{\mu_0}\oint_\Gamma B\cdot d\ell$, where μ_0 is the magnetic constant and $B : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic field. Compute the current through the rectangle with corners $(-1,\pm1,-1)$ and $(1,\pm1,1)$ when
$$B(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}.$$
3. Let $\Gamma \subseteq \mathbb{R}^2$ be a closed, piecewise C^1 curve that does not pass through the origin. Define the *winding number* of Γ about the origin to be
$$W(\Gamma) = \frac{1}{2\pi}\oint_\Gamma \frac{-y\,dx + x\,dy}{x^2+y^2}$$
   (a) Compute the winding number of a circle of radius R and center $(0,0)$, traversed counter clockwise. What happens if you traverse it clockwise instead?
   (b) Compute the winding number of a circle of radius 1 and center $(0,2)$ traversed in either direction.

   The remainder of this problem is devoted to showing that the winding number is an integer. Let $\gamma : [0,1] \to \Gamma$ be a parametrization of Γ and write $\gamma(t) = (x(t),y(t))$. Assume $y(0)=0$ and $x(0)>0$ for simplicity. Let t_1 be the first time when $x(t_1)=0$. Note $y(t_1)\neq 0$, since $(0,0)\notin\Gamma$, and so we let t_2 be the first time after t_1 when $y(t_2)=0$. Again $x(t_2)\neq0$, so let t_3 be the first time after t_2 when $x(t_3)=0$. Continue this, and assume that after finitely many steps we obtain an even number N so that $t_N = 1$ and $y(t_N) = y(0) = 0$.
   (c) Show that $\displaystyle\int_{t_i}^{t_{i+1}} \frac{-y(t)x'(t)+x(t)y'(t)}{x(t)^2+y(t)^2}\,dt = \pm\frac{\pi}{2}$.
   (d) If i is even, show $\displaystyle\int_{t_i}^{t_{i+2}} \frac{-y(t)x'(t)+x(t)y'(t)}{x(t)^2+y(t)^2}\,dt = \pm\pi$ if $\operatorname{sign}(x_ix_{i+2})<0$ and 0 otherwise.
   (e) Show $W(\Gamma)\in\mathbb{Z}$.
4. *(Optional Challenge that is worth a reward)* If $\Gamma \subseteq \mathbb{R}^3$ is a C^2 closed curve, show that $\oint_\Gamma \kappa\,|d\ell|$ is multiple of 2π. Here κ is the curvature defined in your previous homework. [This is called the *turning number* of the curve Γ.]

Assignment 13: Assigned Wed 11/25. Due Wed 12/02

1. Let Γ be the curve parametrized by $\gamma(t) = (a\cos^3 t, a\sin^3 t)$, for $t \in [0, 2\pi]$. Compute the area of the region enclosed by Γ.
2. Let $P \subseteq \mathbb{R}^2$ be a (not necessarily convex) polygon whose vertices, ordered counter clockwise, are (x_1, y_1), ..., (x_N, y_N). Show that

$$\text{area}(P) = \frac{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_Ny_1 - x_1y_N)}{2}.$$

   [This is called the *surveyors formula*. While the statement involves only elementary coordinate geometry, it isn't as easy to prove directly this way. Hint: Use Greens theorem.]
3. Let $U, V \subset \mathbb{R}^2$ be domains, $\varphi : U \to V$ be C^2, and $F : V \to \mathbb{R}^2$ be C^1. Define $G : U \to \mathbb{R}^2$ by $G(x) = (D\varphi_x)^T(F \circ \varphi(x))$. Show that

$$\partial_1 G_2 - \partial_2 G_1 = [(\partial_1 F_2 - \partial_2 F_1) \circ \varphi] \det(D\varphi)$$

   [This was a detail in class used in the proof of Greens theorem.]
4. In each of the following cases, show that

$$\int_U (\partial_x Q - \partial_y P)\, dA \neq \oint_{\partial U} P\, dx + Q\, dy.$$

   Also explain why this does not contradict Greens theorem.
   (a) $U = \{(x, y) \mid y > 0\}$, $P = \dfrac{1}{1 + x^2}$, and $Q = 0$.
   (b) $U = B(0, 1) \subseteq \mathbb{R}^2$, $P = \dfrac{-y}{x^2 + y^2}$, and $Q = \dfrac{x}{x^2 + y^2}$.

Assignment 14: Assigned Wed 12/02. Due Wed 12/09

1. (a) Let $U \subseteq \mathbb{R}^2$ be a domain, and $f : U \to \mathbb{R}$ be C^1, and Σ be the graph of f (i.e. $\Sigma = \{(x, y, z) \mid (x, y) \in U, \ \& \ z = f(x, y)\}$). Show that

$$\text{area}(\Sigma) = \int_U \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dA.$$

   (b) Compute area(Σ) when $U = B(0, r)$, and $f(x, y) = x^2 + y^2$.
2. (a) Let $f : [a, b] \to (0, \infty)$ be C^1, and $\Sigma \subset \mathbb{R}^3$ be the surface formed by rotating the graph of f about the x-axis. Explicitly,

$$\Sigma = \{(x, y, z) \bigm| x \in [a, b] \text{ and } y^2 + z^2 = f(x)^2\}.$$

   Show that $\text{area}(\Sigma) = 2\pi \displaystyle\int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$.
   (b) Find the surface area of a cylinder with base radius r and height h.
   (c) Find the surface area of a cone with base radius r and height h.
3. Let $a > b > 0$, and Σ be the torus obtained by rotating a circle with center $(a, 0, 0)$ and radius b about the z axis. Parametrize the surface, and evaluate the surface integral that computes area(Σ).
4. Let $\Sigma = \{(x, y, z) \mid y^2 + z^2 = 1, -1 < x < 1 \ \& \ z > 0\}$, and $F = e_3$. Compute $\int_\Sigma F \cdot \hat{n}\, dS$, where at any point on Σ, $\hat{n}$ is the upward pointing unit normal.
5. (a) Let A be a 3×3 matrix, and $u, v \in \mathbb{R}^3$. Show $Au \times Av = \text{adj}(A)^T(u \times v)$, where adj($A$) is the adjoint of the matrix A.
   (b) Let $U \subseteq \mathbb{R}^3$ be a domain, $(\Sigma, \hat{n})$ be an oriented surface. Let $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be an injective C^1 function such that $\det(D\psi) > 0$ on all of U, and $F : \psi(\Sigma) \to \mathbb{R}^3$ be a vector field. Show that

$$\int_{\psi(\Sigma)} F \cdot \hat{n}\, dS = \int_\Sigma (\text{adj}(D\psi)(F \circ \psi)) \cdot \hat{n}\, dS$$

Assignment 15: Assigned Wed 12/09. Due Wed 12/16

1. Let $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ be C^1 vector fields, $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 scalar functions. Prove the following identities.
   (a) $\nabla(fg) = f\nabla g + g\nabla f$.
   (b) $\nabla \cdot (fu) = (\nabla f) \cdot u + f\nabla \cdot u$.
   (c) $\nabla \times (fu) = f\nabla \times u + (\nabla f) \times u$
   (d) $\nabla \times (u \times v) = u(\nabla \cdot v) - v(\nabla \cdot u) + (v \cdot \nabla)u - (u \cdot \nabla)v$
2. Let $F = (2x, y^2, z^2)$.
   (a) Compute $\int_\Sigma F \cdot \hat{n}\, dS$, where $\Sigma \subseteq \mathbb{R}^3$ is the sphere of radius 1.
   (b) Compute $\oint_\Gamma F \cdot d\ell$, where Γ is the intersection of Σ above and the plane $x + 2y + 3z = 0$.
3. In each of the following cases show that

$$\int_U (\nabla \cdot v)\, dV \neq \int_{\partial U} v \cdot \hat{n}\, dS.$$

   Also explain why this does not contradict the divergence theorem.
   (a) $U = B(0,1) \subseteq \mathbb{R}^3$, and $v = x/|x|^4$.
   (b) $U = \{x \in \mathbb{R}^3 \mid z > x^2 + y^2\}$, and $v(x, y, z) = (0, 0, e^{-z})$.
4. *(Greens identity)* Let $U \subseteq \mathbb{R}^3$ be a bounded domain whose boundary is a C^1 surface. If $f, g : U \to \mathbb{R}$ be C^2, show

$$\int_U (f\Delta g)\, dV = \int_{\partial U} f\nabla g \cdot \hat{n}\, dS - \int_U (\nabla f \cdot \nabla g)\, dV.$$

   Recall $\Delta g = \sum_i \partial_i^2 g = \nabla \cdot (\nabla g)$.
5. Suppose $U, V \subseteq \mathbb{R}^3$ are domains, $\varphi : U \to V$ is C^2 and $v : V \to \mathbb{R}^3$ is C^1. Define $w = \operatorname{adj}(D\varphi)(v \circ \varphi) : U \to \mathbb{R}^3$. Show that

$$\nabla \cdot w = \det(D\varphi)(\nabla \cdot v) \circ \varphi.$$

   [This was a detail used in the proof of the divergence theorem.]
6. *(Optional challenge)* Suppose $u : \mathbb{R}^3 \to \mathbb{R}^3$ is C^1. Show that $\nabla \cdot u = 0$ if and only if there exists a C^1 vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ such that $u = \nabla \times v$.