21-268 Multidimensional Calculus: Final.

Dec 20, 2015

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 9 questions and 100 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are roughly in order of difficulty. Depending on your intuition, you might find some of the later ones easier than the earlier ones.*
- *Good luck and happy holidays.*

[5] 1. (a) Define what it means for a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ to be differentiable at a point $a \in \mathbb{R}^3$.

[5] (b) If $f(x, y, z) = \Big(\dfrac{y}{x^2 + y^2}\ ,\ x \sin(y + z)\Big)$, then compute the derivative of f at the point $(1, 2, -2)$.

[10] 2. For the function $f(x, y) = x \sin y + \cos y$, classify the point $(0, 0)$ as a local maximum, minimum, saddle or none of the above.

[10] 3. Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$. Compute $\displaystyle\int_U (x^2 + y^2 + z^2)\, dV$.

[10] 4. Let $\Gamma \subseteq \mathbb{R}^2$ be a simple closed curve and U be the region enclosed by Γ. If $\varphi : U \to \mathbb{R}$ is a C^1 function, must

$$\oint_\Gamma \nabla\varphi \cdot d\ell = 0?$$

Prove it, or find a counter example.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function and define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(t, 2t)$.

[5] (a) If $\partial_1 f(0, 0) = a$ and $\partial_2 f(0, 0) = b$, find $g'(0)$ in terms of a and b.

[5] (b) If $\partial_1^2 f(0, 0) = 0$ and $\partial_2^2 f(0, 0) = 0$, must $g''(0) = 0$? Prove it, or find a counter example.

[10] 6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function, and $\Sigma = \{x \in \mathbb{R}^3 \mid f(x) = 0\}$ be a C^1 surface. Let $a \in \mathbb{R}^3$ be some point that is *not* on Σ, and $b \in \Sigma$ be the point on Σ that is closest to a. True or false:

The vector $b - a$ is normal to the surface Σ at the point b.

Prove it, or find a counter example. [You may assume $\nabla f(b) \neq 0$.]

7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$ if $(x, y) \neq (0, 0)$, and $f(x, y) = 0$ otherwise.

[7] (a) For any $v \in \mathbb{R}^2$, does the directional derivative $D_v f(0, 0)$ exist? If yes compute it.

[8] (b) Is f differentiable at the point $(0, 0)$? Justify.

8. Let $U \subseteq \mathbb{R}^2$ be a domain and $f : U \to \mathbb{R}$ be a C^1 function. Let $\Sigma \subseteq \mathbb{R}^3$ be the graph of f defined by

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U \text{ and } z = f(x, y)\}.$$

[7] (a) Express the surface integral $\int_\Sigma 1\, dS$ as an area integral over U of an expression involving f and or derivatives of f. [No justification is required, however, an incorrect answer without justification will receive no partial credit.]

[8] (b) If $f(x, y) = x^2 + y\sqrt{3}$ and U is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$ compute area(Σ).

[10] 9. *(Trickier!)* Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be C^1, let $\Sigma \subseteq \mathbb{R}^3$ be a disk of radius r and center a which is normal to the vector $\hat{n}$. Let Γ be the boundary of Σ, oriented counter-clockwise with respect to the oriented surface $(\Sigma, \hat{n})$. Compute

$$\lim_{r \to 0} \frac{1}{r^2} \oint_\Gamma v \cdot d\ell$$

in terms of $\hat{n}$, $v(a)$, $\partial_1 v(a)$, $\partial_2 v(a)$ and $\partial_3 v(a)$.