CHAPTER 1

Limits and Continuity

1. Open sets in $\mathbb{R}^d$.

- We assume throughout the convention that a vector $x \in \mathbb{R}^d$ has coordinates $(x_1, x_2, \dots, x_d)$.
- Let $|x| = \sqrt{\sum x_i^2}$ be the length of the vector x.
- Let $B(a, r) = \{x \in \mathbb{R}^d \mid |x - a| < r\}$ be the *open ball* of radius r.

Definition 1.1. A set $U \subset \mathbb{R}^d$ is open if for every $a \in U$ there exists $r > 0$ such that $B(a, r) \subset U$.

Definition 1.2. An open set $U \subset \mathbb{R}^d$ is *connected* if it can not be expressed as the union of two non-empty, disjoint open sets.

While this definition is the "official" one, it is a little harder to grasp (e.g. try proving $\mathbb{R}^d$ is connected)! Instead we will use the circular, but intuitive definition to work with instead.

Definition 1.3. A set $U \subset \mathbb{R}^d$ is connected if for any x and y in U, there exists a *continuous* path that connects x and y that stays entirely within the set U.

This definition is circular as we have not yet defined a continuous path.

Definition 1.4. A domain (sometimes called an open domain) is an open connected set.

Most functions we study will have an open connected set as their domain of definition.

2. Limits

Definition 2.1. We say $\lim_{x\to a} f(x) = l$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \varepsilon$.

In English, this roughly translates to the statement *"By making x close enough to a, we can make $f(x)$ arbitrarily close to l"*

Remark 2.2. If f is only defined on an open set U, then we also insist $x, a \in U$ above.

The standard theorems about limits (sums, products, quotients) from one variable calculus still hold in this context. You should make a list and practice a few ε-δ proofs.

Proposition 2.3. *If $\lim_{x\to a} f(x) = l$ then for every $v \in \mathbb{R}^d$ with $v \neq 0$, we must have $\lim_{t\to 0} f(a + tv) = l$.*

Proof. Pick $\varepsilon > 0$. We know $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$. Choose $\delta_1 = \delta/|v|$. Now it immediately follows that $0 < t < \delta$ implies $|f(a + tv) - l| < \varepsilon$. $\square$

The converse (surprisingly) is false!

Example 2.4. Let $f(x) = 1$ if $0 < x_2 < x_1^2$ and $f(x) = 0$ otherwise. Then $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{t\to 0} f(tv) = 0$ for all $v \in \mathbb{R}^2 - \{0\}$.

Example 2.5. Let $f(x) = x_1^2 x_2/(x_1^4 + x_2^2)$, and $f(0) = 0$. Then $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{t\to 0} f(tv) = 0$ for all $v \in \mathbb{R}^2 - \{0\}$.

Proposition 2.3 can be used to show that various limits don't exist.

Example 2.6. Show that $\lim\limits_{x\to 0} \dfrac{x_1 x_2}{|x|^2}$ does not exist.

Proof. Choosing $v_1 = (1, 1)$ and $v_2 = (1, 0)$ we see

$$\lim_{t\to 0} f(tv_1) = \frac{1}{2} \quad \text{and} \quad \lim_{t\to 0} f(tv_2) = 0 \neq \frac{1}{2}.$$

So by Proposition 2.3, $\lim_{x\to 0} x_1 x_2/|x|^2$ can not exist. $\square$

3. Continuity

Definition 3.1. Let $U \subset \mathbb{R}^m$ be a domain, and $f : U \to R^d$ be a function. We say f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Definition 3.2. If f is continuous at every point $a \in U$, then we say f is continuous on U (or sometimes simply f is continuous).

Again the standard results on continuity from one variable calculus hold. Sums, products, quotients (with a non-zero denominator) and composites of continuous functions will all yield continuous functions.

The notion of continuity gives us a generalization of Proposition 2.3 that is useful is computing the limits along arbitrary curves instead.

Proposition 3.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function, and $a \in \mathbb{R}^d$. Let $\gamma : [0, 1] \to \mathbb{R}^d$ be a* any continuous function *with $\gamma(0) = a$, and $\gamma(t) \neq a$ for all $t > 0$. If $\lim_{x\to a} f(x) = l$, then we must have $\lim_{t\to 0} f(\gamma(t)) = l$.*

Corollary 3.4. *If there exists two continuous functions $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{R}^d$ such that for $i \in 1, 2$ we have $\gamma_i(0) = a$ and $\gamma_i(t) \neq a$ for all $t > 0$. If $\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$ then $\lim_{x\to a} f(x)$ can not exist.*