QUANTIFYING THE DISSIPATION ENHANCEMENT OF CELLULAR FLOWS.

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Abstract. We study the dissipation enhancement by cellular flows. Previous work by Iyer, Xu, and Zlatoš produces a family of cellular flows that can enhance dissipation by an arbitrarily large amount. We improve this result by providing quantitative bounds on the dissipation enhancement in terms of the flow amplitude, cell size and diffusivity. Explicitly we show that the *mixing time* is bounded by $C(\varepsilon^2/\kappa + |\ln\delta|^2/(\varepsilon^2 A))$. Here κ is the diffusivity, ε is the cell size, A/ε is the flow amplitude, and $\delta = \sqrt{\kappa/A}$ is the thickness of the boundary layer. The above agrees with the optimal heuristics. We also prove a general result relating the *dissipation time* of incompressible flows to the *mixing time*. The main idea behind the proof is to study the dynamics probabilistically and construct a successful coupling.

1. Introduction

Consider an insoluble dye in an incompressible fluid. Stirring the fluid typically causes filamentation, stretching blobs of die into fine tendrils. Diffusion, on the other hand, efficiently damps these small scales, and the combination of these two effects results in *enhanced dissipation* – the tendency of passive scalars to diffuse faster than in the absence of stirring. This phenomenon has been extensively studied in many contexts, and various authors have established a link between mixing and dissipation enhancement [CKRZ08, Zla10, FI19, CZDE20], studied dissipation enhancement in more general situations [Sei20, ABN21, NP22] and studied it extensively for shear flows [Tay53, BCZ17, Wei19, GZ21, CCZW21]. Enhanced dissipation has also been used to suppress non-linear effects arising in certain situations [FKR06, KX16, FFIT20, IXZ21], and is a subject of active study.

The purpose of this work is to quantify dissipation enhancement for cellular flows, thus providing simple and explicit examples of flows with arbitrarily large dissipation enhancement. Cellular flows arise as a model problem where ambient fluid velocity is a periodic array of opposing vortices. They have been extensively studied in the context of fluid dynamics, homogenization and as random perturbations of dynamical systems [Chi79, CS89, FP94, Kor04, NPR05, DK08, Bak11, HIK+18].

We will use probabilistic techniques to estimate the *mixing time* of a diffusion whose drift is a cellular flow. We then estimate the dissipation enhancement in terms of the mixing time. The bounds we obtain are significantly better than the bounds previously obtained in [IXZ21], and (up to a logarithmic factor) they agree with the optimal heuristic bounds.

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2. Main Result

We will study the concentration of a dye, denoted by ϕ, as a passive scalar, evolving according to the advection diffusion equation

$$\partial_t \phi - (u \cdot \nabla)\phi - \frac{\kappa}{2}\Delta\phi = 0 \qquad \text{in } (0,\infty) \times \mathbb{T}^d\,. \tag{2.1}$$

Here $-u$ represents the velocity field of the ambient fluid, and $\kappa/2 > 0$ is the molecular diffusivity. We restrict our attention to the periodic d-dimensional torus $\mathbb{T}^d$ with side length 1, and we will normalize the initial concentration, ϕ_0, so that

$$\int_{\mathbb{T}^d} \phi_0(x)\, dx = 0\,.$$

As time evolves, the dye spreads uniformly across the torus and $\phi(\cdot, t) \to 0$ as $t \to \infty$. One measure of convergence rate that will interest us is the *dissipation time*: the time required for solutions to (2.1) to lose a constant fraction of their initial energy (see for instance [FNW04, CKRZ08, FI19]). Explicitly, *dissipation time*, denoted by $t_{dis} = t_{dis}(\kappa, u)$ is defined by

$$t_{dis} \stackrel{\text{def}}{=} \inf\Big\{ t \geqslant 0 \;\Big|\; \|\phi(s+t)\|_{L^2} \leqslant \frac{1}{2}\|\phi(s)\|_{L^2} \text{ for all } s \geqslant 0,\ \phi(s) \in \dot{L}^2 \Big\}\,. \tag{2.2}$$

Here $\dot{L}^2$ denotes the space of all mean-zero, square integrable functions on the torus $\mathbb{T}^2$.

The Poincaré inequality and the fact that u is divergence free immediately imply

$$t_{dis}(\kappa, u) \leqslant \frac{1}{4\pi^2\kappa}\,. \tag{2.3}$$

However, this is only an upper bound, and the dissipation time may in fact be much smaller than $O(1/\kappa)$. When this occurs (i.e. when $t_{dis}(u,\kappa) \leqslant o(1/\kappa)$) it is known as *enhanced dissipation.* Intuitively, enhanced dissipation when the stirring velocity field generates small scales (e.g. through filamentation), which are then damped much faster by the diffusion.

Seminal work of Constantin et al. [CKRZ08] provides a spectral characterization of (time independent) velocity fields for which $t_{dis} = o(1/\kappa)$. More explicit, improved bounds were recently obtained in terms of the mixing rate of u. For instance, if u is exponentially mixing then one can show $t_{dis} \leqslant O(|\ln\kappa|^2)$ (see for instance [FI19, Fen19, CZDE20]).

In the context of applications, various authors have shown that sufficiently enhanced dissipation can be used to quench reactions, stop phase separation and prevent singularity formation (see for instance [FKR06, KX16, FFIT20, IXZ21, FM22, FSW22]). Thus finding simple and explicit examples of flows which sufficiently enhance dissipation (i.e. make t_{dis} arbitrarily small) are useful for many applications. While such flows can be found by rescaling velocity fields with strong enough mixing properties (see for instance [FFIT20, IXZ21]), examples of mixing velocity fields on the torus are notoriously hard to construct. The main goal of this work is to provide a simple and explicit family of velocity fields for which t_{dis} can be made arbitrarily small. The family of flows we construct are two dimensional cellular flows. These arise frequently in fluid dynamics as flows around strong arrays of opposing vortices and have been extensively studied [Chi79, RY83, CS89, FP94, Hei03, NPR05, Kor04].

Given $\varepsilon > 0$, consider the cellular flow v defined by

$$v \stackrel{\text{def}}{=} \nabla^\perp(\xi H) = \begin{pmatrix} -\partial_2(\xi H) \\ \partial_1(\xi H) \end{pmatrix}, \qquad \text{where } H(x) \stackrel{\text{def}}{=} \sin\Big(\frac{2\pi x_1}{\varepsilon}\Big) \sin\Big(\frac{2\pi x_2}{\varepsilon}\Big), \tag{2.4}$$

and ξ is a smooth periodic cutoff function function such that

$$\xi(x) = \begin{cases} 1 & |H(x)| \leqslant \frac{1}{4}, \\ 0 & |H(x)| \geqslant \frac{1}{2}. \end{cases}$$

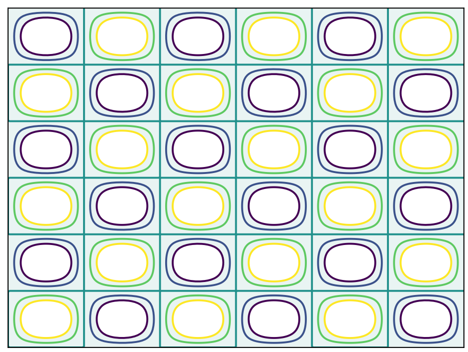

FIGURE 1. Stream lines of the cellular flow defined in equation (2.4). The flow is only non-zero in the shaded region.

This flow has cell size $O(\varepsilon)$, and its stream lines are shown in Figure 1. Our main result chooses $u = Av$ for $A \gg 1$, and estimates the *mixing time* explicitly in terms of the flow amplitude A, cell size ε and diffusivity κ as follows.

Theorem 2.1. *If $u = Av$ with v defined in (2.4), then there exists a finite constant C, independent of ε, A, and κ, such that*

$$t_{dis} \leqslant 3t_{mix} \leqslant C\Big(\frac{\varepsilon^2}{\kappa} + \frac{|\ln \delta|^2}{A\varepsilon^2}\Big), \qquad \text{where } \delta = \sqrt{\frac{\kappa}{A}}. \tag{2.5}$$

Here $t_{mix} = t_{mix}(u, \kappa)$ is the *mixing time*, a notion used to measure the convergence rate of Markov processes [LPW09, MT06]. Roughly speaking, the *mixing time* is the minimum amount of time required for the fundamental solution of (2.1) to be L^1-close to the invariant distribution, which in our case is the uniform distribution. That is, if $\rho(x, s; y, t)$ is the fundamental solution of (2.1), the mixing time is defined by

$$t_{mix} \stackrel{\text{def}}{=} \inf\Big\{t \geqslant 0 \;\Big|\; \sup_{x\in\mathbb{T}^d,\, s\geqslant 0} \int_{\mathbb{T}^d} |\rho(x, s; y, s+t) - 1|\, dy < \frac{1}{2},\ \forall s \geqslant 0\Big\}. \tag{2.6}$$

The mixing time and dissipation time are related to each other: the dissipation time is bounded by three times the mixing time. The mixing time can also be bounded by the dissipation time, up to a logarithmic factor. This is a general result and is not specific to cellular flows.

Proposition 2.2. *Let $u \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{T}^d))$ be a divergence free vector field, and let $t_{mix} = t_{mix}(u, \kappa)$, $t_{dis} = t_{dis}(u, \kappa)$ denote the mixing time and dissipation* *time respectively. There exists a dimensional constant $C = C(d) < \infty$, independent of u and κ such that for all sufficiently small $\kappa > 0$ we have*

$$t_{dis} \leqslant 3t_{mix} \leqslant Ct_{dis} \ln\Big(1 + \frac{1}{\kappa t_{dis}}\Big)\,. \tag{2.7}$$

Remark 2.3. By rescaling we see that on a torus with side length ℓ, the above becomes

$$t_{dis} \leqslant 3t_{mix} \leqslant Ct_{dis} \ln\Big(1 + \frac{\ell^2}{\kappa t_{dis}}\Big)\,. \tag{2.7$'$}$$

for some dimensional constant $C = C(d)$ that is independent of ℓ.

We are presently unaware whether or not the logarithmic factor is necessary. We prove Proposition 2.2 in Section 4, below.

We now compare Theorem 2.1 to previously available bounds. Previous work of the first author, Xu and Zlatoš [IXZ21] has already shown that the dissipation time of a sufficiently strong and fine cellular flow can be made arbitrarily small. The estimates in [IXZ21], however, are neither explicit nor optimal! In particular, Theorem 1.3 in [IXZ21] only asserts the existence of sufficiently strong and fine cellular flows with arbitrarily small t_{dis}, without providing a quantitative bound. A more explicit bound is provided in [IXZ21, Remark 6.6] which yields a sub-optimal bound of the form $t_{dis} \leqslant C \log(A/\kappa) A^{-1/64}\kappa^{-1}$ after rescaling. This is much weaker than (2.5).

We next compare Theorem 2.1 to the well known homogenization results that estimate the *effective diffusivity*. Recall standard results (see for instance [BLP78, PS08]) show that the long time behavior of (2.1) is effectively that of the purely diffusive equation

$$\partial_t \bar\phi - \frac{1}{2} D_{\mathit{eff}} \Delta \bar\phi = 0\,,$$

with an enhanced diffusion coefficient D_{eff}, known as the *effective diffusivity*. The effective diffusivity of cellular flows has been extensively studied [Chi79,FP94,Kor04] and is known to asymptotically be

$$D_{\mathit{eff}} \approx O(\varepsilon^2 \sqrt{\kappa A})\,.$$

Given this one would expect from (2.3) that

$$t_{dis} \leqslant \frac{1}{4\pi^2 D_{\mathit{eff}}} = O\Big(\frac{1}{\varepsilon^2 \sqrt{\kappa A}}\Big)$$

For the above to be valid, however, one would need $t_{dis} \geqslant O(\varepsilon^2/\kappa)$, the time the dye takes to diffuse through one cell. This heuristic leads one to choose $\varepsilon = O(\kappa/A)^{1/2}$, leading to the bound $t_{dis} \leqslant O(1/\kappa)$. Not only is this much weaker than (2.5), but it also provides no significant improvement over (2.3). Thus the mixing time bound in Theorem 2.1 captures an effect on time scales *much smaller* than the time scales at which standard homogenization results are applicable.

We now discuss optimizing the choice of parameters ε, A in Theorem 2.1.

Remark 2.4. One strategy is to simply minimize the right hand side of (2.5) in the cell size ε, and choose

$$\varepsilon = \sqrt{\delta |\ln \delta|} = \frac{1}{\sqrt{2}} \Big(\frac{\kappa}{A}\Big)^{1/4} \Big(\ln\Big(\frac{A}{\kappa}\Big)\Big)^{1/2}\,.$$

This choice of ε leads to

$$t_{mix} \leqslant \frac{C\delta|\ln\delta|}{\kappa} = \frac{C|\ln\delta|}{\sqrt{\kappa A}}\,.$$

For a fixed κ the right hand side vanishes as $A \to \infty$, providing a simple family of explicit flows with arbitrarily small dissipation time.

Remark 2.5 (Fixed energy). Another strategy to optimize the choice of parameters in (2.5) is to minimize the right hand side while holding the *energy* $\|u\|_{L^2}^2 = O(A^2/\varepsilon^2)$ constant. Physically the energy $\|u\|_{L^2}^2$ is proportional to the total kinetic energy of the ambient fluid. In this case we choose

$$\varepsilon \approx \Big(\frac{\kappa|\ln\delta|^2}{\|u\|_{L^2}}\Big)^{1/5} = \frac{|\ln\delta|^{2/5}}{\mathrm{Pe}^{1/5}}\,.$$

Here $\mathrm{Pe} = A/(\varepsilon\kappa)$ is the *Péclet number* — a non-dimensional ratio measuring the relative strength of the convection and diffusion. This choice of cell size results in

$$t_{mix} \leqslant \frac{C|\ln\delta|^{4/5}}{\kappa^{3/5}\|u\|_{L^2}^{2/5}} = \frac{C|\ln\delta|^{4/5}}{\kappa\mathrm{Pe}^{2/5}} \tag{2.8}$$

For any fixed molecular diffusivity $\kappa > 0$, the right hand side vanishes as $\|u\|_{L^2}^2 \to \infty$, and (2.8) provides the optimal asymptotic behavior of the mixing time of cellular flows with high energy.

Remark 2.6 (Fixed power). Finally we choose the parameters ε, A to minimize the right hand side of (2.5) while holding the *enstrophy* $\|\nabla u\|^2 = O(A^2/\varepsilon^4)$ constant. Physically the enstrophy $\|\nabla u\|_{L^2}^2$ is proportional to the power dissipated by the ambient fluid (see [LTD11]). In this case we choose

$$\varepsilon \approx \Big(\frac{|\ln\delta|^2\kappa}{\|\nabla u\|}\Big)^{1/6}, \qquad \text{which implies} \qquad t_{mix} \leqslant \frac{C|\ln\delta|^{2/3}}{\kappa^{2/3}\|\nabla u\|^{1/3}}\,. \tag{2.9}$$

For any fixed molecular diffusivity $\kappa > 0$, the right hand side vanishes as $\|u\|_{L^2}^2 \to \infty$, and (2.9) provides the optimal asymptotic behavior of the mixing time of cellular flows with high enstrophy.

We now describe the main idea behind our proof. The Ito diffusion associated to (2.1) is defined by the SDE

$$dX_t = Av(X_t)\,dt + \sqrt{\kappa}\,dB_t\,, \tag{2.10}$$

on the 2-dimensional torus $\mathbb{T}^2$. Here B is a standard 2-dimensional Brownian motion. Since $\nabla\cdot v = 0$, the invariant measure of X is the Lebesgue measure on the torus.

We will estimate the mixing time of X by constructing a successful coupling (see for instance [LPW09, MT06]). To do this, we first project the process to one cell (of side length ε, where we enforce $1/\varepsilon$ to be integer), and couple the projections. The drift doesn't help at this stage and it takes on average $O(\varepsilon^2/\kappa)$ time to couple using the diffusion. Next, following the Freidlin Wentzell point of view [FW93, FW94, IN16, HIK⁺18] we view the dynamics of X as a random walk on a lattice of $O(1/\varepsilon^2)$ cells. It is known that the expected coupling time of a discrete random walk on such lattice is bounded above by $O(1/\varepsilon^2)$, and that roughly takes one step of a random walk every time it crosses a *boundary layer* of thickness $\varepsilon\delta$. Thus we can construct a successful coupling after the number of boundary layer crossings is at least $O(1/\varepsilon^2)$. Crossings of the boundary layer happen through the diffusion alone, and estimating their frequency yields Theorem 2.1. We carry out the details in Section 3, below.

Before delving into the details we make three remarks: First, the extra logarithmic factor $|\ln \delta|$ in (2.5) arises due to the logarithmic slow down of Hamiltonian systems as they approach hyperbolic critical points (all cell corners, in our case). Second, the smooth cutoff ξ in (2.4) is used to initiate the coupling of the projected processes in a time that is independent of A. Third, the explicit formula for H in (2.4) is used to construct a simple coupling in subsequent steps using symmetry. While the logarithmic factor $|\ln \delta|$ is unavoidable, both the smooth cutoff ξ and the explicit formula for H are mainly used to simplify technicalities in the proof.

Plan of this paper. In the next section (Section 3) we prove Theorem 2.1, modulo several technical lemmas bounding certain hitting times. In Section 4 we prove Proposition 2.2, relating the dissipation time and mixing time for general incompressible flows. In Section 5 we prove an $O(\varepsilon^2/\kappa)$ bound on the coupling time when the process is projected to a torus of side length ε. Finally in Section 6 we prove the remaining lemmas stated in Section 3 by counting boundary layer crossings.

3. Proof of the Mixing Time Bound (Theorem 2.1)

The goal of this section is to prove Theorem 2.1. In light of Proposition 2.2, we only need to bound the mixing time. We will do this by a *coupling construction.* To fix notation, we will subsequently assume X and $\tilde X$ are solutions of the SDEs

$$dX_t = Av(X_t)dt + \sqrt{\kappa}\, dB_t\,, \tag{3.1}$$

$$d\tilde X_t = Av(\tilde X_t)dt + \sqrt{\kappa}\, d\tilde B_t\,. \tag{3.2}$$

with initial data

$$X_0 = x\,, \quad \tilde X_0 = \tilde x \qquad \boldsymbol{P}^{(x,\tilde x)}\text{-almost surely}\,.$$

Here B and $\tilde B_t$ are both 2D Brownian motions. We will choose $\tilde B$ in terms of B in a manner that ensures a suitable bound on the *coupling time.* Recall the coupling time

$$\tau_{cpl} \stackrel{\text{def}}{=} \inf\{t \geqslant 0 \mid X_t = \tilde X_t\}\,,$$

is the first time X and $\tilde X$ meet, and standard results (see for instance [LPW09, Ch. 5]) guarantee

$$t_{mix} \leqslant C \sup_{(x,\tilde x)\in\mathbb{T}^2\times\mathbb{T}^2} \boldsymbol{E}^{(x,\tilde x)} \tau_{cpl}\,. \tag{3.3}$$

Thus our task is now to choose the Brownian motion $\tilde B$ and bound $\boldsymbol{E}\tau_{cpl}$. The construction of $\tilde B$ can be described quickly, however, the bound on $\boldsymbol{E}\tau_{cpl}$ requires several technical lemmas. For clarity of presentation we will describe the construction of $\tilde B$ below, and momentarily postpone the lemmas bounding the coupling time.

Proof of Theorem 2.1. The coupling construction is divided into several stages, which we describe individually.

Step 1: Coupling projections. Observe first that the drift v is periodic with period ε, and thus both X and $\tilde X$ can be viewed as diffusions on a torus with side length ε. Let

$\mathbb{T}^2_\varepsilon = [0,\varepsilon)^2$ be the two dimensional torus with side length ε, and let $\Pi_\varepsilon \colon \mathbb{T}^2 \to \mathbb{T}^2_\varepsilon$ be the projection defined by

$$\Pi_\varepsilon(x_1, x_2) = \bigl(x_1 \pmod{\varepsilon},\ x_2 \pmod{\varepsilon}\bigr)\,.$$

We will subsequently assume that $1/\varepsilon \in \mathbb{N}$, so the above projection is well defined. (We also clarify that $\mathbb{T}^2$ above denotes the standard two dimensional torus with side length 1.)

Consider the projected diffusions

$$Y = \Pi_\varepsilon X\,, \qquad \tilde{Y} = \Pi_\varepsilon \tilde{X}\,, \tag{3.4}$$

on the torus $\mathbb{T}^2_\varepsilon$. Since the drift v is divergence free one can use PDE methods to show that the mixing time of Y is bounded by $O(\varepsilon^2/\kappa)$. This, however, is not sufficient for our purposes as we need a coupling between Y and $\tilde{Y}$ for subsequent steps, and we need the coupling time to be bounded independent of A. We will couple Y and $\tilde{Y}$ by waiting until they enter the central region of cells where $u = 0$. In this region Y and $\tilde{Y}$ are simply Brownian motions, and we can couple them by reflection (see for instance [LR86]), in time τ^Y_{cpl} that is bounded independent of A. Explicitly, we will show (Lemma 3.1, below) that

$$\boldsymbol{E}^{(x,\tilde{x})} \tau^Y_{cpl} \leqslant \frac{C\varepsilon^2}{\kappa}\,, \tag{3.5}$$

for some finite constant C. Here, and subsequently, we will assume that the constant C is independent of the parameters ε, A, κ, the initial data $x, \tilde{x}$, and may increase from line to line.

Step 2: Moving to vertical cell boundaries. By the Markov property, we may now restart time and assume that at time 0 we have $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$. In this step we will now choose $B = \tilde{B}$ and wait until X and $\tilde{X}$ hit a vertical cell boundary. That is, we set

$$\sigma_v \stackrel{\text{def}}{=} \inf\Bigl\{t \geqslant 0 \Bigm| X^1_t \in \frac{\varepsilon}{2}\mathbb{Z}\Bigr\}\,, \qquad \tilde{\sigma}_v \stackrel{\text{def}}{=} \inf\Bigl\{t \geqslant 0 \Bigm| \tilde{X}^1_t \in \frac{\varepsilon}{2}\mathbb{Z}\Bigr\}\,, \tag{3.6}$$

where $X^1, \tilde{X}^1$ denote the first coordinates process of X, and $\tilde{X}$ respectively. Periodicity of v will ensure $\sigma_v = \tilde{\sigma}_v$, and we will show (Lemma 3.2, below) that

$$\boldsymbol{E}^x \sigma_v \leqslant \frac{C\varepsilon^2}{\kappa} \tag{3.7}$$

Step 3: Vertical Coupling. By the Markov property again, we restart time and assume $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$, and $X^1_0, \tilde{X}^1_0 \in \frac{\varepsilon}{2}\mathbb{Z}$. We will now choose $\tilde{B}^1 = -B^1$ and $\tilde{B}^2 = B^2$, and wait until time τ_v defined by

$$\tau_v \stackrel{\text{def}}{=} \inf\{t \geqslant 0 \mid X^1_t = \tilde{X}^1_t\}\,. \tag{3.8}$$

Note that by symmetry of v we will have[1] $(\Pi_\varepsilon X_t)^2 = (\Pi_\varepsilon \tilde{X}_t)^2$ for all $t \leqslant \tau_v$, and thus at time τ_v we will have $\Pi_\varepsilon X_t = \Pi_\varepsilon \tilde{X}_t$. (See Figure 2, below, for an illustration of trajectories of X and $\tilde{X}$ under this choice of noise.) We will show (Lemma 3.3, below) that

$$\boldsymbol{E}\tau_v \leqslant C\left(\frac{\varepsilon^2}{\kappa} + \frac{|\ln\delta|^2}{A\varepsilon^2}\right)\,. \tag{3.9}$$

[1]We clarify here that $(\Pi_\varepsilon X_t)^2$ refers to the second coordinate of $\Pi_\varepsilon X_t$.

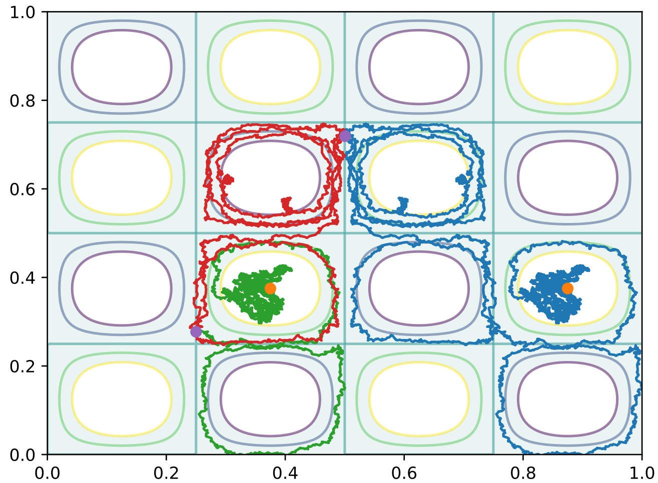

FIGURE 2. Sample trajectories illustrating the coupling in steps 2 and 3. Here $X_0 = (0.75, 0.25)$, $\tilde{X}_0 = (0.25, 0.25)$, and the trajectory of X is shown in blue. Until $\tilde{X}$ hits a vertical cell boundary the trajectory of $\tilde{X}$ (shown in green) is simply a shift of the trajectory of X. After this time, the trajectory of $\tilde{X}$ (shown in red) is a mirror image of the trajectory of X until they hit the same vertical line ($x = 0.5$ in this case).

The proof of (3.9) requires a estimates on the number of times the flow crosses the boundary layer; this is technical, but has been well studied by numerous authors and the proofs can be readily adapted to our situation.

Step 4: Horizontal hitting and coupling. At this point we have arranged for $\Pi_\varepsilon X = \Pi_\varepsilon \tilde{X}$, and $X^1 = \tilde{X}^1 \in \frac{\varepsilon}{2}\mathbb{Z}$. As usual, we restart time and assume that the above happens at time 0. We will now repeat steps 2 and 3 in the horizontal direction: First choose $\tilde{B} = B$ until $X^2, \tilde{X}^2 \in \frac{\varepsilon}{2}\mathbb{Z}$, then choose $\tilde{B}^1 = B^1$, $\tilde{B}^2 = -B^2$, and then wait until $X^2 = \tilde{X}^2$. The time taken for each of these steps is bounded in Lemmas 3.4 and 3.5, below. The symmetry of v will ensure that when $X^2 = \tilde{X}^2$, we will also have $X^1 = \tilde{X}^1$, thus giving a successful coupling of $X, \tilde{X}$.

Using Chebychev's inequality, the above guarantees us a coupling of $X, \tilde{X}$ with probability at least 1/16 in time at most twice the expected value of the stopping times in each of the above steps. Thus using the Markov property and Lemmas 3.1–3.5, below, we obtain a successful coupling with the coupling time bounded by

$$\boldsymbol{E}^{(x,\tilde{x})}\tau_{cpl} \leqslant C\Big(\frac{\varepsilon^2}{\kappa} + \frac{|\ln\delta|^2}{A\varepsilon^2}\Big)\,. \tag{3.10}$$

Using (3.3), concludes the proof. □

It remains to bound the stopping times in each of the above steps. For clarity of presentation we state each bound as a Lemma below, and prove the lemmas in subsequent sections.

Lemma 3.1 (Coupling of projections)**.** *There exists a Brownian motion $\tilde{B}$ such that $(Y, \tilde{Y})$ is a coupling of Y (on the torus $\mathbb{T}^2_\varepsilon$), and the coupling time satisfies* (3.5)*.*

Lemma 3.2 (Vertical boundary hitting time)**.** *Suppose* $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$*. Choose* $\tilde{B} = B$*, and let* σ_v *and* $\tilde{\sigma}_v$ *(equation (3.6)) be the first hitting times of* X *and* $\tilde{X}$ *to the vertical cell boundaries, respectively. Then* $\sigma_v = \tilde{\sigma}_v$ *and equation (3.7) holds.*

Lemma 3.3 (Vertical coupling)**.** *Suppose* $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$*, and* $X_0^1 \in \frac{\varepsilon}{2}\mathbb{Z}$*. Let* $\tilde{B} = (-B^1, B^2)$*, and let* τ_v *(equation (3.8)) be the first time first time* X *and* $\tilde{X}$ *are on the same vertical line. Then* $(\Pi_\varepsilon X_t)^2 = (\Pi_\varepsilon \tilde{X}_t)^2$ *for all* $t \leqslant \tau_v$*, the expected value of* τ_v *is bounded by (3.9).*

Lemma 3.4 (Horizontal boundary hitting time)**.** *Suppose that* $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$*, and* $X_0^1 = \tilde{X}_0^1 \in \frac{\varepsilon}{2}\mathbb{Z}$*. Choose* $\tilde{B} = B$*, and let*

$$\sigma_h \stackrel{\text{def}}{=} \inf\Big\{ t \geqslant 0 \;\Big|\; X_t^2 \in \frac{\varepsilon}{2}\mathbb{Z} \Big\}, \qquad \tilde{\sigma}_h \stackrel{\text{def}}{=} \inf\Big\{ t \geqslant 0 \;\Big|\; \tilde{X}_t^2 \in \frac{\varepsilon}{2}\mathbb{Z} \Big\},$$

be the first hitting time to the horizontal cell boundaries. Then

$$\sigma_h = \tilde{\sigma}_h\,, \qquad X^1_{\sigma_h} = \tilde{X}^1_{\tilde{\sigma}_h}\,, \qquad \textit{and} \qquad \boldsymbol{E}\sigma_h = \boldsymbol{E}\tilde{\sigma}_h \leqslant \frac{C\varepsilon^2}{\kappa}\,.$$

Lemma 3.5 (Horizontal coupling)**.** *Suppose* $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}_0$*,* $X_0^1 = \tilde{X}_0^1$*, and* $X_0^2 \in \frac{\varepsilon}{2}\mathbb{Z}$*. Choose* $\tilde{B} = (B^1, -B^2)$ *and let*

$$\tau_h = \inf\Big\{ t \geqslant 0 \;\Big|\; X_t^2 = \tilde{X}_t^2 \Big\},$$

be the first time X *and* $\tilde{X}$ *are on the same horizontal line. Then*

$$\tau_h = \tilde{\tau}_h\,, \qquad X_{\tau_h} = \tilde{X}_{\tau_h}\,, \qquad \textit{and} \qquad \boldsymbol{E}\tau_h \leqslant \frac{C|\ln\delta|^2}{A\varepsilon^2}\,.$$

Each of these lemmas will be proved in subsequent sections.

4. Relationship between the dissipation time and mixing time

In this section we prove Proposition 2.2 which relates the mixing time and the dissipation time of general incompressible flows. Throughout this section we will assume $u \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{T}^d))$ is a divergence free vector field, and let X be the (time inhomogeneous) Markov process on $\mathbb{T}^d$ defined by the SDE

$$dX_t = u(X_t)\,dt + \sqrt{\kappa}\,dB_t\,. \tag{4.1}$$

Here B is a standard d-dimensional Brownian motion on the torus.

Let $\rho(x,s;y,t)$ be the transition density of X. By the Kolmogorov equations, we know that ρ is the fundamental solution to (2.1), and thus the mixing time of X is given by (2.6). Using the Kolmogorov equations again, the dissipation time (defined in (2.2)) can be equivalently defined by

$$t_{dis} \stackrel{\text{def}}{=} \inf\Big\{ t \geqslant 0 \;\Big|\; \sup_{x\in\mathbb{T}^d,\, s\geqslant 0} \|\boldsymbol{E}^{(\cdot,s)}\vartheta(X_{s+t})\|_{L^2} < \frac{1}{2}\|\vartheta\|_{L^2}\,,\ \forall \vartheta \in \dot{L}^2(\mathbb{T}^d) \Big\}. \tag{4.2}$$

Recall $\dot{L}^2(\mathbb{T}^d)$ is the set of all mean zero L^2 functions on the torus $\mathbb{T}^d$, and $\boldsymbol{E}^{(x,s)}$ denotes the expected value under the probability measure $\boldsymbol{P}^{(x,s)}$ under which $X_s = x$ almost surely. The constant $1/2$ in (2.2), (2.6) and (4.2) is chosen for convenience. Replacing it by any constant that is strictly smaller than 1 will only change t_{mix} and t_{dis} by a constant factor that is independent of u, κ and d.

The first inequality in (2.6) can be proved elementarily, and we do that first.

Lemma 4.1. *The dissipation time and mixing time satisfy the inequality*

$$t_{dis} \leqslant 3t_{mix}\,.$$

Proof. For simplicity, and without loss of generality, we will assume that $s = 0$ in both (2.6) and (4.2). Let $\theta_t(x) \stackrel{\text{def}}{=} \boldsymbol{E}^{(x,0)}\theta_0(X_t) = \boldsymbol{E}^x\theta_0(X_t)$ for some θ_0 in $\dot{L}^2$. Since θ_0 is mean 0, we note

$$\theta_t(x) = \int_{\mathbb{T}^d} \rho(x,0;y,t)\theta_0(y)\,dy = \int_{\mathbb{T}^d} (\rho(x,0;y,t) - 1)\theta_0(y)\,dy\,.$$

and hence

$$\|\theta_t\|_{L^2}^2 \leqslant \Big(\int_{\mathbb{T}^d\times\mathbb{T}^d} |\rho(x,0;y,t) - 1|\,dy\,dx\Big)\cdot$$
$$\Big(\int_{\mathbb{T}^d\times\mathbb{T}^d} \theta_0(y)^2(\rho(x,0;y,t) + 1)\,dx\,dy\Big)\,. \tag{4.3}$$

Since the Lebesgue measure is invariant, we note

$$\int_{\mathbb{T}^d} \rho(x,0;y,t)\,dx = 1\,, \qquad \text{for every } y \in \mathbb{T}^d\,,$$

and hence (4.3) implies

$$\|\theta_t\|_{L^2}^2 \leqslant 2\|\theta_0\|_{L^2}^2 \Big(\sup_{x\in\mathbb{T}^d} \int_{\mathbb{T}^d} |\rho(x,0;y,t) - 1|\,dy\Big)\,. \tag{4.4}$$

The Chapman Kolmogorov equations and invariance of the Lebesgue measure immediately imply

$$\sup_x \int_{\mathbb{T}^d} |\rho(x,0;y,nt_{mix}) - 1| \leqslant \frac{1}{2^n}\,, \tag{4.5}$$

for any natural number $n \in \mathbb{N}$. Choosing $t = 3t_{mix}$ and using (4.4), we see that (4.5) immediately implies $\|\theta_{3t_{mix}}\|_{L^2}^2 \leqslant \frac{1}{4}\|\theta_0\|_{L^2}^2$, which finishes the proof. ∎

The proof of the second inequality in (2.7) follows from Proposition 4.2 in [IXZ21], which provides an L^1 to L^∞. We reproduce this here for convenience, and then go on to prove the second inequality in (2.7).

Lemma 4.2 (Proposition 4.2 in [IXZ21])**.** *There exists a constant $C = C(d)$, independent of u, such that for all $\vartheta \in \dot{L}^2$, and all sufficiently small $\kappa > 0$ we have*

$$\|\boldsymbol{E}^{(x,s)}\vartheta(X_{s+t})\|_{L^\infty} \leqslant \frac{1}{2}\|\vartheta\|_{L^1}\,, \quad \textit{for all } s \geqslant 0\,,\ t \geqslant Ct_{dis}\ln\Big(1 + \frac{1}{\kappa t_{dis}}\Big)\,. \tag{4.6}$$

Remark 4.3. Since $\|\vartheta\|_{L^1} \leqslant \|\vartheta\|_{L^\infty}$, we can iterate (4.6) to yield

$$\|\boldsymbol{E}^{(x,s)}\vartheta(X_{s+t})\|_{L^\infty} \leqslant 2^{-n}\|\vartheta\|_{L^1}\,, \quad \text{for all } s \geqslant 0\,,\ t \geqslant nCt_{dis}\ln\Big(1 + \frac{1}{\kappa t_{dis}}\Big)\,.$$

Proof. For simplicity, and without loss of generality we assume $s = 0$. Using well known drift independent estimates (see for instance Lemma 5.6 in [CKRZ08], Lemmas 3.1, 3.3 in [FKR06], and Lemma 5.4 in [Zla10]) we know

$$\|\theta_{t+2t_{dis}}\|_{L^\infty} \leqslant \frac{c}{(\kappa t_{dis})^{\frac{d}{4}}}\|\theta_{t+t_{dis}}\|_{L^2}\,, \qquad \text{and} \qquad \|\theta_{t_{dis}}\|_{L^2} \leqslant \frac{c}{(\kappa t_{dis})^{\frac{d}{4}}}\|\theta_0\|_{L^1}\,,$$

for some dimensional constant $c = c(d)$. Now iterating (4.2) we see

$$\|\theta_{t+t_{dis}}\|_{L^2} \leqslant 2^{-\lfloor t/t_{dis}\rfloor}\|\theta_{t_{dis}}\|_{L^2} \leqslant 2^{1-t/t_{dis}}\|\theta_{t_{dis}}\|_{L^2}\,,$$

and hence

$$\|\theta_{t+2t_p^*}\|_{L^\infty} \leqslant \frac{c}{(\kappa t_{dis})^{\frac{d}{4}}}\|\theta_{t+t_{dis}}\|_{L^2} \leqslant \frac{c2^{1-t/t_{dis}}}{(\kappa t_{dis})^{\frac{d}{4}}}\|\theta_{t_{dis}}\|_{L^2} \leqslant \frac{c^2 2^{1-t/t_{dis}}}{(\kappa t_{dis})^{\frac{d}{2}}}\|\theta_0\|_{L^1}$$

Thus if we choose $t \geqslant Ct_{dis}\ln(1 + 1/(\kappa t_{dis}))$ for some sufficiently large constant $C = C(p, d)$, we obtain

$$\|\theta_{t+2t_{dis}}\|_{L^\infty} \leqslant \frac{1}{2}\|\theta_0\|_{L^1}\,.$$

This finishes the proof of Lemma 4.2. ☐

We can now prove the second inequality in (2.7).

Lemma 4.4. *There exists a dimensional constant $C = C(d)$, independent of u and κ such that*

$$t_{mix} \leqslant Ct_{dis}\ln\Big(1 + \frac{1}{\kappa t_{dis}}\Big)\,.$$

Proof. For simplicity, and without loss of generality we assume $s = 0$. Choose large enough so that (4.6) holds. By standard regularity theory, we know that for any $\varepsilon > 0$, the density $\rho(x, 0; y, \varepsilon)$ is integrable in y. Since $\rho \geqslant 0$, we note

$$\int_{\mathbb{T}^d} |\rho(x, 0; y, \varepsilon) - 1|\,dy \leqslant \int_{\mathbb{T}^d} (\rho(x, 0; y, \varepsilon) + 1)\,dy = 2\,.$$

Let C be the constant from (4.6) and choose

$$t = 2Ct_{dis}\ln\Big(1 + \frac{1}{\kappa t_{dis}}\Big)\,.$$

Iterating Lemma 4.2 (as in Remark 4.3), we note that for every $x \in \mathbb{T}^d$

$$\begin{aligned}\|\rho(x, 0; y, 2t + \varepsilon) - 1\|_{L^1(y)} &\leqslant \|\rho(x, 0; y, 2t + \varepsilon) - 1\|_{L^\infty(y)}\\ &\leqslant \frac{1}{4}\|\rho(x, 0; y, \varepsilon) - 1\|_{L^1(y)} \leqslant \frac{1}{2}\,.\end{aligned}$$

This shows $t_{mix} \leqslant t$, finishing the proof. ☐

The proof of Proposition 2.2 follows immediately from Lemmas 4.1 and 4.4.

5. Coupling of Projections (Proof of Lemma 3.1)

In this section we prove Lemma 3.1 showing that the projected processes Y, $\tilde{Y}$ (defined in (3.4)) will couple in time $O(\varepsilon^2/\kappa)$ in expectation. Coupling of diffusions have been studied by many authors, dating back to Lindvall and Rogers [LR86]. In their original work, Lindvall and Rogers [LR86] provide a method to couple diffusions in $\mathbb{R}^d$ by "reflecting" the noise. Unfortunately, if we use their methods directly the bound we obtain on the coupling time will depend on the Lipschitz constant of the drift; in our case, this is $O(A/\varepsilon)$ which is unbounded. It is for this reason that we modify the cellular flows using the cutoff function ξ. With the cutoff, we have a central region in each square where there is no drift. Once $Y, \tilde{Y}$ enter this region, they can be successfully coupled by reflection.

To carry out the details of the above plan, define

$$Q = [0, \varepsilon/2]^2\,, \quad U = Q \cap \{H > 1/2\}\,, \quad U' = Q \cap \{H > h_0\}\,,$$

for some $h_0 \in (3/4, 1)$ that is independent of ε, A, κ and will be chosen shortly. We will run Y and $\tilde{Y}$ independently until they both enter U', and then reflect the noise until they couple. To estimate the time taken by each of these steps we use the following results.

Lemma 5.1. *Let $u \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{T}^d)$ be a general (not necessarily cellular) divergence free drift, and consider the SDE (4.1) on the d-dimensional torus $\mathbb{T}^d$. The mixing time of X is bounded by*

$$t_{mix}(X) \leqslant \frac{C}{\kappa}\,, \tag{5.1}$$

for some dimensional constant $C = C(d)$ that is independent of u and κ.

Remark 5.2. By rescaling, on a torus with side length ℓ, the bound (5.1) becomes

$$t_{mix}(X) \leqslant \frac{C\ell^2}{\kappa}\,, \tag{5.1$'$}$$

for some dimensional constant C that is independent of ℓ, u and κ.

Remark 5.3. We believe that in this generality there exists a coupling for which $\boldsymbol{E}\tau_{cpl} \leqslant C/\kappa$, however we are presently unable to produce such a coupling.

Lemma 5.4. *Let $\tilde{B}$ be a Brownian motion that is independent of B. There exists a time $t_1 \leqslant O(\varepsilon^2/\kappa)$ such that for all $t \geqslant t_1$, we have*

$$\inf_{y,\tilde{y}\in\mathbb{T}^2_\varepsilon} \boldsymbol{P}^{(y,\tilde{y})}(Y_t,\ \tilde{Y}_t \in U') \geqslant \frac{|U'|^2}{4\varepsilon^4}. \tag{5.2}$$

Here $|U'| = \operatorname{Leb}(U')$ denotes the Lebesgue measure of U'.

Lemma 5.5. *Choose the Brownian motion $\tilde{B}$ to be the Brownian motion B reflected about the line perpendicular to $y - \tilde{y}$. Explicitly, choose $\tilde{B} = MB$, where*

$$M = I - 2\hat{n}\hat{n}^T\,, \qquad \textit{and} \qquad \hat{n} = \frac{y-\tilde{y}}{|y-\tilde{y}|}\,.$$

There exists a time $t_2 \leqslant O(\varepsilon^2/\kappa)$ and a constant $c > 0$ such that

$$\inf_{y,\tilde{y}\in U'} \boldsymbol{P}^{(y,\tilde{y})}(\tau^Y_{cpl} \leqslant t_2) \geqslant c\,.$$

Momentarily postponing the proofs of Lemmas 5.4 and 5.5, we prove Lemma 3.1.

Proof of Lemma 3.1. Choose t_1 according to Lemma 5.4 and run Y and $\tilde{Y}$ independently until time t_1. Lemma 5.4 guarantees that at time t_1 we have (5.2). (Note that $|U'| = O(\varepsilon^2)$, and so $\frac{|U'|^2}{4\varepsilon^4} = O(1)$.)

Now choose t_2 and $\tilde{B}$ according to Lemma 5.5. This construction will guarantee

$$\inf_{y,\tilde{y}\in U'} \boldsymbol{P}^{(y,\tilde{y})}(\tau^Y_{cpl} \geqslant (t_1+t_2)) \leqslant 1 - c'\,, \quad \text{where} \quad c' \stackrel{\text{def}}{=} \frac{c|U'|^2}{4\varepsilon^4} > 0\,,$$

and c is the constant in Lemma 5.5.

In the event that $\tau^Y_{cpl} > t_1 + t_2$, we simply repeat the above two steps. The Markov property will guarantee

$$\inf_{y,\tilde{y}\in U'} \boldsymbol{P}^{(y,\tilde{y})}(\tau^Y_{cpl} \geqslant n(t_1+t_2)) \leqslant (1-c')^n\,.$$

Thus, for any $y, \tilde{y} \in U'$ we see

$$\boldsymbol{E}^{(y,\tilde{y})}\tau^Y_{cpl} = \int_0^\infty \boldsymbol{P}^{(y,\tilde{y})}(\tau^Y_{cpl} \geqslant t)\,dt \leqslant (t_1+t_2)\sum_{n=0}^\infty \boldsymbol{P}^{(y,\tilde{y})}(\tau^Y_{cpl} \geqslant n(t_1+t_2))$$

$$\leqslant (t_1 + t_2) \sum_{n=0}^{\infty} (1 - c')^n \leqslant \frac{t_1 + t_2}{c'} \,,$$

concluding the proof. □

It remains to prove Lemmas 5.1, 5.4 and 5.5.

Proof of Lemma 5.1. Let ϕ solve (2.1). Multiplying by ϕ, using the fact that $\nabla \cdot u = 0$ and the Poincaré inequality shows

$$\|\phi(s+t)\|_{L^2} \leqslant e^{\lambda_1 \kappa t} \|\phi(s)\|_{L^2} \,,$$

where λ_1 is the first non-zero eigenvalue of the Laplacian on $\mathbb{T}^d$. This immediately implies $t_{dis}(X) \leqslant 1/(\lambda_1 \kappa)$, and using Proposition 2.2 concludes the proof. □

Proof of Lemma 5.4. Using Lemma 5.1 and rescaling (see Remark 5.2), we see

$$t_{mix}(Y) \leqslant \frac{C\varepsilon^2}{\kappa} \,.$$

Now choose $N = \log_2(2\varepsilon^2/|U'|)$, and note that N is independent of ε. For every $t \geqslant N t_{mix}(Y)$ and $y \in \mathbb{T}^2_\varepsilon$, we have

$$\Big| \boldsymbol{P}^y(Y_t \in U') - \frac{|U'|}{\varepsilon^2} \Big| \leqslant 2^{-N} \frac{|U'|}{2\varepsilon^2} \quad \text{and hence} \quad \boldsymbol{P}^y(Y_t \in U') \geqslant \frac{|U'|}{2\varepsilon^2} \,,$$

Since $\tilde{Y}$ is independent of Y and satisfies the same bound we obtain (5.2) as claimed. □

Proof of Lemma 5.5. Notice that as long as Y, $\tilde{Y}$ remain in U, they are simply rescaled standard Brownian motions. Let ℓ_b be perpendicular bisector of the line segment joining y and $\tilde{y}$, and τ_ℓ be the hitting time of Y to ℓ_b. The choice of $\tilde{B}$ ensures that if $Y, \tilde{Y}$ remain inside U until they ℓ_b, then they couple at time τ_ℓ.

In order to estimate the hitting time to ℓ_b before exiting U, let $R = |y - \tilde{y}|/2$ be the distance of y to ℓ_b. Let K be the square with center y, side length $2R$, and one pair of sides parallel to ℓ_b. Note that if h_0 is sufficiently closed to 1, this square lies entirely in U. Let τ_K be the exit time of Y from K, and note that

$$\boldsymbol{P}^{y,\tilde{y}}(\tau^Y_{cpl} \leqslant t) \geqslant \boldsymbol{P}^y(\tau_K \leqslant t,\ Y_{\tau_K} \in \ell_b) = \frac{1}{4} \boldsymbol{P}^y(\tau_K \leqslant t) \,.$$

The last equality followed by symmetry, as at time τ_K it is equally likely that Y_{τ_K} belongs to any of the four sides of K.

The last term on the right can be bounded by Chebyshev's inequality, and the fact that the expected exit time of Brownian motion from a square is known. Namely,

$$\boldsymbol{P}^y(\tau_K \leqslant t) \geqslant 1 - \frac{\boldsymbol{E}^y \tau_K}{t} \geqslant 1 - \frac{R^2}{\kappa t} \geqslant 1 - \frac{\varepsilon^2}{\kappa t} \geqslant \frac{1}{2} \,,$$

provided $t \geqslant 2\varepsilon^2/\kappa$. Choosing $t_2 = 2\varepsilon^2/\kappa$ concludes the proof. □

6. Synchronization and Reflection

In this section we prove Lemmas 3.3–3.5. In order to prove these lemmas we will bounds on the boundary layer crossing time. These have been studied previously by various authors (see for instance [Kor04, FW12, IN16, HIK+18]), and the version we quote here can be obtained by a direct rescaling of those in [IN16]. Define the *boundary layer* by $\mathcal{B}_\delta$ by

$$\mathcal{B}_\delta \stackrel{\text{def}}{=} \{|H| < \delta\} \subset \mathbb{T}^2\,, \tag{6.1}$$

where we recall from (2.5) that $\delta = \sqrt{\kappa/A}$. The middle of the boundary layer is the level set $\{H = 0\}$, and is known as the *separatrix*.

We will now study repeated exits from the boundary layer, followed by returns to the separatrix. Define the sequences of stopping times σ_n and τ_n inductively by starting with $\tau_0 = 0$. For $n \geqslant 1$, define

$$\sigma_n = \inf\{t \geqslant \tau_{n-1} \mid X_t \notin \mathcal{B}_\delta\} \tag{6.2}$$

$$\tau_n = \inf\{t \geqslant \sigma_n \mid H(X_t) = 0\}\,. \tag{6.3}$$

That is, σ_n is the first exit from the boundary layer $\mathcal{B}_\delta$ after time τ_n, and τ_n is the first return to the separatrix after time σ_n.

At time τ_n we must have either $X^1 \in \frac{\varepsilon}{2}\mathbb{Z}$, or $X^2 \in \frac{\varepsilon}{2}\mathbb{Z}$. We now separate the times when $X^1 \in \frac{\varepsilon}{2}\mathbb{Z}$, and when $X^2 \in \frac{\varepsilon}{2}\mathbb{Z}$. Given $i \in \{1, 2\}$, let $\tau_0^i = 0$ and inductively define

$$\tau_n^i = \inf\Big\{\tau_k > \tau_{n-1}^i \;\Big|\; X^i_{\tau_k} \in \frac{\varepsilon}{2}\mathbb{Z}\Big\}\,.$$

We claim that up to a logarithmic factor, the chance that $\tau_n^i \leqslant t$ is comparable to the number of crossings of a standard Brownian motion over an interval of size $\varepsilon/\sqrt{A}$. This is the first lemma we state.

Lemma 6.1. *There exists a constant $c > 0$ such that, for $n \in \mathbb{N}$, $i \in \{1, 2\}$, we have*

$$\inf_{|H(x)|<\delta} \boldsymbol{P}^x(\tau_n^i \leqslant t) \geqslant \Big(1 - \frac{cn\varepsilon\delta|\ln\delta|}{\sqrt{\kappa t}}\Big)^+\,. \tag{6.4}$$

This lemma is simply a rescaling of Lemma 2.2 in [IN16], and we refer the reader there for the proof. We remark, however, that the proof in [IN16] uses PDE techniques from [CS89, FP94, NPR05, IKNR14]. Lemma 6.1 can also be proved directly using probabilistic techniques, and we refer the reader to [FW12, Kor04, DK08, HIK+18] for related crossing estimates.

In order to apply Lemma 6.1, we need the process X to enter the boundary layer $\mathcal{B}_\delta$. This happens in time at most $O(\varepsilon^2|\ln\delta|/\kappa)$, and is the content of our next lemma.

Lemma 6.2. *Let σ_e be the first hitting time of X to the level set $\{H = \delta\}$ (i.e. $\sigma_e = \inf\{t \geqslant 0 \mid H(X_t) = \delta\}$). Then*

$$\sup_{x\in\mathbb{T}^2} \boldsymbol{E}^x\sigma_e \leqslant \frac{C\varepsilon^2}{\kappa}\,. \tag{6.5}$$

Proof. We first project to the torus of side length ε, and note that $\sigma_e = \inf\{t \geqslant 0 \mid H(Y_t) = \delta\}$. Note $\{H > \delta\}$ contains two connected components, each occupying an area of at most $1/4$ of the torus $\mathbb{T}^2_\varepsilon$. For any $x \in \mathbb{T}^2_\varepsilon$, let U be the connected component of $\{H > \delta\}$ that contains x. Thus, for any $t \geqslant t_{mix}(Y)$ we know

$$\Big|\boldsymbol{P}^x(X_t \in U^c) - \frac{|U^c|}{\varepsilon^2}\Big| \leqslant \frac{1}{4}\,, \quad \text{and hence} \quad \boldsymbol{P}^{x_0}(X_t \in U) \geqslant \frac{1}{2}\,.$$

By continuity of trajectories we note that the event $\{\sigma_e \leqslant t\} \supseteq \{X_t \in U^c\}$, and so $\boldsymbol{P}^{x_0}(\sigma_e < t) \geqslant 1/2$. In the event that $\sigma_e > t$, we use the Markov property and repeat the above argument to yield

$$\boldsymbol{E}^x \sigma_e \leqslant 2t_{mix}(Y)\,.$$

By (5.1′) with $\ell = \varepsilon$ we know $t_{mix}(Y) \leqslant C\varepsilon^2/\kappa$, concluding the proof. □

6.1. **Proofs of the hitting time estimates (Lemmas 3.2 and 3.4).** Then we may estimate the first hit at vertical boundary lines.

Proof of Lemma 3.2. Notice that, periodicity of v and the synchronous choice $\tilde{B} = B$, implies $\sigma_v = \tilde{\sigma}_v$. Thus we only have to prove (3.7). Without loss of generality assume $(\Pi_\varepsilon X_0)^1 \in [0, \varepsilon/2)$. (We clarify that $(\Pi_\varepsilon X_0)^1$ refers to the first coordinate of $\Pi_\varepsilon X_0$.) If $(\Pi_\varepsilon X_0)^1 = 0$, then $\sigma_v = 0$, and there is nothing to prove, and thus we may assume $(\Pi_\varepsilon X_0)^1 \in (0, \varepsilon/2)$. Let $V \subseteq \mathbb{T}^2_\varepsilon$ be the set of all points y such that $y_1 \in [\varepsilon/2, \varepsilon]$, and note that V occupies half the area of $\mathbb{T}^2_\varepsilon$. Thus for any $t \geqslant 2t_{mix}(Y)$ we see

$$\Big|\boldsymbol{P}(Y_t \in V) - \frac{1}{2}\Big| \leqslant \frac{1}{4} \quad \text{and hence} \quad \boldsymbol{P}(Y_t \in V) \geqslant \frac{1}{4}\,.$$

By continuity of trajectories, $\{\sigma_v \leqslant t\} \supseteq \{Y_t \in V\}$, and so $\boldsymbol{P}(\sigma_v \leqslant t) \geqslant 1/4$. If $\sigma_v > t$, then we use the Markov property and repeat the above argument to show

$$\boldsymbol{E}\sigma_v \leqslant 8t_{mix}(Y) \leqslant \frac{C\varepsilon^2}{\kappa}\,,$$

as desired. □

Proof of Lemma 3.4. The proof is identical to the proof of Lemma 3.2. □

6.2. **Coupling time estimates (Lemmas 3.3 and 3.5).** We now turn our attention to Lemma 3.3. Note first that by definition v is ε periodic and

$$\begin{aligned} v_1(-x_1, x_2) &= -v_1(x_1, x_2)\,, & v_2(-x_1, x_2) &= v_2(x_1, x_2)\,, \\ v_1(x_1, -x_2) &= v_1(x_1, x_2)\,, & v_2(x_1, -x_2) &= -v_2(x_1, x_2)\,, \\ v(x_1 + \tfrac{\varepsilon}{2}, x_2) &= -v(x_1, x_2)\,, & v(x_1, x_2 + \tfrac{\varepsilon}{2}) &= -v(x_1, x_2)\,. \end{aligned}$$

As a result choosing $\tilde{B} = (-B_1, B_2)$ and the assumptions $\Pi_\varepsilon X_0 = \Pi_\varepsilon \tilde{X}$ and $X_0^1, \tilde{X}_0^1 \in \frac{\varepsilon}{2}\mathbb{Z}$ imply

$$X_t^1 = -\tilde{X}_t^1 \pmod{\tfrac{\varepsilon}{2}}\,, \quad \text{and} \quad X_t^2 = \tilde{X}_t^2 \pmod{\varepsilon}\,. \tag{6.6}$$

Let

$$\ell_1 \stackrel{\text{def}}{=} \Big\{\frac{X_0^1 + \tilde{X}_0^1}{2}\Big\} \times \mathbb{T}^1 \subseteq \mathbb{T}^2$$

be the vertical line half way between X_0 and $\tilde{X}_0$. Note $(X_0^1 + \tilde{X}_0^1)/2 \in \frac{\varepsilon}{2}\mathbb{Z}$ and so ℓ_1 is contained in the separatrix. By (6.6) we see that τ_v is exactly the hitting time of X to ℓ_1. Thus we may now ignore $\tilde{X}$ and simply estimate the hitting time of X to ℓ_1.

Note that $X_{\tau_n^1} \in (\frac{\varepsilon}{2}\mathbb{Z}) \times \mathbb{R}$ for all n and behaves like a random walk on the collection of vertical lines $(\frac{\varepsilon}{2}\mathbb{Z}) \times \mathbb{T}^1 \subseteq \mathbb{T}^2$. There are $2/\varepsilon$ such vertical lines in the torus $\mathbb{T}^2$, and so we expect that after $O(1/\varepsilon^2)$ steps of this random walk, $X_{\tau_n^1}$ will land in our desired line segment ℓ_1. This is our next result.

Lemma 6.3. *Note that $\tau_v = \inf\{t \geqslant 0 \mid X_t \in \ell_1\}$. There exists $p_0 > 0$, and a constant C_1, independent of A, ε, κ, such that, for $n = C_1/\varepsilon^2$, and $x \in \mathbb{T}^2$ such that $x_1 \in \frac{\varepsilon}{2}\mathbb{Z}$,*

$$\boldsymbol{P}^x(\tau_v \leqslant \tau_n^1) \geqslant p_0\,.$$

Postponing the proof of Lemma 6.3 to Section 6.3, we prove Lemma 3.3.

Proof of Lemma 3.3. As explained above, τ_v is the hitting time of X to the bisector ℓ_1. Using Lemmas 6.1 and 6.3 we see that

$$\boldsymbol{P}(\tau_n^1 \leqslant t_1) \geqslant 1 - \frac{p_0}{2}\,, \quad \text{where } t_1 = \frac{4c^2 C_1^2 |\ln \delta|^2}{p_0^2 \varepsilon^2 A}\,, \quad n = \frac{C_1}{\varepsilon^2}\,. \tag{6.7}$$

Here c is the constant from equation (6.4), and p_0, C_1 are constants from Lemma 6.3. With Lemma 6.3, we also see that

$$\boldsymbol{P}^x(\tau_v \leqslant \tau_n^1) \geqslant p_0\,. \tag{6.8}$$

Combining (6.7) and (6.8) gives

$$\boldsymbol{P}^x\big(\tau_v \leqslant \tau_n^1 \leqslant t_1\big) \geqslant \frac{p_0}{2}\,,$$

which implies $\boldsymbol{P}(\tau_v \leqslant t_1) \geqslant \frac{p_0}{2}$. Using the Markov property and iterating this implies

$$\boldsymbol{P}(\tau_v > kt_1) \leqslant \Big(1 - \frac{p_0}{2}\Big)^k\,, \quad \text{and hence } \boldsymbol{E}\tau_v \leqslant \frac{2t_1}{p_0}\,,$$

finishing the proof. □

Proof of Lemma 3.5. The proof is identical to the proof of Lemma 3.3. Note that at times when $X_t^2 = \tilde{X}_t^2$, we actually have $X_t = \tilde{X}_t$ and hence $X_{\tau_h} = \tilde{X}_{\tau_h}$. □

6.3. **The hitting time to the bisector (Lemma 6.3).** In order to prove Lemma 6.3 we will lift trajectories of X from the torus $\mathbb{T}^2$ to the covering space $\mathbb{R}^2$. For clarity, we will denote the lifted process by $\hat{X}$. Define the family of lines

$$\hat{\ell}_1 = \Big\{x \in \mathbb{R}^2 \;\Big|\; x_1 = n + \frac{n_0\varepsilon}{2}\,, n \in \mathbb{Z}\Big\}\,,$$

where $n_0 \in \mathbb{Z}$ is chosen such that

$$\ell_1 = \Big\{x \in \mathbb{T}^2 \;\Big|\; x_1 = \frac{n_0\varepsilon}{2}\Big\}\,.$$

Note that the event of X hitting ℓ_1 on $\mathbb{T}^2$ is exactly the same as the event of $\hat{X}$ hitting $\hat{\ell}_1$ on $\mathbb{R}^2$. Moreover, if $\hat{X}$ travels a horizontal distance of at least 1, then it must pass through one of the lines in $\hat{\ell}_1$. We will use this to estimate $\boldsymbol{P}(\tau_v \leqslant \tau_n^1)$.

Lemma 6.4. *Suppose $\hat{X}$ satisfies the SDE (2.10) in $\mathbb{R}^2$, with $\hat{X}_0 = \hat{x} \in \mathbb{R}^2$ such that $\hat{x}_1 = 0$. There exist constants C_1, $p_0 > 0$, independent of A, ε, κ, such that, for $n = C_1/\varepsilon^2$ we have*

$$\boldsymbol{P}^{\hat{x}}(|\hat{X}^1_{\tau_n^1}| > 1) \geqslant p_0\,. \tag{6.9}$$

Proof. Let $S_n = \hat{X}^1_{\tau_n^1}$, and observe that by symmetry of v we must have $\boldsymbol{E}^{\hat{x}} S_n = 0$. If $\boldsymbol{E}^{\hat{x}} S_n^2 \geqslant 1$ we note

$$\Big(\boldsymbol{E}^{\hat{x}} S_n^2 - 1\Big)^2 \leqslant \Big(\boldsymbol{E}^{\hat{x}} S_n^2 \boldsymbol{1}_{\{|S_n| \geqslant 1\}}\Big)^2 \leqslant \boldsymbol{E}^{\hat{x}} S_n^4 \boldsymbol{P}^{\hat{x}}(|S_n| \geqslant 1)$$

and hence

$$\boldsymbol{P}^{\hat{x}}(|S_n| \geqslant 1) \geqslant \frac{(\boldsymbol{E}^{\hat{x}} S_n^2 - 1)^2}{\boldsymbol{E}^{\hat{x}} S_n^4}\,. \tag{6.10}$$

whenever $\mathrm{Var}(S_n) > 1$.

To use (6.10), we need to show $\boldsymbol{E}^{\hat{x}} S_n^2 \geqslant 1$, and find a suitable upper bound for $\boldsymbol{E}^{\hat{x}} S_n^4$. For the first part we note [IN16] shows that the variance of S_n is comparable to that of a random walk with steps of size ε. That is, we know

$$\boldsymbol{E}^{\hat{x}} S_n^2 \geqslant c_1 n \varepsilon^2\,, \tag{6.11}$$

for some constant $c_1 > 0$, that is independent of ε, A and κ. Thus choosing $n = 2/(c_1 \varepsilon^2)$ will guarantee $\boldsymbol{E}^{\hat{x}} S_n^2 \geqslant 1$.

For the second part we need to find an upper bound for $\boldsymbol{E} S_n^4$. For simplicity, let $\xi_m = \hat{X}^1_{\tau^1_{m+1}} - \hat{X}^1_{\tau^1_m}$, with $\tau^1_0 = 0$, so that $S_n = \xi_0 + \cdots + \xi_{n-1}$. Notice

$$\boldsymbol{E}^x S_n^4 = \sum_{m=0}^{n-1} \boldsymbol{E}^x |\xi_m|^4 + 6 \sum_{m'=1}^{n-1} \sum_{m=0}^{m'-1} \boldsymbol{E}^x |\xi_m|^2 |\xi_{m'}|^2\,,$$

since the cross terms vanish by symmetry.

From Lemma 2.1 in [IN16], we know that

$$\boldsymbol{E}^{\hat{x}} |\xi_m|^2 \leqslant c_2 \varepsilon^2\,,$$

for some finite constant c_2 that is independent of ε, A and κ. The same proof (Section 5 in [IN16]) also shows that

$$\boldsymbol{E}^{\hat{x}} |\xi_m|^4 \leqslant c_2 \varepsilon^4\,.$$

Moreover, for $m < m'$, by tower property,

$$\boldsymbol{E}^{\hat{x}}(|\xi_m|^2 |\xi_{m'}|^2) = \boldsymbol{E}^{\hat{x}}\Big(|\xi_m|^2 \boldsymbol{E}^{\hat{X}_{\tau^1_{m+1}}} |\xi_{m'}|^2\Big) \leqslant \boldsymbol{E}^{x}\left[|\xi_m|^2 c_2 \varepsilon^2\right] \leqslant (c_2 \varepsilon^2)^2\,.$$

Thus

$$\boldsymbol{E}^{\hat{x}} S_n^4 \leqslant c_2 n \varepsilon^4 + c_2^2 n^2 \varepsilon^4 \leqslant c n^2 \varepsilon^4\,, \tag{6.12}$$

where $c = 2c_2(1 + c_2)$.

Combining (6.10), (6.11) and (6.12) we see

$$\boldsymbol{P}^{\hat{x}}(|S_n| > 1) \geqslant \frac{(c_1 n \varepsilon^2 - 1)^2}{c n^2 \varepsilon^4}\,. \tag{6.13}$$

Choosing $n = C_1/\varepsilon^2$ for some large constant C_1, we obtain (6.9) as desired. ☐

Using this, we prove Lemma 6.3.

Proof of Lemma 6.3. Note that $x_1 \in \frac{\varepsilon}{2}\mathbb{Z}$. Using symmetry and periodicity, we may, without loss of generality, assume $x_1 = 0$.

Lifting the process X to $\mathbb{R}^2$, we recall that when $|\hat{X}^1_t| \geqslant 1$, the trajectory of $\hat{X}$ must have passed through one of the lines in $\hat{\ell}_1$. This implies

$$\boldsymbol{P}^x\big(\tau_v \leqslant \tau^1_n\big) \geqslant \boldsymbol{P}^{\hat{x}}\big(|\hat{X}^1_{\tau^1_n}| \geqslant 1\big)\,,$$

and applying Lemma 6.4 concludes the proof. ☐

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