Growth of Sobolev norms and loss of regularity in transport equations

Gianluca Crippa* Tarek Elgindi† Gautam Iyer‡ Anna L. Mazzucato§

October 3, 2021

Abstract

We consider transport of a passive scalar advected by an irregular divergence free vector field. Given any non-constant initial data $\bar{\rho} \in H^1_{\text{loc}}(\mathbb{R}^d)$, $d \geqslant 2$, we construct a divergence free advecting velocity field v (depending on $\bar{\rho}$) for which the unique weak solution to the transport equation does not belong to $H^1_{\text{loc}}(\mathbb{R}^d)$ for any positive positive time. The velocity field v is smooth, except at one point, controlled uniformly in time, and belongs to almost every Sobolev space $W^{s,p}$ that does not embed into the Lipschitz class. The velocity field v is constructed by pulling back and rescaling an initial data dependent sequence of sine/cosine shear flows on the torus. This loss of regularity result complements that in *Ann. PDE*, 5(1):Paper No. 9, 19, 2019.

In memory of Charles "Charlie" Doering.

1 Introduction

This article concerns the effect of transport by an irregular vector field on a passive scalar. In what follows, we refer to *irregular transport* as transport by a vector field that does not possess Lipschitz regularity in the space variable.

It is well known that, if the advecting vector field is Lipschitz uniformly in time, the Cauchy-Lipschitz theory applies and the flow is well-defined pointwise in space and time. The flow and its inverse are then also Lipschitz and, at least, Lipschitz regularity of the initial data is preserved under the action of the flow. In this case, the unique solution to the linear transport equation is obtained by composing the initial data with the inverse of the flow map.

In this work, we are interested in loss of regularity for the weak solution of the transport equation, when the advecting vector field is, in some sense to be made precise, as close as possible to being Lipschitz in space. We therefore consider vector fields that belong to a suitable Sobolev space in space, uniformly in time. Informally, we then show that given any (non-constant) initial data in $\mathbb{R}^d$, $d \geqslant 2$, with square integrable derivative, there exists a divergence-free vector field that is almost Lipschitz uniformly in time such that the solution of the associated transport equation loses its regularity instantaneously in time. The loss of regularity is due to an amplification effect on the derivative of the solution by the action of the advecting flow.

To fix notation, denote the passive scalar by $\rho = \rho(x,t)$, and the advecting field by $v = v(x,t)$. Here $t \geqslant 0$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. We assume ρ is a weak solution of the linear transport equation:

$$\partial_t \rho + v \cdot \nabla \rho = 0, \tag{1}$$

on $\mathbb{R}^d \times [0,\infty)$, with initial data $\bar{\rho}(x)$.

We also use standard notation for the function spaces we use. We employ the Sobolev space $W^{k,p}(\mathbb{R}^d)$, where $k \in \mathbb{Z}_+$ and $1 \leqslant p \leqslant \infty$, defined as:

$$W^{k,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) \mid \partial^\alpha f \in L^p(\mathbb{R}^d),\ |\alpha| \leqslant k\},$$

where we used multi-index notation for derivatives. The spaces $W^{r,p}$, where $r > 0$ and $1 \leqslant p \leqslant \infty$, are then defined by interpolation. For $p = 2$, the space $W^{r,2}$ coincides with the space H^r, defined via the Fourier Transform. (We refer the reader to [1] for a comprehensive introduction).

We assume that the advecting vector field v is compactly supported in $\mathbb{R}^d$ and divergence-free, while we assume that the initial condition $\bar{\rho} \in H^1_{\text{loc}}(\mathbb{R}^d)$. The vector field we construct to prove loss of regularity belongs to all Sobolev spaces $W^{r,p}(\mathbb{R}^d)$, where $1 \leqslant p < \infty$ and $1 \leqslant r < d/p + 1$, uniformly in time. By the Sobolev Embedding Theorem, these spaces give essentially all Sobolev classes that do not embed in the Lipschitz class $W^{1,\infty}(\mathbb{R}^d)$.

The loss of regularity result presented here extends the results by some of the authors in [2]. There, it was proved that there exists a smooth, compactly supported initial data $\bar{\rho}$ and a vector field $v \in L^\infty([0,\infty); W^{1,p}(\mathbb{R}^d))$, for $1 < p < \infty$, such that the weak solution ρ of (1) does not belong to H^s for any $s > 0$ instantaneously in time. (In [3], the authors prove, non-constructively, that loss of regularity is a generic phenomenon in the sense of Baire's Category

*Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland

†Mathematics Department, Duke University, 120 Science Drive, Durham, NC 27708-0320, U.S.A.

‡Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, U.S.A.

§Mathematics Department, Penn State University, University Park, PA, 16802, U.S.A.

Theorem.) By contrast, we are able to show loss of regularity for all non-constant initial data in $H^1_{\rm loc}$ (with v depending on the initial data), but we can only prove $\rho(\cdot,t)\notin H^s_{\rm loc}$, for any $s\geqslant 1$ and for all $t>0$.

In both [2] and this work, we construct at the same time the vector field v and the advected scalar ρ via an iterative procedure starting from a pair u^0, θ^0 (where θ^0 solves the transport equation with advecting field u^0) which acts as a building block, and applying a suitable sequence of rescalings, where each rescaling produces a pair u^n, θ^n. In [2], u^0 is a vector field that mixes a certain initial tracer configuration optimally in time, and one can control the growth of the H^s norm of θ^0 from below for all $s>0$ via interpolation, since u^0 drives all negative Sobolev norms of the tracer to zero exponentially fast. The action of each rescaling is to accelerate the growth of the H^s-norms of θ^n as $n\to\infty$. The different u^n and θ^n are combined to give rise to the vector field v and associated weak solution ρ of (1), the Sobolev norms of which blow up for any $t>0$. This result is optimal from the point of view of the loss of regularity, in the sense that the only regularity that is propagated generically by a velocity field with the same regularity as v is essentially a "logarithm" of a derivative [4, 5]. We mention also the related work [6], where the author gives an example of a divergence-free vector field in H^1 such that its flow is not in any Sobolev space with positive regularity. His construction is random at its core, while the one in [2] is deterministic and explicit.

In this note, we also use a suitable sequence of rescalings of basic flows. These flows are constructed in such a way to lead to growth in time of the H^1 Sobolev norm of any initial data for the passive scalar. Although the vector field depends on the initial data, it enjoys universal bounds. The vector fields are constructed using shear flows and, after rescaling, the growth happens on certain cubes that depend on the initial data $\bar\rho$ for (1).

Our main result is the following.

Theorem 1.1. *Let $\bar\rho\in H^1_{loc}(\mathbb{R}^d)$ be a non-constant function. There exists a compactly supported divergence-free vector field $v\in L^\infty([0,\infty)\times\mathbb{R}^d)$, depending on $\bar\rho$, such that the following hold:*

(a) The velocity field v is smooth except at one point in $\mathbb{R}^d$. Moreover,

$$v\in L^\infty([0,\infty);W^{r,p}(\mathbb{R}^d))\qquad \text{for every } 1\leqslant p<\infty\,,\ \text{and } 1\leqslant r<\frac{d}{p}+1\,.$$

(b) The unique weak solution of (1) in $L^\infty([0,\infty);L^2_{loc}(\mathbb{R}^d))$ with initial data $\bar\rho$ is such that

$$\rho(\cdot,t)\notin H^1_{loc}(\mathbb{R}^d)\qquad \text{for every } t>0\,.$$

As mentioned earlier, if $r>d/p+1$ and $v\in L^\infty([0,\infty);W^{r,p}(\mathbb{R}^d))$, then the Sobolev embedding theorem implies that v is Lipschitz in space, uniformly in time. This in turn implies that H^1 regularity of the initial data is preserved and so the threshold $r<d/p+1$ above can not be improved.

The main idea behind the proof is as follows:

1. The first step is an elementary observation about periodic functions. Take any non-constant periodic function $\bar\phi$. Then, we claim at least one sine or cosine shear flow parallel to one of the coordinate axis must increase the H^1 norm of $\bar\phi$ by a constant factor (see Lemma 2.1, below).

2. By localizing and rescaling the above flow, we can obtain a countable (shrinking) family of separated cubes that cluster at one point, so that in each cube the flow increases the H^1 norm of the advected scalar by a larger and larger factor (see Section 3, below).

3. Now we need to ensure that the rescaling factors and the location of the cubes can be chosen so that the H^1 norm of the solution diverges at any positive time, but the velocity field remains sufficiently regular. Our choice ensures $v\in W^{r,p}$ for every r below the critical Sobolev embedding threshold (i.e. $r<d/p+1$).

The rest of the paper is organized as follows. In Section 2, we introduce the basic building block in the construction and show how the building block leads to growth of the Sobolev norms for solutions of the transport equation (1). Then, in Section 3 we conclude the proof of loss of regularity. Lastly, in Section 4 we draw some conclusions.

Throughout the paper, we denote the total mass of any measurable (with respect to the d-dimensional Lebesgue measure) set Ω by $|\Omega|$, while $\mathbf{1}_\Omega$ denotes the indicator function of the set Ω, if not empty. We employ the notation $\lesssim$ to denote a bound which holds up to a generic constant that may change from line to line, and similarly for $\gtrsim$.

2 Construction of the basic flow and growth of Sobolev norms

The aim of this section is to carry out the first step in the proof of the main theorem. We first prove the elementary observation (Lemma 2.1, below) that for any non-constant periodic function, at least one sine or cosine shear along a coordinate axis can be used to increase its H^1 norm by a constant factor. Next we lift this construction to compactly supported cubes in $\mathbb{R}^d$, and iterate to obtain exponential growth in time (Proposition 2.2, below). This will be the basic building block that will be rescaled and used in subsequent steps in Section 3.

To notationally separate the construction of our building block from the actual rescaled flow in Theorem 1.1, in this section we use u to denote the advecting velocity field on the torus and ϕ to denote the passively advected (periodic) scalar with initial data $\bar\phi$. For convenience we will work with 8-periodic functions on the d-dimensional torus $\mathbb{T}^d$ obtained by identifying parallel faces of the cube $[0,8]^d$.

Lemma 2.1. *Let $A > 0$ and define $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ by*

$$f_1(z) = A\sin(2\pi z) \qquad \text{and} \qquad f_2(z) = A\cos(2\pi z)\,,$$

and let $\Omega_0 \subseteq \mathbb{T}^d$ be a piecewise C^1 domain. For any $\bar\phi \in H^1(\mathbb{T}^d)$, $T > 0$, there exists a divergence-free velocity field U (depending on $\mathbf{1}_{\Omega_0}\bar\phi$ and T) such that the following hold:

1. *The velocity field U is a shear flow of the form*

$$U(x) = \pm f_i(x_j) e_{j'}\,, \qquad \text{where } j' = \begin{cases} j+1 & j < d \\ 1 & j = d\,. \end{cases} \tag{2}$$

   Here $e_j \in \mathbb{R}^d$ is the j^{th} standard basis vector, and x_j denotes the j^{th} coordinate of $x \in \mathbb{T}^d$.

2. *The solution to the transport equation*

$$\partial_t \phi + U \cdot \nabla\phi = 0 \tag{3}$$

   on $\mathbb{T}^d$ with initial data $\bar\phi$ satisfies

$$\|\nabla\phi(\cdot, T)\|^2_{L^2(\Omega_T)} \geqslant \Big(1 + \frac{2\pi^2 A^2 T^2}{d}\Big) \|\nabla\bar\phi\|^2_{L^2(\Omega_0)}\,. \tag{4}$$

   Here Ω_T is the image of Ω_0 under the flow map of the shear flow U after time T.

Proof. Given $i, i' \in \{1, 2\}$ and $j \in \{1, \dots, d\}$, we let

$$u_{i,i',j}(x) = (-1)^i f_{i'}(x_j) e_{j'}\,,$$

and we let $\phi_{i,i',j}$ be the solution of the transport equation (3) with vector field $u_{i,i',j}$. We denote by $\Omega_{T,i,i',j}$ the image of Ω_0 under the flow map of the shear flow $u_{i,i',j}$ after time T. Since

$$\phi_{i,i',j}(x,t) = \bar\phi(x - (-1)^i f_{i'}(x_j) t e_{j'})\,,$$

we compute

$$\partial_k \phi_{i,i',j} = \begin{cases} \partial_k\bar\phi - (-1)^i f_i'(x_j) t \partial_{j'}\bar\phi & k = j\,, \\ \partial_k \bar\phi & k \neq j\,. \end{cases}$$

We square the expression above and sum over i, i'. Using the fact that $\sum_{i'} f_{i'}^2 = A^2$, integrating over $\Omega_{T,i,i',j}$, and changing variables back to the original domain Ω_0 gives

$$\sum_{i,i'} \|\partial_k \phi_{i,i',j}\|^2_{L^2(\Omega_{T,i,i',j})} = \begin{cases} 4\|\partial_j\bar\phi\|^2_{L^2(\Omega_0)} + 8\pi^2 A^2 t^2 \|\partial_{j'}\bar\phi\|^2_{L^2(\Omega_0)} & k = j\,. \\ 4\|\partial_k\bar\phi\|^2_{L^2(\Omega_0)} & k \neq j\,, \end{cases}$$

Summing over $k \in \{1, \dots, d\}$ and $j \in \{1, \dots, d\}$ then shows that

$$\sum_{i,i',j} \|\nabla\phi_{i,i',j}\|^2_{L^2(\Omega_{T,i,i',j})} = 4d\|\nabla\bar\phi\|^2_{L^2(\Omega_0)} + 8\pi^2 A^2 t^2 \|\nabla\bar\phi\|^2_{L^2(\Omega_0)}\,.$$

Since there are $4d$ terms on the sum on the left, there must exist one term that is at least a $1/(4d)$ fraction of the right hand side. This immediately yields (4) as claimed. ∎

Our next task is to show that for any (non-constant) initial datum, we can find a smooth compactly supported divergence-free vector field in $\mathbb{R}^d$ for which the solution to the transport equation grows exponentially in H^1. This is the main result of this section, and is what will be used in the proof of Theorem 1.1.

Proposition 2.2. *Let $\bar\theta \in H^1_{\text{loc}}(\mathbb{R}^d)$ and fix $\alpha > 0$. There exists a constant $C(\alpha, d)$ (independent of $\bar\theta$) and a divergence-free vector field $u \colon \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ (depending on $\bar\theta$) such that u is piecewise constant in time, supported on the cube $\tilde\Omega_0 = (-3, 4)^d$, satisfies the bound*

$$\sup_{0 \leqslant \tau < \infty} \|u(\cdot, \tau)\|_{C^1(\mathbb{R}^d)} \leqslant C(\alpha, d)\,,$$

and the following two assertions hold.

1. *The unique solution of the transport equation*

$$\partial_t \theta + u \cdot \nabla\theta = 0 \tag{5}$$

   in $\mathbb{R}^d$ with initial data $\bar\theta$, satisfies

$$\|\nabla\theta(\cdot, n)\|_{L^2(\Omega_0)} \geqslant e^{\alpha n} \|\nabla\bar\theta\|_{L^2(\Omega_0)}\,,$$

   for all non-negative integer times $n \in \mathbb{N}$. Here Ω_0 is the cube $(0,1)^d$ in $\mathbb{R}^d$;

2. *For all times $t \geqslant 0$, the above solution θ satisfies*

$$\|\nabla\theta(\cdot, t)\|_{L^2(\tilde\Omega_0)} \geqslant e^{\alpha t - \beta} \|\nabla\bar\theta\|_{L^2(\Omega_0)}\,. \tag{6}$$

   Here β is a constant that depends on α and d, but not on $\bar\theta$.

Remark 2.3. With minor modifications to the proof one can ensure that the velocity field u in Proposition 2.2 is in fact smooth, and satisfies $\|u(\cdot, t)\|_{C^k} \leqslant C(\alpha, d, k)$ for all $t \geqslant 0$.

The proof of Proposition 2.2 consists of two steps. The first step involves pulling back the shear flow on the torus from Lemma 2.1 to a compactly supported flow in $\mathbb{R}^d$. We do this in Lemma 2.4, below. Once this is established, we simply iterate this procedure to obtain exponential growth at integer times. Since the norm of u is controlled uniformly in time, the H^1 norm at non-integer times can be estimated by giving up a small factor.

Lemma 2.4. *Let $\bar\theta \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, and fix $T>0$, $\alpha'>1$. There exists a divergence-free vector field u on $\mathbb{R}^d \times [0,\infty)$ (depending on $\bar\theta$, α' and T) such that the following hold:*

1. *The vector field u is piecewise constant in time, supported on the cube $\tilde\Omega_0 = (-3,4)^d$, and satisfies*
$$\sup_{0\leqslant\tau\leqslant T}\|u(\cdot,\tau)\|_{C^1(\mathbb{R}^d)} \leqslant C(d)\Big(1+\frac{\alpha'}{T}\Big)\,,$$
for some dimensional constant $C(d)>0$, that is independent of $\bar\theta$.
2. *The weak solution of the transport equation (5) in $\mathbb{R}^d$ with initial data $\bar\theta$ satisfies*
$$\|\nabla\theta(\cdot,T)\|_{L^2(\Omega_0)} \geqslant \alpha'\,\|\nabla\bar\theta\|_{L^2(\Omega_0)}\,,$$
where $\Omega_0=(0,1)^d\subseteq\mathbb{R}^d$.

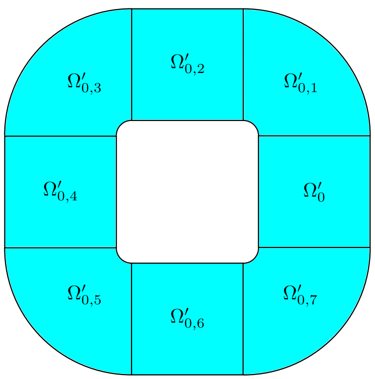

Figure 1: The rounded octagonal track $\mathcal{A}'_1$

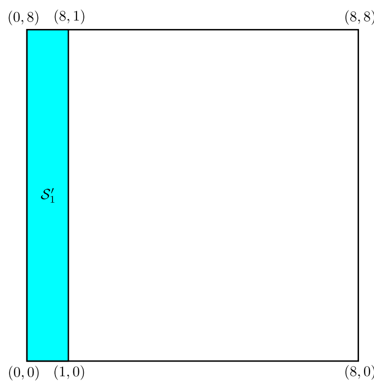

Figure 2: The strip $\mathcal{S}'_1 \subseteq \mathbb{T}^2$.

The main idea behind the proof of Lemma 2.4 is as follows. Momentarily suppose $d=2$ and view Ω_0 as a subset of the two-dimensional torus $\mathbb{T}^2$ obtained by identifying parallel sides of the square $[0,8]^2$. Now, by Lemma 2.1, there is a horizontal or vertical trigonometric shear, U, that increases the H^1 norm by a constant factor. Suppose this shear was vertical. In this case the flow would spread out the initial data over the vertical strip $\mathcal{S}'_1$, shown in Figure 2. The strip $\mathcal{S}'_1\subseteq\mathbb{T}^2$ is topologically an annulus, and so we can find an annulus $\mathcal{A}'_1\subseteq\mathbb{R}^2$ (see Figure 1) and an area preserving diffeomorphism $\varphi_1\colon\mathcal{A}'_1\to S'_1$ such that φ_1 is the identity on Ω_0. We use φ_1 to pullback U to a vector field u on $\mathcal{A}'_1$. This velocity field will spread the initial data out in the track $\mathcal{A}'_1$. However, since the area of Ω_0 is one eighth the area of $\mathcal{A}'_1$, one can give up a factor of 8, perform a radial rotation along the track and ensure that the H^1 norm in Ω_0 itself grows as desired. We now carry out the details.

Proof of Lemma 2.4. Let $\mathcal{A}'_1\subseteq\mathbb{R}^2$ be the rounded octagonal track constructed as follows (see Figure 1): the region Ω'_0 is the square $(0,1)^2\subseteq\mathbb{R}^2$, the regions $\Omega'_{0,2}$, $\Omega'_{0,4}$ and $\Omega'_{0,6}$ are squares of side length 1. The remaining four regions are quarter annuli with inner radius $\frac{2}{\pi}-\frac12$ and outer radius $\frac{2}{\pi}+\frac12$. These radii are chosen so that the area of each piece is 1. We observe that $\mathcal{A}'_1\subset(-3,4)^2$.

Let $\mathcal{S}'_1=(0,1)\times(0,8)\subseteq\mathbb{T}^2$ be the strip of width 1 parallel to the x_2 axis (see Figure 2). Let $\varphi'_1\colon\bar{\mathcal{A}}'_1\to\bar{\mathcal{S}}'_1\subseteq\mathbb{T}^2$ be an area preserving diffeomorphism such that
$$\varphi'_1(x')=x' \qquad \text{for all } x'\in\Omega'_0\,.$$
This map can be explicitly constructed by simply deforming each of the quarter annuli into unit squares, and performing the appropriate rotation on each of the squares $\Omega'_{0,2}$, $\Omega'_{0,4}$ and $\Omega'_{0,6}$.

In d-dimensions, we define $\mathcal{A}_1=\mathcal{A}'_1\times(0,1)^{d-2}\subseteq\mathbb{R}^d$, and $\mathcal{S}_1=\mathcal{S}'_1\times(0,1)^{d-2}\subseteq\mathbb{T}^d$. We observe that $\mathcal{A}_1\subset(-3,4)^d$. We define $\varphi_1\colon\bar{\mathcal{A}}_1\to\bar{\mathcal{S}}_1$ by
$$\varphi_1(x_1,\dots,x_d)=(\varphi'_1(x_1,x_2),x_3,\dots,x_d)\,,$$
and note that $\varphi_1(x)=x$ for all $x\in(0,1)^d$. Finally, for each $j\in\{2,\dots,d-1\}$ we repeat the above procedure along the j^{th} and $(j+1)^{\text{th}}$ axis, and for $j=d$ we do the same along the j^{th} and 1^{st} axis. This yields the regions $\mathcal{A}_j$, and corresponding maps $\varphi_j\colon\bar{\mathcal{A}}_j\to\mathbb{T}^d$.

Now, we let $\bar\phi$ be an H^1 extension of $(\mathbf{1}_{\Omega_0}\bar\theta)\circ\varphi_1^{-1}$ to $\mathbb{T}^d$. We note that our choice of φ_j implies $(\mathbf{1}_{\Omega_0}\bar\theta)\circ\varphi_1^{-1}=(\mathbf{1}_{\Omega_0}\bar\theta)\circ\varphi_j^{-1}$ for all $j\in\{1,\dots,d\}$. Let $A>0$ be a large constant that will be chosen shortly. By Lemma 2.1 there exists $j\in\{1,\dots,d\}$ and a shear flow U on $\mathbb{T}^d$, directed along the j^{th} coordinate axis, such that U is the form (2) and
$$\|\nabla\phi(\cdot,T)\|^2_{L^2(\Omega_T)}\geqslant\Big(1+\frac{2\pi^2A^2T^2}{d}\Big)\|\nabla\bar\phi\|^2_{L^2(\Omega_0)}\,.$$
Here ϕ is the solution of the transport equation (3) on $\mathbb{T}^d$ with initial data $\bar\phi$. For simplicity, and without loss of generality, we will now assume $j=1$.

Next, we let $\tilde u\colon\mathcal{A}_1\to\mathbb{R}^d$ be the pullback of U under φ_1. That is, we define
$$\tilde u=(\varphi_1^{-1})^*(U)=(D\varphi_1^{-1}U)\circ\varphi_1\,.$$
Since φ_1 preserves the Lebesgue measure, and $\nabla\cdot U=0$ we must also have $\nabla\cdot\tilde u=0$. Now extend $\tilde u$ to be a C^1 divergence-free vector field supported in $(-3,4)^d$, and let $\tilde\theta$ be the solution to the transport equation
$$\partial_t\tilde\theta+\tilde u\cdot\nabla\theta=0$$
in $\mathbb{R}^d$ with initial data $\bar\theta$. By the construction of $\tilde u$ and the fact that $\bar\theta=\bar\phi\circ\varphi_1$ on Ω_0, we must have
$$\tilde\theta(x,t)=\phi(\varphi_1(x),t)\qquad\text{for all } x\in\Omega_t\,,$$

where Ω_t is the image of Ω_0 under the flow map of $\tilde{u}$ after time t. Hence,

$$\begin{aligned}\|\nabla\tilde{\theta}(\cdot,T)\|_{L^2(\mathcal{A}_1)}^2 &\geqslant \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\|\nabla\phi(\cdot,T)\|_{L^2(\mathcal{S}_1)}^2 \geqslant \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\|\nabla\phi(\cdot,T)\|_{L^2(\Omega_T)}^2\\ &\geqslant \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\Big(1+\frac{2\pi^2A^2T^2}{d}\Big)\|\nabla\bar{\phi}\|_{L^2(\Omega_0)}^2 \geqslant \alpha_0'\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}^2,\end{aligned}\tag{7}$$

where

$$\alpha_0' = \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\|\nabla\varphi_1\|_{L^\infty}^{-2}\Big(1+\frac{2\pi^2A^2T^2}{d}\Big).$$

To finish the proof, we need to replace the left hand side of the above with $\|\nabla\tilde{\theta}(\cdot,T)\|_{L^2(\Omega_0)}$. To do this we divide $\mathcal{A}_1$ into eight regions of equal measure, and note that on at least one of these regions we must have $\|\nabla\tilde{\theta}(\cdot,T)\|_{L^2(\Omega_{0,i})}^2 \geqslant \frac{1}{8}\|\tilde{\theta}(\cdot,T)\|_{L^2(\mathcal{A}_1)}^2$. If we now use a flow, $\tilde{w}$, that shifts this region back to Ω_0, then we will have the desired inequality. We elaborate on this below.

The flow $\tilde{w}$ above can be constructed as follows: Let $U=-e_2$, and view U as a flow on the strip $\mathcal{S}_1\subseteq\mathbb{T}^d$. Let $\tilde{w}$ be the pullback of U_2 under the map φ_1. By construction of φ_1 we note that for every $i\in\{0,7\}$, the flow of $\tilde{w}$ will map the region $\Omega_{0,i}$ to the region $\Omega_{0,0}=\Omega_0$ in time i. (Here $\Omega_{0,i}=\Omega_{0,i}'\times(0,1)^{d-2}\subseteq\mathcal{A}_1$, where $\Omega_{0,i}'$ is shown in Figure 1 and described at the beginning of the proof.)

From (7), there must exist $i\in\{0,\dots,7\}$ such that

$$\|\nabla\tilde{\theta}(\cdot,T)\|_{L^2(\Omega_{0,i})}^2 \geqslant \frac{1}{8}\|\nabla\tilde{\theta}(\cdot,T)\|_{L^2(\mathcal{A}_1)}^2 \geqslant \frac{\alpha_0'}{8}\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}^2.$$

With this i we define the desired velocity field u by

$$u(x,t)=\begin{cases}\tilde{u}(x) & 0\leqslant t\leqslant T,\\ \tilde{w}(x) & T<t\leqslant T+i,\end{cases}$$

and let θ solve (5) with initial data $\bar{\theta}$. Notice $\theta(\cdot,t)=\tilde{\theta}(\cdot,t)$ for all $t\in[0,T]$, and

$$\theta(x,T+i)=\tilde{\theta}(\tilde{\varphi}_{\tilde{w}}^{-1}(x,i)),$$

where $\tilde{\varphi}_{\tilde{w}}(\cdot,t)$ is the flow map of $\tilde{w}$ after time t. Consequently,

$$\begin{aligned}\|\nabla\theta(\cdot,T+i)\|_{L^2(\Omega_0)} &\geqslant \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\|\nabla\varphi_1\|_{L^\infty}^{-2}\|\nabla\theta(\cdot,T)\|_{L^2(\Omega_{0,i})}\\ &\geqslant \|\nabla\varphi_1^{-1}\|_{L^\infty}^{-2}\|\nabla\varphi_1\|_{L^\infty}^{-2}\frac{\alpha_0'}{8}\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}^2 \geqslant \alpha'\|\nabla\bar{\theta}\|_{L^2(\Omega_0)},\end{aligned}$$

provided we choose $A=\alpha'C(d)/T$, for some large dimensional constant $C(d)$ that only depends on d. Note that

$$\sup_{0\leqslant t\leqslant T+i}\|u\|_{C^1}\leqslant \max\{C_1(d)A, C_2(d)\}$$

for some dimensional constants $C_1(d)$ and $C_2(d)$. Thus rescaling time by a factor of $T/(T+i)$ the velocity field u satisfies all the conditions in the statement of Lemma 2.4. This concludes the proof. □

We conclude this section by repeatedly applying Lemma 2.4 to prove Proposition 2.2.

Proof of Proposition 2.2. We first apply Lemma 2.4 with $T=1$ and $\alpha'=e^\alpha$ to obtain a velocity field u such that

$$\|\nabla\theta(\cdot,1)\|_{L^2(\Omega_0)}\geqslant e^\alpha\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}, \qquad\text{and}\qquad \sup_{0\leqslant t\leqslant 1}\|u(\cdot,t)\|_{C^1(\mathbb{R}^d)}\leqslant C(\alpha).$$

Now we apply Lemma 2.4 starting at time 1 with initial data $\theta(\cdot,1)$ to obtain a velocity field u (defined for $1\leqslant t\leqslant 2$) such that

$$\sup_{1\leqslant t\leqslant 2}\|u(\cdot,t)\|_{C^1(\mathbb{R}^d)}\leqslant C(\alpha),$$

and

$$\|\nabla\theta(\cdot,2)\|_{L^2(\Omega_0)}\geqslant e^\alpha\|\nabla\theta(\cdot,1)\|_{L^2(\Omega_0)}\geqslant e^{2\alpha}\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}.$$

Note that the constant $C(\alpha)$ remained unchanged, as we still applied Lemma 2.4 for a time interval of length 1. Proceeding inductively we obtain the first assertion in Proposition 2.2.

For the second assertion, we let $n\in\mathbb{N}$ and $t\in[n,n+1)$. Since the flow of the velocity field u preserves the domain $\tilde{\Omega}_0$, and since $\sup_{0\leqslant t<\infty}\|u\|_{C^1}\leqslant C(\alpha)$, we must have

$$\begin{aligned}\|\nabla\theta(\cdot,t)\|_{L^2(\tilde{\Omega}_0)}^2 &\geqslant \frac{1}{C_1(\alpha)}\|\nabla\theta(\cdot,n)\|_{L^2(\tilde{\Omega}_0)}^2\\ &\geqslant \frac{1}{C_1(\alpha)}\|\nabla\theta(\cdot,n)\|_{L^2(\Omega_0)}^2 \geqslant \frac{e^{\alpha n}}{C_1(\alpha)}\|\nabla\bar{\theta}\|_{L^2(\Omega_0)}^2,\end{aligned}$$

for some constant $C_1(\alpha)$ that depends on α but not $\bar{\theta}$. This immediately implies the second assertion, finishing the proof. □

3 Loss of regularity for the transport equation

In this section we conclude the proof of Theorem 1.1. The basic idea of the proof resembles very closely that in [2], but with some important differences.

Both proofs entail an iterative construction in which some "building block" is replicated on a disjoint family of cubes at smaller spatial scales. The building block in [2] is an optimal mixer from [7], which enjoys uniform-in-time bounds on the first-order derivatives and decreases the negative norms of a specific advected scalar exponentially in time. By interpolation, the positive norms of the scalar increase exponentially in time, and roughly speaking the iterative construction entails a rescaling in time that makes the exponential increase an instantaneous blow up, still keeping under control the $W^{1,p}$ norm of the vector field for every $p<\infty$. By contrast, in the present proof we rely on the velocity field constructed in Section 2, which increases the H^1 norm of the advected scalar exponentially in time, but in general it is not mixing. The advantage of this approach is that higher regularity norms of the velocity field are controlled uniformly in time, and that the growth of the Sobolev norm holds for every (nontrivial) advected scalar with initial data in H^1. We will therefore be able to keep under control higher $W^{r,p}$ norms of the vector field uniformly in time, and to show loss of H^1 regularity for every such initial data. In fact, since the construction is local, we need only assume that the initial data is locally in $H^1(\mathbb{R}^d)$.

The iterative construction becomes however less explicit, since the location and the spatial scale of the family of cubes depend on the initial data, as we need to select the cubes in such a way that the derivative of the initial data is large enough in all of the cubes.

Proof of Theorem 1.1. We divide the proof in three steps.

Step 1. Set-up of the geometric construction. We need to determine a sequence of cubes in $\mathbb{R}^d$ on which we replicate rescaled constructions based on Proposition 2.2. We denote by Q_n a cube of side-length λ_n (both the location of the cubes and the side-lengths are to be determined), and we denote by $\tilde{Q}_n$ the cube with the same center as Q_n and side-length $7\lambda_n$. We will make sure that $\{\tilde{Q}_n\}$ is a disjoint family contained in a bounded set and it clusters to a point.

On every Q_n and $\tilde{Q}_n$ we replicate the construction of the velocity field u_n in Proposition 2.2 (we make explicit the dependence of u_n on the index n, since the velocity field in Proposition 2.2 depends on the initial data), rescaling in space by a factor λ_n and in time by a factor τ_n (which is also to be determined). We neglect a rigid motion, needed to make the cube Q_n concentric and aligned with the cube Ω_0 in Proposition 2.2, which is irrelevant to compute all needed norms of velocity field and advected scalar. Then we can define the velocity field as a rescaling of the vector field u_n in Proposition 2.2, namely

$$v_n(x,t) = \frac{\lambda_n}{\tau_n}\, u_n\left(\frac{x}{\lambda_n}, \frac{t}{\tau_n}\right), \tag{8}$$

and we observe that v_n is supported in the cube $\tilde{Q}_n$. Next, we let

$$v = \sum_{n=1}^{\infty} v_n\,.$$

Because the v_n are supported in disjoint cubes, it is straightforward to show that v is divergence-free and that v is C^1 in space outside of a point in $\mathbb{R}^d$, which is given by the limit (in the sense of sets) of the cubes $\tilde{Q}_n$ as $n \to \infty$. By Remark 2.3, v can be taken smooth outside of this point. We let ρ be the unique weak solution in $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^d))$ of the transport equation (1) with advecting field v and initial data $\bar{\rho}$ (notice that v has compact support).

By a scaling computation (as in Section 3.2 of [2]) and using Remark 2.3 we see that

$$\|v(\cdot,t)\|_{\dot{W}^{r,p}(\mathbb{R}^d)} \lesssim \sum_{n=1}^{\infty} \frac{\lambda_n^\gamma}{\tau_n}, \qquad \forall\, t>0\,,$$

where

$$\gamma = 1 - r + \frac{d}{p} > 0\,.$$

But, thanks to the bound (6) provided by Proposition 2.2, for every $n \in \mathbb{N}$ we have

$$\|\nabla\rho(\cdot,t)\|_{L^2(\tilde{Q}_n)} \geqslant \exp\left(\frac{\alpha t}{\tau_n} - \beta\right) M_n, \qquad \forall\, t>0\,,$$

where we have set

$$M_n = \|\nabla\bar{\rho}\|_{L^2(Q_n)}\,,$$

Therefore, using the fact that we will select the cubes $\tilde{Q}_n$ to be disjoint, it follows that

$$\|\nabla\rho(\cdot,t)\|_{L^2(\mathbb{R}^d)} \gtrsim \sum_{n=1}^{\infty} \exp\left(\frac{\alpha t}{\tau_n}\right) M_n, \qquad \forall\, t>0\,.$$

We conclude that our task is to determine the location of the disjoint cubes Q_n and choose the sequences $\{\lambda_n\}$ and $\{\tau_n\}$ in such a way that

$$\sum_{n=1}^{\infty} e^{t/\tau_n} M_n = \infty, \qquad \forall\, t>0, \tag{9}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n^\gamma}{\tau_n} < \infty, \qquad \forall\, \gamma>0\,. \tag{10}$$

Step 2. Choice of the cubes. We set $f = |\nabla\bar{\rho}|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, which clearly entails $M_n = \|f\|^{1/2}_{L^1(Q_n)}$. We set

$$A_r(x) = \frac{1}{|\mathcal{Q}_r(x)|}\int_{\mathcal{Q}_r(x)} f(y)\,dy\,,$$

where we denote by $\mathcal{Q}_r(x)$ the cube of side-length $r>0$ centered at $x \in \mathbb{R}^d$, and we set

$$\tilde{D} = \left\{ x \in \mathbb{R}^d \;:\; \exists \lim_{r\downarrow 0} A_r(x) = f(x) \right\}.$$

By the Lebesgue differentiation theorem we have $|\mathbb{R}^d \setminus \tilde{D}| = 0$. The assumption that $\bar{\rho}$ is not a constant function translates into $f \not\equiv 0$, which in turn guarantees the existence of $\bar{\delta} > 0$ and of a bounded set $D \subset \tilde{D}$, with $|D|>0$, such that

$$\forall\, x \in D, \qquad \exists \lim_{r\downarrow 0} A_r(x) = f(x) \geqslant \bar{\delta} > 0\,.$$

This means that, for every $x \in D$, there exists $\bar{r}_x > 0$ with the property:

$$\int_{\mathcal{Q}_r(x)} f(y)\,dy \geqslant \frac{\bar{\delta}}{2} r^d, \qquad \forall\, 0 < r \leqslant \bar{r}_x\,.$$

We can therefore iteratively pick a monotonic sequence $\{\lambda_n\}$ satisfying

$$0 < \lambda_n \leqslant e^{-n}, \qquad \lambda_n \downarrow 0, \tag{11}$$

and choose the centers $x_n \in D$ of the cubes in such a way that the cubes $\mathcal{Q}_{7\lambda_n}(x_n)$ are disjoint and, setting $Q_n = \mathcal{Q}_{\lambda_n}(x_n)$, we have

$$M_n \geqslant C\lambda_n^{d/2}, \qquad \forall n\,. \tag{12}$$

The existence of the sequences $\{x_n\}$ and $\{\lambda_n\}$ as above is guaranteed by the fact that we can inductively choose x_n and $\lambda_n > 0$ (small enough) to have

$$\left| D \setminus \bigcup_{k=1}^{n} \mathcal{Q}_{7\lambda_k}(x_k) \right| > 0, \qquad \forall n\,.$$

The fact that D has been chosen to be bounded guarantees that $\{x_n\}$ can be chosen to be a convergent sequence, and $\{\mathcal{Q}_{7\lambda_n}(x_n)\}$ to be contained in a bounded set. We conclude that $\{Q_n\}$ is our desired sequence of cubes.

Step 3. Choice of the sequence τ_n and conclusion. The lower bound (12) shows that the condition (9) for the loss of regularity of the solution holds if

$$\sum_{n=1}^{\infty} e^{t/\tau_n} \lambda_n^{d/2} = \infty, \qquad \forall t > 0\,. \tag{13}$$

We recall condition (10) for the regularity of the velocity field:

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{\gamma}}{\tau_n} < \infty, \qquad \forall \gamma > 0\,. \tag{14}$$

The sequence $\{\lambda_n\}$ has been implicitly chosen in the previous step to satisfy (11). We now show how it is possible to choose the sequence $\{\tau_n\}$ in such a way that (13) and (14) hold. To this end, we set

$$\tau_n = \left(\log \frac{1}{\lambda_n}\right)^{-2}.$$

The series in condition (13) becomes

$$\begin{aligned}\sum_{n=1}^{\infty} e^{t/\tau_n} \lambda_n^{d/2} &= \sum_{n=1}^{\infty} \left(e^{\log \frac{1}{\lambda_n}}\right)^{t \log \frac{1}{\lambda_n}} \lambda_n^{d/2} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^{t \log \frac{1}{\lambda_n}} \lambda_n^{d/2} = \sum_{n=1}^{\infty} \lambda_n^{t \log \lambda_n + d/2},\end{aligned}$$

which diverges since $\lambda_n^{t\log\lambda_n + d/2} \to +\infty$ as $n \to \infty$ for every $t > 0$.

On the other hand, choosing $N = N(\gamma)$ so that

$$\left(\log \frac{1}{\lambda_n}\right)^2 \leqslant \left(\frac{1}{\lambda_n}\right)^{\gamma/2}, \qquad \forall n \geqslant N(\gamma)$$

(recall that $\lambda_n \downarrow 0$), the series in condition (14) can be estimated using (11) as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\lambda_n^{\gamma}}{\tau_n} &= \sum_{n=1}^{\infty} \left(\log \frac{1}{\lambda_n}\right)^2 \lambda_n^{\gamma} \leqslant \sum_{n=1}^{N(\gamma)-1} \left(\log \frac{1}{\lambda_n}\right)^2 \lambda_n^{\gamma} + \sum_{n=N(\gamma)}^{\infty} \lambda_n^{\gamma/2} \\ &\leqslant \sum_{n=1}^{N(\gamma)-1} \left(\log \frac{1}{\lambda_n}\right)^2 \lambda_n^{\gamma} + \sum_{n=N(\gamma)}^{\infty} e^{-\gamma n/2},\end{aligned}$$

which is finite for any $\gamma > 0$. This concludes the proof of the theorem. □

4 Conclusion

In this work, we study properties of weak solutions to a linear transport equation, when the advecting velocity is rough, i.e., it has only Sobolev regularity in space.

We extend the results in [2] to show that, given any non-constant initial data with square integrable derivative, it is possible to choose the advecting vector field in such a way that the solution loses its regularity instantaneously. To be more precise, we measure the regularity of the passive scalar in Sobolev spaces and show that all derivatives of the solution of order greater or equal to 1 blow up in L^2 for any $t > 0$. This result shows severe ill-posedness in the sense of Hadamard for the transport equation in Sobolev spaces. This result is sharp in the scale of Sobolev spaces, that is, the vector field in our example belongs to all Sobolev spaces that do not embed in the Lipschitz class.

Although the construction is not as explicit as in [2], this example is based on a judicious choice of shear flows acting on the torus, then extended to the full space. Our construction is not universal, in the sense that the advecting field depends on the choice of initial data. It is an open question whether one can construct one single vector field that make the norm of derivatives of the solution blow up for (almost) all initial data. Even though the vector field depends in a strong way on the initial data, the blow-up mechanism described in this work is distinctively linear, since it is based on rescaling and superposing basic flows and solutions.

Acknowledgements

The authors thank Giovanni Alberti for stimulating discussions on loss of regularity for transport equations that led to the problem addressed in this work.

References

[1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003. ISBN 0-12-044143-8.

[2] Giovanni Alberti, Gianluca Crippa, and Anna L. Mazzucato. Loss of regularity for the continuity equation with non-Lipschitz velocity field. *Ann. PDE*, 5 (1):Paper No. 9, 19, 2019. ISSN 2524-5317. doi: 10.1007/s40818-019-0066-3. URL `https://doi.org/10.1007/s40818-019-0066-3`.

[3] Marina Ghisi and Massimo Gobbino. Residual pathologies. Preprint arXiv:1908.09496, 2019.

[4] Elia Brué and Quoc-Hung Nguyen. On the Sobolev space of functions with derivative of logarithmic order. *Adv. Nonlinear Anal.*, 9(1): 836–849, 2020. ISSN 2191-9496. doi: 10.1515/anona-2020-0027. URL `https://doi.org/10.1515/anona-2020-0027`.

[5] Gianluca Crippa and Camillo De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.*, 616:15–46, 2008. ISSN 0075-4102. doi: 10.1515/CRELLE.2008.016. URL `https://doi.org/10.1515/CRELLE.2008.016`.

[6] Pierre-Emmanuel Jabin. Critical non-Sobolev regularity for continuity equations with rough velocity fields. *J. Differential Equations*, 260(5): 4739–4757, 2016. ISSN 0022-0396. doi: 10.1016/j.jde.2015.11.028. URL `https://doi.org/10.1016/j.jde.2015.11.028`.

[7] Giovanni Alberti, Gianluca Crippa, and Anna L. Mazzucato. Exponential self-similar mixing by incompressible flows. *J. Amer. Math. Soc.*, 32(2):445–490, 2019. ISSN 0894-0347. doi: 10.1090/jams/913. URL `https://doi.org/10.1090/jams/913`.