

PHASE SEPARATION IN THE ADVECTIVE CAHN–HILLIARD EQUATION

YU FENG, YUANYUAN FENG, GAUTAM IYER, AND JEAN-LUC THIFFEAULT

ABSTRACT. The Cahn–Hilliard equation is a classic model of phase separation in binary mixtures that exhibits spontaneous coarsening of the phases. We study the Cahn–Hilliard equation with an imposed advection term in order to model the stirring and eventual mixing of the phases. The main result is that if the imposed advection is sufficiently mixing then no phase separation occurs, and the solution instead converges exponentially to a homogeneous mixed state. The mixing effectiveness of the imposed drift is quantified in terms of the dissipation time of the associated advection-hyperdiffusion equation, and we produce examples of velocity fields with a small dissipation time. We also study the relationship between this quantity and the dissipation time of the standard advection-diffusion equation.

1. Introduction

Spinodal decomposition refers to the phase separation of a binary mixture, such as alloys that are quenched below their critical temperature. A well-studied model is the Cahn–Hilliard equation [CH58, Cah61], where the evolution of the normalized concentration difference c between the two phases is governed by the equation

$$(1.1) \quad \partial_t c + \gamma D \Delta^2 c = D \Delta (c^3 - c).$$

Here $D > 0$ is a mobility parameter, and $\sqrt{\gamma}$ is the Cahn number, which is related to the surface tension at the interface between phases. The coefficient γD is a hyperdiffusion that regularizes the equation at small length scales by overcoming the destabilizing $-D \Delta c$ term. The concentration c is normalized such that the regions $\{c = 1\}$ and $\{c = -1\}$ represent domains that are pure in each phase. For simplicity, we will only consider (1.1) on the d -dimensional torus \mathbb{T}^d .

When γ is small, solutions to (1.1) spontaneously form domains with $c = \pm 1$ separated by thin transition regions (see Figure 1). This has been well studied by many authors (see for instance [ES86, Ell89, Peg89]), and the underlying mechanism can be understood as follows. The free energy of this system, \mathcal{E} , can be decomposed into the sum of the chemical free energy, $\mathcal{E}_{\text{chem}}$, and the interfacial free energy, \mathcal{E}_{int} , where

$$\mathcal{E}_{\text{chem}} \stackrel{\text{def}}{=} \frac{1}{4} \int_{\mathbb{T}^d} (c^2 - 1)^2 dx \quad \text{and} \quad \mathcal{E}_{\text{int}} \stackrel{\text{def}}{=} \frac{1}{2} \gamma \int_{\mathbb{T}^d} |\nabla c|^2 dx.$$

Using (1.1), one can directly check that \mathcal{E} decreases with time, and hence solutions should approach minimizers of \mathcal{E} after a long time. Minimizing the chemical free energy $\mathcal{E}_{\text{chem}}$ favors forming domains where $c = \pm 1$. Minimizing the interfacial free

2010 *Mathematics Subject Classification.* Primary 76F25; Secondary 37A25, 76R50.

Key words and phrases. Cahn–Hilliard equation, enhanced dissipation, mixing.

This material is based upon work partially supported by the National Science Foundation, under grant DMS-1814147, and the Center for Nonlinear Analysis.

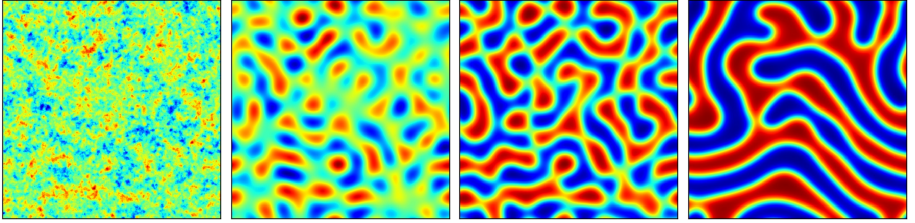


FIGURE 1. The solution of the Cahn–Hilliard equation (1.1) on the 2D torus with $D = 0.001$, $\gamma = .01/(2\pi)^2$ at times $t = 0, 2, 5$ and 20 . The phases $c = 1$ and $c = -1$ are red and blue, respectively, and $c = 0$ is green.

energy \mathcal{E}_{int} favors interfaces of thickness $\sqrt{\gamma}$ separating the domains. As a result, the typical behavior of equation (1.1) is to spontaneously phase-separate as in Figure 1.

In this paper we study the effect of stirring on spontaneous phase separation. When subjected to an incompressible stirring velocity field $u(t, x)$, equation (1.1) is modified to

$$(1.2) \quad \partial_t c + u \cdot \nabla c + \gamma \Delta^2 c = \Delta(c^3 - c).$$

For simplicity, we have set the mobility parameter D to be 1. The advective Cahn–Hilliard equation (1.2) has been studied by many authors [CPB88, LLG95, ONT07a, ONT07b, ONT08, LDE+13] for both passive and active advection. Under a strong shear flow, for instance, it is known that solutions to (1.2) equilibrate along the flow direction and spontaneously phase separates in the direction perpendicular to the flow.

Our main result is to show that if the stirring velocity field is sufficiently mixing, then no phase separation occurs. More precisely, we show that if the *dissipation time* of u is small enough, then c converges exponentially to the total concentration $\bar{c} = \int_{\mathbb{T}^d} c_0 dx$. This is illustrated by the numerical simulations in Figure 2, where the velocity field u was chosen to be alternating horizontal and vertical shear flows with randomized phases (see [Pie94, ONT07a, ONT08]). When the shear amplitude, A , is small, the norms of the solution settle to some non-zero value after a large time. As the amplitude is increased, the flow mixes faster, and we see the solution decays exponentially to $\bar{c} = 0$.

Decay of the advective Cahn–Hilliard equation. To state our main result, we need to first introduce the notion of *dissipation time*. Let u be a divergence-free vector field and consider the equation

$$(1.3) \quad \partial_t \theta + u \cdot \nabla \theta + \gamma(-\Delta)^\alpha \theta = 0,$$

with $\alpha > 0$, periodic boundary conditions, and mean-zero initial data. For $\alpha = 1$ this is the *advection-diffusion equation*; for $\alpha = 2$ it is the *advection-hyperdiffusion equation*. Incompressibility of u and the Poincaré inequality immediately imply that $\|\theta_t\|_{L^2}$ is decreasing as a function of t , and

$$(1.4) \quad \|\theta_{s+t}\|_{L^2} \leq e^{-(2\pi)^{2\alpha} \gamma t} \|\theta_s\|_{L^2}.$$

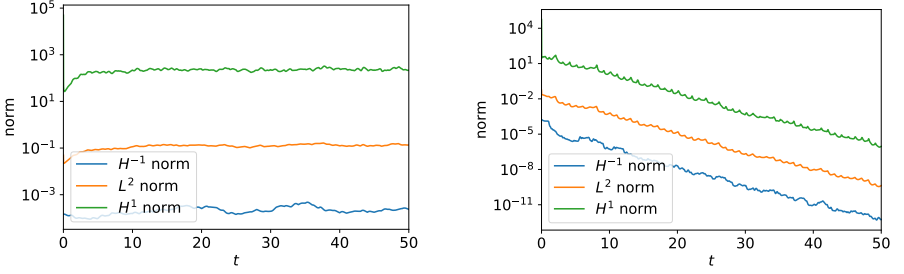


FIGURE 2. Decay of H^1 , L^2 , and H^{-1} norms for the random shear flow for $A = 0.5$ (left) and $A = 2$ (right). On the left the norms settle to equilibrium values; on the right they decay exponentially.

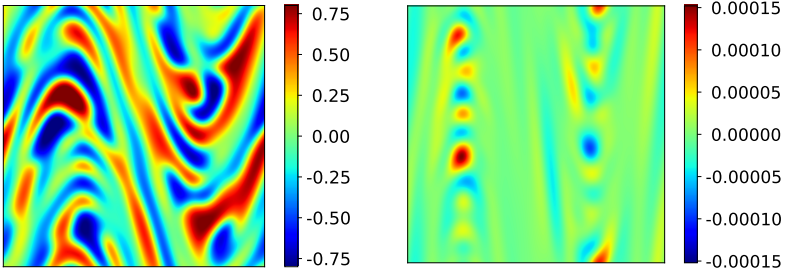


FIGURE 3. Final concentration c for the two cases in Figure 2.

Thus, we are guaranteed

$$(1.5) \quad \|\theta_{s+t}\|_{L^2} \leq \frac{1}{2} \|\theta_s\|_{L^2}, \quad \text{for every } t \geq \frac{\ln 2}{(2\pi)^{2\alpha}\gamma},$$

and every $s \geq 0$. However, u generates gradients through filamentation, which causes solutions to dissipate $\|\theta_t\|_{L^2}$ faster. This may result in the lower bound in (1.5) being attained at much smaller times, and the smallest time t at which this happens is known as the *dissipation time* (see for instance [FW03, FI19]).

Definition 1.1 (Dissipation time). Let $\mathcal{S}_{s,t}^{u,\alpha}$ be the solution operator to (1.3) on $\mathbb{T}^d \times (0, \infty)$. That is, for any $f \in L^2(\mathbb{T}^d)$, the function $\theta_t = \mathcal{S}_{s,t}^{u,\alpha} f$ solves (1.3) with initial data $\theta_s = f$, and periodic boundary conditions. The *dissipation time* of u is

$$(1.6) \quad \tau_\alpha^*(u, \gamma) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 \mid \|\mathcal{S}_{s,s+t}^{u,\alpha}\|_{\dot{L}^2 \rightarrow \dot{L}^2} \leq \frac{1}{2} \text{ for all } s \geq 0 \right\}.$$

Here \dot{L}^2 is the space of all mean-zero, square integrable functions on the torus \mathbb{T}^d .

While this definition makes sense for any $\alpha > 0$, we are mainly interested in the case when α is either 1 or 2. Note that (1.5) implies $\tau_\alpha^*(u, \gamma) \leq O(1/\gamma)$ as $\gamma \rightarrow 0$. If, however, u is mixing, then this can be dramatically improved (see for instance [CKRZ08, Zla10, CZDE18, Wei18, FI19, Fen19]). In fact [FI19] bound $\tau_1^*(u, \gamma)$ explicitly in terms of the mixing rate of u . Moreover, when u is exponentially mixing, [CZDE18, Wei18, Fen19] show that $\tau_1^*(u, \gamma) \leq O(|\ln \gamma|^2)$ as $\gamma \rightarrow 0$.

With this notion, we can now state our main result.

Theorem 1.2. *Let $d \in \{2, 3\}$, $u \in L^\infty((0, \infty); W^{1,\infty}(\mathbb{T}^d))$, and c be the solution of (1.2) with initial data $c_0 \in H^2(\mathbb{T}^d)$.*

(1) *When $d = 2$, for any $\beta > 1$, $\mu > 0$, there exists a time*

$$T_0 = T_0(\|c_0 - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$$

such that if $\tau_2^(u, \gamma) < T_0$, then for every $t \geq 0$, we have*

$$(1.7) \quad \|c(t) - \bar{c}\|_{L^2} \leq \beta e^{-\mu t} \|c_0 - \bar{c}\|_{L^2}.$$

(2) *When $d = 3$, for any $\beta > 1$, $\mu > 0$, there exists a time*

$$T_1 = T_1(\|c_0 - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$$

such that if

$$(1.8) \quad (1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma) < T_1,$$

then (1.7) still holds for every $t \geq 0$.

Remark 1.3. The times T_0 and T_1 can be computed explicitly, as can be seen from the proof of the theorem and equations (2.16) and (2.27) respectively.

Several authors have used mixing properties of the advection term to quench reactions, prevent blow-up, and stem the growth of non-linear PDEs (see for instance [FKR06, HL09, BKNR10, KX16, BH17, IXZ19]). Our results are similar in spirit to those in [IXZ19], where the authors used related ideas to prove decay of solutions to a large class of nonlinear parabolic equations. These results were formulated for second-order PDEs where the diffusive term is the Laplacian, but they can easily be generalized to apply when the diffusive term is the bi-Laplacian as we have in (1.2). Unfortunately, the assumptions required for these results to apply are not satisfied by the nonlinear term, even when $d = 2$. Thus we cannot directly use the results in [IXZ19] here.

Our 3D result is qualitatively different (and weaker) from the 2D case, or the results in [IXZ19]. Indeed, in [IXZ19], the authors only rely on smallness of the dissipation time τ_1^* . Here, in 2D we rely on the smallness of τ_2^* . However, in 3D we now require smallness of $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*$. In the next section we produce velocity fields where this is arbitrarily small. We remark, however, that while we can find velocity fields for which $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*$ is arbitrarily small, it appears impossible to produce velocity fields for which $(1 + \|\nabla u\|_{L^\infty}) \tau_2^*$ is arbitrarily small. To see this, the proof in [Poo96] (see also equation (9) in [MD18]) can be easily adapted to obtain the lower bound

$$\tau_2^*(u, \gamma) \geq \frac{1}{C\|u\|_{C^2}} \ln\left(1 + \frac{C\|u\|_{C^2}}{\gamma}\right)$$

for some explicit dimensional constant C . When $\tau_2^*(u, \gamma)$ is small, we expect $\|u\|_{C^2}$ to be large, and in this case the above shows $(1 + \|u\|_{C^2}) \tau_2^*(u, \gamma)$ grows at least logarithmically with $\|u\|_{C^2}$.

Incompressible velocity fields with small dissipation time. In order to apply Theorem 1.2, we need to produce incompressible velocity fields u for which $\tau_2^*(u, \gamma)$ is arbitrarily small when $d = 2$, and for which $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma)$ is arbitrarily small when $d = 3$. We do this here by rescaling *mixing flows*. This has been studied previously by [CKRZ08, KSZ08, Zla10, CZDE18, FI19, Fen19] when the diffusive term is the standard Laplacian. With minor modification, the proofs can be adapted to our context, where the diffusive term is the bi-Laplacian.

Proposition 1.4. *Let $v \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, and define $u_A(x, t) = Av(x, At)$. If v is weakly mixing with rate function h , then*

$$\tau_2^*(u_A, \gamma) \xrightarrow{A \rightarrow \infty} 0.$$

If further v is strongly mixing with rate function h , and

$$(1.9) \quad th(t) \xrightarrow{t \rightarrow \infty} 0,$$

then

$$(1 + \|\nabla u_A\|_{L^\infty})^{1/2} \tau_2^*(u_A, \gamma) \xrightarrow{A \rightarrow \infty} 0.$$

For ease of presentation, we defer the definition of weak and strong mixing used above to Section 3 (see Definition 3.1, below). The deficiency of Proposition 1.4 is that it is not easy to construct examples of mixing flows. The simplest example we are aware of is alternating flows with randomized phases introduced by Pierrehumbert [Pie94] and used to produce our Figure 2. More recently, a variety of other examples were constructed in [YZ17, ACM19, EZ19, BBPS19].

This shortcoming was addressed in [IXZ19], where the authors showed that for any $\tau_0 > 0$, there exists a sufficiently strong and fine cellular flow, u , for which $\tau_1^*(u, \gamma) < \tau_0$. This provides a simple and explicit family of velocity fields with arbitrarily small $\tau_1^*(u, \gamma)$, and in [IXZ19] the authors used it to prevent blow up in the Keller–Segel and other second-order, non-linear, parabolic PDEs. We expect that for any $\tau_0 > 0$, one can also construct sufficiently strong and fine cellular flows for which $\tau_2^*(u, \gamma) < \tau_0$. Unfortunately the proof in [IXZ19] does not generalize easily, and thus we are presently unable to produce cellular flows for which $\tau_2^*(u, \gamma)$ is small enough, or for which (1.8) holds.

Relationships between the various dissipation times. Since for any $\alpha, \gamma > 0$, the quantity $\tau_\alpha^*(u, \gamma)$ is a measure of the rate at which u mixes, it is natural to study its behavior as α and γ vary. When $\alpha = 1$, the behavior of $\tau_\alpha^*(u, \gamma)$ as $\gamma \rightarrow 0$ was recently studied in [CZDE18, FI19, Fen19] and quantified in terms of the mixing rate. We will instead study the behavior of $\tau_\alpha^*(u, \gamma)$ when γ is fixed and α varies. Moreover, since $\tau_1^*(u, \gamma)$ and $\tau_2^*(u, \gamma)$ are particularly interesting from a physical point of view, we focus our attention on the relationship between these two quantities. Our first result is an upper bound for $\tau_2^*(u, \gamma)$ in terms of $\tau_1^*(u, \gamma)$.

Lemma 1.5. *There exists an explicit dimensional constant C such that for every divergence-free $u \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, and every $\gamma > 0$, we have*

$$(1.10) \quad \tau_2^*(u, \gamma) \leq C \tau_1^*(u, \gamma) (1 + \|u\|_{C^2} \tau_1^*(u, \gamma)).$$

Since velocity fields with small $\tau_1^*(u, \gamma)$ are known, one use of Lemma 1.5 is to produce velocity fields for which $\tau_2^*(u, \gamma)$ and $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma)$ are small. For instance, if u is mixing at a sufficiently fast rate, then results of [Wei18, CZDE18, FI19, Fen19] along with Lemma 1.5 can be used to produce velocity fields for which $\tau_2^*(u, \gamma)$ and $(1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma)$ are arbitrarily small. Lemma 1.5, however, cannot be used to produce cellular flows for which $\tau_2^*(u, \gamma)$ is arbitrarily small. Indeed, with the τ_1^* bound in [IXZ19], or even the best expected heuristic for cellular flows, the right-hand side of (1.10) diverges.

Plan of the paper. In Section 2 we prove our main result (Theorem 1.2). In Section 3 we recall the definition of weak and strong mixing and prove Proposition 1.4. In Section 4 we prove Lemma 1.5 bounding τ_2^* in terms of τ_1^* . Finally, for completeness, we conclude with an appendix estimating the dissipation time τ_2^* in terms of the mixing rate of the advecting velocity field.

2. Decay of the advective Cahn–Hilliard equation

This section is devoted to the proof of Theorem 1.2. We begin by recalling the well-known existence of global strong solutions to equation (1.2). Elliott and Songmu [ES86] proved well-posedness in the absence of advection. Since the advection is a first-order linear term, their proof can easily be adapted to our setting. We state the result here for convenience.

Proposition 2.1. *Let $\gamma > 0$, $u \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{T}^d))$ be divergence-free and $c_0 \in H^2(\mathbb{T}^d)$. There exists a unique strong solution to (1.2) in the space*

$$c(t, x) \in L_{\text{loc}}^2([0, \infty); H^4(\mathbb{T}^d)) \cap L_{\text{loc}}^\infty([0, \infty); H^2(\mathbb{T}^d)) \cap H_{\text{loc}}^1([0, \infty); L^2(\mathbb{T}^d)).$$

For the remainder of this section let $\beta > 1$, $\gamma > 0$, and $\mu > 0$ be as in the statement of Theorem 1.2. Without loss of generality we may further assume $\beta \in (1, 2]$. We also fix a divergence-free velocity field $u \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{T}^d))$, $c_0 \in H^2(\mathbb{T}^d)$ and let c be the unique strong solution to equation (1.2) with initial data c_0 . The existence of such a solution is guaranteed by Proposition 2.1.

The main idea behind the proof of Theorem 1.2 is to split the analysis into two cases. First, when the time average of $\|\Delta c\|_{L^2}$ is large, standard energy estimates will show that the variance of c decreases exponentially. Second, when the time average of $\|\Delta c\|_{L^2}^2$ is small, we will use the advection term to show that the variance of c still decreases exponentially, at a comparable rate.

We begin with a lemma handling the first case.

Lemma 2.2. *For any $t_0 > 0$, we have*

$$(2.1) \quad \sup_{0 \leq \tau \leq \gamma \ln \beta} \|c(t_0 + \tau) - \bar{c}\|_{L^2}^2 \leq \beta \|c(t_0) - \bar{c}\|_{L^2}^2.$$

Moreover, if for some $\tau \in (0, \gamma \ln \beta)$ we have

$$(2.2) \quad \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} \|\Delta c\|_{L^2}^2 ds \geq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2,$$

then

$$(2.3) \quad \|c(t_0 + \tau) - \bar{c}\|_{L^2} \leq e^{-\mu\tau} \|c(t_0) - \bar{c}\|_{L^2}.$$

For clarity of presentation, we momentarily postpone the proof of Lemma 2.2. We will now treat the two- and three-dimensional cases separately.

2.1. The two-dimensional case. Suppose the time average of $\|\Delta c\|_{L^2}^2$ is small. In this case, we will show that if $\tau_2^*(u, \gamma)$ is small enough, then the variance of c still decreases by a constant fraction after time $\tau_2^*(u, \gamma)$.

Lemma 2.3. *For any $t_0 \geq 0$, there exists a time*

$$T'_0 = T'_0(\|c(t_0) - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu) \in (0, \gamma \ln \beta]$$

such that if

$$(2.4a) \quad \tau_2^*(u, \gamma) \leq T'_0(\|c(t_0) - \bar{c}\|_{L^2}, \beta, \gamma, \mu, \bar{c}),$$

$$(2.4b) \quad \frac{1}{\tau_2^*(u, \gamma)} \int_{t_0}^{t_0 + \tau_2^*(u, \gamma)} \|\Delta c\|_{L^2}^2 ds \leq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2,$$

then (2.3) still holds at time $\tau = \tau_2^*(u, \gamma)$. Moreover, the time T'_0 can be chosen to be decreasing as a function of $\|c(t_0) - \bar{c}\|_{L^2}$.

Remark. The time T'_0 can be computed explicitly in terms of $\|c(t_0) - \bar{c}\|_{L^2}$, β , γ , μ , and \bar{c} , as can be seen from (2.16), below.

Momentarily postponing the proof of Lemma 2.3, we prove Theorem 1.2 in 2D.

Proof of Theorem 1.2 when $d = 2$. Define

$$T_0 = \min \left\{ T'_0, \frac{\ln \beta}{2\mu} \right\},$$

where T'_0 is the time given by Lemma 2.3 with $t_0 = 0$. For conciseness, let $\tau_2^* = \tau_2^*(u, \gamma)$, and suppose $\tau_2^* < T'_0$. If

$$(2.5) \quad \frac{1}{\tau_2^*} \int_0^{\tau_2^*} \|\Delta c\|_{L^2}^2 ds \geq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2,$$

and since $T'_0 < \gamma \ln \beta$ by choice, Lemma 2.2 applies and we must have

$$(2.6) \quad \|c(\tau_2^*) - \bar{c}\|_{L^2} \leq e^{-\mu\tau_2^*} \|c_0 - \bar{c}\|_{L^2}.$$

If on the other hand (2.5) does not hold, then Lemma 2.3 applies and (2.6) still holds.

Since T'_0 is a decreasing function of $\|c - \bar{c}\|_{L^2}$, we may restart the above argument at time τ_2^* . Proceeding inductively, we find

$$\|c(n\tau_2^*) - \bar{c}\|_{L^2} \leq e^{-\mu n\tau_2^*} \|c_0 - \bar{c}\|_{L^2},$$

for all $n \in \mathbb{N}$.

Now for any time $t \geq 0$, let $n \in \mathbb{N}$ be such that $t \in (n\tau_2^*, (n+1)\tau_2^*)$. Since $t - n\tau_2^* \leq \tau_2^* \leq \gamma \ln \beta$, Lemma 2.2 applies and (2.1) yields

$$\begin{aligned} \|c(t) - \bar{c}\|_{L^2} &\leq \sqrt{\beta} \|c(n\tau_2^*) - \bar{c}\|_{L^2} \leq \sqrt{\beta} e^{-\mu n\tau_2^*} \|c_0 - \bar{c}\|_{L^2} \\ &\leq \sqrt{\beta} e^{-\mu t + \mu\tau_2^*} \|c_0 - \bar{c}\|_{L^2} \leq \beta e^{-\mu t} \|c_0 - \bar{c}\|_{L^2}. \end{aligned}$$

The last inequality follows from $\tau_2^* \leq \ln \beta / (2\mu)$. This completes the proof. \square

2.2. The three-dimensional case. In this case, in order to prove the analog of Lemma 2.3, we need a stronger assumption on $\tau_2^*(u, \gamma)$.

Lemma 2.4. *For any $t_0 \geq 0$, there exists a time $T'_1 = T'_1(\|c(t_0) - \bar{c}\|_{L^2}, \bar{c}, \beta, \gamma, \mu)$ such that if*

$$(2.7) \quad (1 + \|\nabla u\|_{L^\infty})^{1/2} \tau_2^*(u, \gamma) \leq T'_1,$$

$$(2.8) \quad \frac{1}{2\tau_2^*(u, \gamma)} \int_{t_0}^{t_0 + 2\tau_2^*(u, \gamma)} \|\Delta c\|_{L^2}^2 ds \leq \frac{\beta + 2\gamma\mu}{\gamma^2} \|c(t_0) - \bar{c}\|_{L^2}^2,$$

then

$$(2.9) \quad \|c(t_0 + 2\tau_2^*(u, \gamma)) - \bar{c}\|_{L^2} \leq e^{-2\mu\tau_2^*(u, \gamma)} \|c(t_0) - \bar{c}\|_{L^2}.$$

Moreover, the time T'_1 can be chosen to be decreasing as a function of $\|c(t_0) - \bar{c}\|_{L^2}$.

Remark. The time T'_1 can be computed explicitly in terms of $\|c(t_0) - \bar{c}\|_{L^2}$, β , γ , μ , and \bar{c} , as can be seen from (2.27) below.

Momentarily postponing the proof of Lemma 2.4, we prove Theorem 1.2 in 3D.

Proof of Theorem 1.2 when $d = 3$. Let T'_1 be the time given by Lemma 2.4 with $t_0 = 0$, and define

$$T_1 = \min \left\{ T'_1, \frac{\ln \beta}{4\mu} \right\}.$$

The remainder of the proof is now identical to the proof when $d = 2$ (page 7) with Lemma 2.3 replaced with Lemma 2.4. \square

2.3. Variance decay in 2D (Lemmas 2.2 and 2.3). It now remains to prove the lemmas. The variance decay when $\|\Delta c\|_{L^2}$ is large follows directly from the energy inequality in both 2D and 3D. We prove this first.

Proof of Lemma 2.2. For simplicity and without loss of generality we assume $t_0 = 0$. Multiplying equation (1.2) by $c - \bar{c}$ and integrating over \mathbb{T}^d , we obtain

$$\begin{aligned} \partial_t \|c - \bar{c}\|_{L^2}^2 &= 2\langle \Delta(c^3 - c - \gamma \Delta c), c - \bar{c} \rangle \\ (2.10) \quad &\leq -6\|c \nabla c\|_{L^2}^2 + 2\|c - \bar{c}\|_{L^2} \|\Delta c\|_{L^2} - 2\gamma \|\Delta c\|_{L^2}^2. \end{aligned}$$

Here the notation $\langle f, g \rangle = \int_{\mathbb{T}^d} fg \, dx$ denotes the standard L^2 inner-product on \mathbb{T}^d . Drop the first term in (2.10) and apply Young's inequality to find

$$(2.11) \quad \partial_t \|c - \bar{c}\|_{L^2}^2 \leq -\gamma \|\Delta c\|_{L^2}^2 + \frac{1}{\gamma} \|c - \bar{c}\|_{L^2}^2,$$

and hence

$$(2.12) \quad \|c(t) - \bar{c}\|_{L^2}^2 \leq \|c_0 - \bar{c}\|_{L^2}^2 e^{t/\gamma} \quad \text{for all } t \geq 0.$$

In particular, if $t \in (0, \gamma \ln \beta)$, we see that (2.1) holds with $t_0 = 0$.

For (2.3), note that integration of (2.11) from 0 to τ with (2.1) and (2.2) gives

$$\|c(\tau) - \bar{c}\|_{L^2}^2 \leq \|c_0 - \bar{c}\|_{L^2}^2 \left(1 + \frac{\beta\tau}{\gamma}\right) - \gamma \int_0^\tau \|\Delta c\|_{L^2}^2 \, ds \leq \|c_0 - \bar{c}\|_{L^2}^2 (1 - 2\mu\tau).$$

Since $1 - 2\mu\tau \leq e^{-2\mu\tau}$, this proves (2.3) as desired. \square

We now turn to Lemma 2.3, where the time integral of $\|\Delta c\|_{L^2}^2$ is assumed small. In this case, by definition of τ_2^* , the linear terms halve the variance of c in time τ_2^* . If τ_2^* is small enough, then we show that the nonlinear terms cannot increase the variance too much in this time interval.

Proof of Lemma 2.3. For notational convenience, we use $\mathcal{S}_{s,t}$ to denote $\mathcal{S}_{s,t}^{u,2}$, the solution operator in Definition 1.1 with $\alpha = 2$. As before, we also use τ_2^* to denote $\tau_2^*(u, \gamma)$. For simplicity, and without loss of generality, we will again assume $t_0 = 0$.

By Duhamel's principle, we know

$$c(\tau_2^*) - \bar{c} = \mathcal{S}_{0,\tau_2^*}(c_0 - \bar{c}) + \int_0^{\tau_2^*} \mathcal{S}_{s,\tau_2^*}(\Delta(c^3(s) - c(s))) \, ds.$$

By definition of $\tau_2^* = \tau_2^*(u, \gamma)$, and the fact that $\mathcal{S}_{s,t}^{u,\alpha}$ is an L^2 -contraction, we have

$$(2.13) \quad \|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{B}{2} + \int_0^{\tau_2^*} \|\Delta(c^3 - c)\|_{L^2} \, ds,$$

where $B \stackrel{\text{def}}{=} \|c_0 - \bar{c}\|_{L^2}$. We now estimate the second term on the right of (2.13). First note

$$\begin{aligned}
 \|\Delta(c^3 - c)\|_{L^2} &= \|6c|\nabla c|^2 + 3c^2\Delta c - \Delta c\|_{L^2} \\
 &\leq 6\|c - \bar{c}\|_{L^\infty} \|\nabla c\|_{L^4}^2 + 6|\bar{c}| \|\nabla c\|_{L^4}^2 + 3\|c - \bar{c}\|_{L^\infty}^2 \|\Delta c\|_{L^2} \\
 &\quad + 6|\bar{c}| \|c - \bar{c}\|_{L^\infty} \|\Delta c\|_{L^2} + (3\bar{c}^2 + 1) \|\Delta c\|_{L^2}.
 \end{aligned}
 \tag{2.14}$$

By the Gagliardo–Nirenberg inequality we know

$$\begin{aligned}
 \|c - \bar{c}\|_{L^\infty} &\leq C \|\Delta c\|_{L^2}^{d/4} \|c - \bar{c}\|_{L^2}^{1-d/4}, \\
 \|\nabla c\|_{L^4} &\leq C \|\Delta c\|_{L^2}^{(4+d)/8} \|c - \bar{c}\|_{L^2}^{(4-d)/8},
 \end{aligned}$$

for some dimensional constant C . Here, and subsequently, we assume C is a purely dimensional constant that may increase from line to line. Substituting this in (2.14) when $d = 2$ we find

$$\begin{aligned}
 \|\Delta(c^3 - c)\|_{L^2} &\leq C \|\Delta c\|_{L^2}^2 \|c - \bar{c}\|_{L^2} \\
 &\quad + C |\bar{c}| \|\Delta c\|_{L^2}^{3/2} \|c - \bar{c}\|_{L^2}^{1/2} + (3\bar{c}^2 + 1) \|\Delta c\|_{L^2} \\
 &\leq C(1 + \bar{c}^2)(1 + \|c - \bar{c}\|_{L^2})(\|\Delta c\|_{L^2} + \|\Delta c\|_{L^2}^2).
 \end{aligned}
 \tag{2.15}$$

If we choose T'_0 small enough to ensure $T'_0 < \gamma \ln \beta$, then (2.1), (2.4b), (2.13) and (2.15) yield

$$\|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{B}{2} + \frac{C_{\beta,\mu} \tau_2^*}{\gamma^2} (1 + \bar{c}^2)(1 + B^2)B.$$

Here, $C_{\beta,\mu}$ is a constant that only depends on β, μ that may increase from line to line. Now choosing

$$T'_0 = \min \left\{ \frac{\gamma^2}{4C_{\beta,\mu}(1 + \bar{c}^2)(1 + B^2)}, \gamma \ln \beta, \frac{1}{4\mu} \right\}
 \tag{2.16}$$

we see that whenever $\tau_2^* \leq T'_0$ we must have

$$\|c(\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{3B}{4} \leq (1 - \mu\tau_2^*)B \leq e^{-\mu\tau_2^*} \|c_0 - \bar{c}\|_{L^2},$$

as claimed. Clearly the choice of T'_0 above is decreasing in B , finishing the proof. \square

2.4. Variance decay in 3D (Lemma 2.4). To prove variance decay in 3D, we first need an H^1 bound. For the remainder of this subsection we assume $d = 3$.

Lemma 2.5. *Define the free energy, \mathcal{E} , by*

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \frac{1}{4} \int_{\mathbb{T}^3} (c^2 - 1)^2 dx + \frac{1}{2} \gamma \int_{\mathbb{T}^3} |\nabla c|^2 dx.$$

Then, for any $t_0, \tau \geq 0$ we have

$$\|\nabla c(t_0 + \tau)\|_{L^2}^2 \leq \frac{2\mathcal{E}(t_0)}{\gamma} + \frac{\|\nabla u\|_{L^\infty}}{2\pi^2\gamma} e^{\tau/\gamma} \|c(t_0) - \bar{c}\|_{L^2}^2.
 \tag{2.17}$$

Proof. Without loss of generality assume $t_0 = 0$. Multiplying (1.2) by $c^3 - c - \gamma\Delta c$ and integrating over \mathbb{T}^3 , we have

$$\partial_t \mathcal{E} + \langle u \cdot \nabla c, c^3 - c - \gamma\Delta c \rangle = -\|\nabla(c^3 - c - \gamma\Delta c)\|_{L^2}^2.
 \tag{2.18}$$

Since u is divergence free,

$$|\langle u \cdot \nabla c, c^3 - c - \gamma\Delta c \rangle| = |\langle u \cdot \nabla c, \gamma\Delta c \rangle| \leq \gamma \|\nabla u\|_{L^\infty} \|\nabla c\|_{L^2}^2.$$

Use this in (2.18) and integrate in time to get

$$(2.19) \quad \begin{aligned} \int_0^\tau \|\nabla(c^3 - c - \gamma \Delta c)\|^2 ds + \mathcal{E}(\tau) &\leq \mathcal{E}(0) + \gamma \|\nabla u\|_{L^\infty} \int_0^\tau \|\nabla c\|_{L^2}^2 ds \\ &\leq \mathcal{E}(0) + \frac{\gamma \|\nabla u\|_{L^\infty}}{4\pi^2} \int_0^\tau \|\Delta c\|_{L^2}^2 ds. \end{aligned}$$

Time-integrating (2.11) and using (2.12), we find

$$(2.20) \quad \gamma \int_0^\tau \|\Delta c\|_{L^2}^2 ds \leq \frac{1}{\gamma} \int_0^\tau \|c - \bar{c}\|_{L^2}^2 ds + \|c(t_0) - \bar{c}\|_{L^2}^2 \leq e^{\tau/\gamma} \|c_0 - \bar{c}\|_{L^2}^2.$$

Finally, we substitute (2.20) in (2.19) to obtain

$$\frac{1}{2} \gamma \|\nabla c(\tau)\|_{L^2}^2 \leq \mathcal{E}(\tau) \leq \mathcal{E}(0) + \frac{\|\nabla u\|_{L^\infty}}{4\pi^2} e^{\tau/\gamma} \|c_0 - \bar{c}\|_{L^2}^2,$$

which immediately implies (2.17) as claimed. \square

We now prove Lemma 2.4.

Proof of Lemma 2.4. As before, we assume without loss of generality that $t_0 = 0$. In the 3D case, we will express $c(2\tau_2^*)$ using Duhamel's principle. However, for reasons that will be explained below, we need to use a starting time of $t_1 \in [0, \tau_2^*]$, which might not be 0. Note that for any $t_1 \in [0, \tau_2^*]$, we have

$$c(2\tau_2^*) - \bar{c} = \mathcal{S}_{t_1, 2\tau_2^*}(c(t_1) - \bar{c}) + \int_{t_1}^{2\tau_2^*} \mathcal{S}_{s, 2\tau_2^*}(\Delta(c^3 - c)) ds.$$

Since $2\tau_2^* - t_1 \geq \tau_2^*$, the above implies

$$(2.21) \quad \|c(2\tau_2^*) - \bar{c}\|_{L^2} \leq \frac{1}{2} \|c(t_1) - \bar{c}\|_{L^2} + \int_{t_1}^{2\tau_2^*} \|\Delta(c^3 - c)\| ds.$$

To bound the first term on the right, we note that if $2T_1' \leq \gamma \ln \beta$, then (2.1) implies

$$(2.22) \quad \|c(t_1) - \bar{c}\|_{L^2} \leq \sqrt{\beta} B,$$

where $B \stackrel{\text{def}}{=} \|c_0 - \bar{c}\|_{L^2}$.

To bound the second term on the right-hand side, recall the Gagliardo–Nirenberg interpolation inequalities in 3D guarantee

$$\begin{aligned} \|c - \bar{c}\|_{L^\infty} &\leq C \|\nabla c\|_{L^2}^{1/2} \|\Delta c\|_{L^2}^{1/2}, \\ \|\nabla c\|_{L^4} &\leq C \|\nabla c\|_{L^2}^{1/4} \|\Delta c\|_{L^2}^{3/4}. \end{aligned}$$

Expanding $\|\Delta(c^3 - c)\|_{L^2}$ as in (2.14), and using these inequalities, we see

$$(2.23) \quad \begin{aligned} \|\Delta(c^3 - c)\|_{L^2} &\leq C \|\Delta c\|_{L^2}^2 \|\nabla c\|_{L^2} + C |\bar{c}| \|\Delta c\|_{L^2}^{3/2} \|\nabla c\|_{L^2}^{1/2} \\ &\quad + (3\bar{c}^2 + 1) \|\Delta c\|_{L^2}, \\ &\leq C(1 + \bar{c}^2)(1 + \|\nabla c\|_{L^2})(\|\Delta c\|_{L^2} + \|\Delta c\|_{L^2}^2). \end{aligned}$$

The difference from the 2D case is precisely at this step, as the above estimate does not allow us to bound the second term on the right of (2.21) using (2.8) and (2.1) alone. Indeed, to bound this term, we now need a time-uniform bound on $\|\nabla c\|_{L^2}$, in combination with (2.8) and (2.1). Unfortunately, the only such bounds we can obtain depend on u , and thus our criterion in 3D involves both $\|\nabla u\|_{L^\infty}$ and τ_2^* .

To carry out the details, note first that by Chebyshev's inequality and (2.8) we can choose $t_1 \in [0, \tau_2^*]$ so that

$$(2.24) \quad \|\Delta c(t_1)\|_{L^2}^2 \leq \frac{2\beta + 4\gamma\mu}{\gamma^2} B^2.$$

Using the Gagliardo–Nirenberg inequality and (2.24) we note that the free energy \mathcal{E} at time t_1 can be bounded by

$$\begin{aligned} \mathcal{E}(t_1) &\leq \frac{1}{4} \|c(t_1)\|_{L^4}^4 + \frac{1}{2} \gamma \|\nabla c(t_1)\|_{L^2}^2 + \frac{1}{4} \\ &\leq 2 \|c(t_1) - \bar{c}\|_{L^4}^4 + \frac{1}{2} \gamma \|\nabla c\|_{L^2}^2 + 2\bar{c}^4 + \frac{1}{4} \\ &\leq C \|c(t_1) - \bar{c}\|_{L^2}^{5/2} \|\Delta c(t_1)\|_{L^2}^{3/2} + \frac{\gamma}{8\pi^2} \|\Delta c(t_1)\|_{L^2}^2 + 2\bar{c}^4 + \frac{1}{4} \\ &\leq \frac{C\beta^{5/4}(2\beta + 4\gamma\mu)^{3/4}}{\gamma^{3/2}} B^4 + \frac{\beta + 2\gamma\mu}{4\pi^2\gamma} B^2 + 2\bar{c}^4 + \frac{1}{4} \\ &\leq \frac{C_{\beta,\mu} B^4}{\gamma^2} + 2(\bar{c}^4 + 1). \end{aligned}$$

Thus, for any time $t \in [t_1, 2\tau^*]$, we use Lemma 2.5 and obtain

$$(2.25) \quad \begin{aligned} \|\nabla c(t)\|_{L^2}^2 &\leq \frac{2\mathcal{E}(t_1)}{\gamma} + \frac{\|\nabla u\|_{L^\infty}}{2\pi^2\gamma} e^{(t-t_1)/\gamma} \|c(t_1) - \bar{c}\|_{L^2}^2 \\ &\leq \frac{C_{\beta,\mu}}{\gamma^3} (1 + \|\nabla u\|_{L^\infty}) (1 + \bar{c}^4) (B^4 + 1). \end{aligned}$$

The use of (2.8), (2.23) and (2.25) in (2.21) yields

$$(2.26) \quad \begin{aligned} \|c(2\tau_2^*) - \bar{c}\|_{L^2} &\leq \frac{\sqrt{\beta}B}{2} + (1 + \bar{c}^2) \left(1 + \frac{C_{\beta,\mu}}{\gamma^{3/2}} (1 + \|\nabla u\|_{L^\infty})^{1/2} (1 + \bar{c}^2) (1 + B^2) \right) \\ &\quad \cdot \left(\frac{C_{\beta,\mu}}{\gamma^2} B^2 + \frac{C_{\beta,\mu}}{\gamma} B \right) \tau_2^* \\ &\leq \frac{\sqrt{\beta}B}{2} + \frac{C_{\beta,\mu}\tau_2^*}{\gamma^{7/2}} (1 + \|\nabla u\|_{L^\infty})^{1/2} (1 + \bar{c}^4) (1 + B^3) B. \end{aligned}$$

Thus if we choose

$$(2.27) \quad T_1' \stackrel{\text{def}}{=} \min \left\{ \left(\frac{3}{4} - \frac{\sqrt{\beta}}{2} \right) \frac{\gamma^{7/2}}{(1 + B^3)(1 + \bar{c}^4)C_{\beta,\mu}}, \frac{\gamma \ln \beta}{2}, \frac{1}{8\mu} \right\},$$

then our assumption (2.7) and the bound (2.26) imply (2.9) as claimed. Note that, since we have previously assumed $\beta \leq 2$, the choice of T_1' will be strictly positive. Finally, the fact that T_1' is decreasing in $\|c_0 - \bar{c}\|_{L^2}$ follows directly from (2.27). \square

3. The dissipation time of mixing flows

In this section we prove Proposition 1.4. Since working on closed Riemannian manifolds introduces almost no added complexity, we will prove Proposition 1.4 in this setting. Let M be a d -dimensional, smooth, closed Riemannian manifold, with metric normalized so that $\text{vol}(M) = 1$. Let Δ denote the Laplace–Beltrami operator on M , and $u \in L^\infty([0, \infty); W^{1,\infty}(M))$ be a divergence-free vector field. We begin by recalling the definition of *weakly mixing* and *strongly mixing* that we use.

Definition 3.1. Let $h: [0, \infty) \rightarrow (0, \infty)$ be a continuous decreasing function that vanishes at ∞ . Given $\phi_0 \in \dot{L}^2(M)$, let ϕ denote the solution of

$$(3.1) \quad \partial_t \phi + u(t, x) \cdot \nabla \phi = 0,$$

on M , with initial data ϕ_0 .

- (1) We say u is *weakly mixing* with rate function h if for every $\phi_0, \psi \in \dot{H}^1(M)$ and every $s, T \geq 0$ we have

$$\left(\frac{1}{T} \int_0^T |\langle \phi(s+t), \psi \rangle|^2 dt \right)^{1/2} \leq h(T) \|\phi(s)\|_{H^1} \|\psi\|_{H^1}.$$

- (2) We say u is *strongly mixing* with rate function h if for every $\phi_0, \psi \in \dot{H}^1(M)$ and every $s, t \geq 0$ we have

$$|\langle \phi(s+t), \psi \rangle| \leq h(t) \|\phi(s)\|_{H^1} \|\psi\|_{H^1}.$$

The use of H^1 norms in Definition 3.1 is purely for convenience, and is motivated by [LTD11, Thi12, FI19]. The traditional choice in the dynamical systems literature is to use C^1 norms instead. This difference, however, is not significant as varying the norms used in Definition 3.1 only changes the mixing rate function (see for instance Appendix A in [FI19]).

In [FI19, Fen19] the authors estimated the dissipation time $\tau_1^*(u, \gamma)$ in terms of the weak (or strong) mixing rate function h . With minor modifications, their work can be modified to give the following estimate for τ_2^* .

Theorem 3.2. Let $u \in L^\infty([0, \infty); C^2(M))$ be a divergence-free vector field, and $h: [0, \infty) \rightarrow (0, \infty)$ be a continuous decreasing function that vanishes at ∞ .

- (1) There exists constants $C_1, C_2 > 0$ such that if u is weakly mixing with rate function h , then for all sufficiently small γ we have

$$\tau_2^*(u, \gamma) \leq t_* + C_1 \|u\|_{C^2} t_*^2. \quad (3.2)$$

Here t^* is the unique solution of

$$\gamma \|u\|_{C^2} t_*^2 = C_2 (h(t_*/\sqrt{2}))^{8/(4+d)}. \quad (3.3)$$

- (2) There exists constants $C_1, C_2 > 0$ such that if u is strongly mixing with rate function h , then for all sufficiently small γ , we have (3.2), where t_* is the unique solution of

$$\gamma \|u\|_{C^2} t_*^2 = C_2 h^2(t_*/2\sqrt{2}). \quad (3.4)$$

The proof of Theorem 3.2 is very similar to that in [Fen19, Chapter 4], and we provide a sketch in Appendix A. We now prove Proposition 1.4 using Theorem 3.2.

Proof of Proposition 1.4. Rescaling time by a factor of A we immediately see that

$$(3.5) \quad \tau_2^*(u_A, \gamma) = \frac{1}{A} \tau_2^*\left(v, \frac{\gamma}{A}\right).$$

For the first assertion in Proposition 1.4, we assume v is weakly mixing with rate function h . Using (3.2) and (3.5) we see that

$$(3.6) \quad \tau_2^*(u_A, \gamma) \leq \frac{1}{A} (t_*(A) + C_1 \|v\|_{C^2} t_*^2(A)),$$

where $t_*(A)$ solves

$$(3.7) \quad \frac{\gamma}{A} \|v\|_{C^2} t_*^2(A) = C_2 (h(\frac{t_*(A)}{\sqrt{2}}))^{8/(4+d)}.$$

Clearly this implies $t_*(A) \rightarrow \infty$ as $A \rightarrow \infty$. Since h vanishes at ∞ , this in turn implies that $t_*^2(A)/A \rightarrow 0$ as $A \rightarrow \infty$. Consequently, the right hand side of (3.6) vanishes as $A \rightarrow \infty$, proving first assertion of Proposition 1.4

For the second assertion, we assume v is strongly mixing with rate function h satisfying (1.9). In this case Theorem 3.2 and (3.5) imply (3.6) still holds, provided $t_*(A)$ is defined by

$$(3.8) \quad \frac{\gamma}{A} \|v\|_{C^2} t_*^2(A) = C_2 h^2 \left(\frac{t_*(A)}{2\sqrt{2}} \right).$$

Note that this still implies $t_*(A) \rightarrow \infty$ as $A \rightarrow \infty$. Using this along with (1.9) we see that

$$\frac{t_*^2(A)}{A} \leq \frac{\varepsilon}{t_*^2(A)}$$

for any $\varepsilon > 0$, and all sufficiently large A . Using this in (3.6) yields $A^{1/2} \tau_2^*(u_A, \gamma) \rightarrow 0$ as $A \rightarrow \infty$, concluding the proof. \square

4. Relationship between τ_1^* and τ_2^* (Lemma 1.5)

In this section we prove Lemma 1.5 bounding $\tau_2^*(u, \gamma)$ in terms of $\tau_1^*(u, \gamma)$. Throughout we fix $u \in L^\infty([0, \infty); C^2(\mathbb{T}^d))$, and assume θ is a solution of (1.3) with $\alpha = 2$ and mean-zero initial data $\theta_0 \in \dot{L}^2(\mathbb{T}^d)$. As before, we abbreviate $\tau_\alpha^*(u, \gamma)$ to τ_α^* .

The proof of Lemma 1.5 is similar to that of Theorem 1.2 in 3D. We divide the analysis into two cases: the first where the time average of $\|\Delta\theta\|_{L^2}^2$ is large (Lemma 4.1), and the second where the time average of $\|\Delta\theta\|_{L^2}^2$ is small (Lemma 4.2). Lemma 1.5 will be proved after these two lemmas.

Lemma 4.1. *If for some $t_0 \geq 0$, $\lambda, \tau > 0$ we have*

$$(4.1) \quad \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \|\Delta\theta\|_{L^2}^2 ds \geq \lambda \|\theta(t_0)\|_{L^2}^2,$$

then

$$(4.2) \quad \|\theta(t_0 + \tau)\|_{L^2} \leq e^{-\lambda\gamma\tau} \|\theta(t_0)\|_{L^2}.$$

Proof. Multiplying (1.3) by θ and integrating, we obtain

$$\|\theta(t_0 + \tau)\|_{L^2}^2 = \|\theta(t_0)\|_{L^2}^2 - 2\gamma \int_{t_0}^{t_0+\tau} \|\Delta\theta\|_{L^2}^2 ds.$$

Inequalities (4.1) and $1 - x \leq e^{-x}$ yield (4.2) as desired. \square

Lemma 4.2. *There exists an explicit dimensional constant C_1 such that if*

$$\lambda \stackrel{\text{def}}{=} \frac{1}{4\gamma\tau_1^*(20C_1\|u\|_{C^2}\tau_1^* + 11)},$$

and for some $t_0 \geq 0$ we have

$$(4.3) \quad \frac{1}{2\tau_1^*} \int_{t_0}^{t_0+2\tau_1^*} \|\Delta\theta\|_{L^2}^2 ds \leq \lambda \|\theta(t_0)\|_{L^2}^2,$$

then (4.2) still holds at time $\tau = 2\tau_1^*$.

Proof. Without loss of generality assume $t_0 = 0$. By Chebyshev's inequality, there exists $t_1 \in [0, \tau_1^*]$ such that

$$(4.4) \quad \|\Delta\theta(t_1)\|_{L^2}^2 \leq 2\lambda\|\theta_0\|_{L^2}^2.$$

Since

$$\partial_t\theta + u \cdot \nabla\theta - \gamma\Delta\theta = -\gamma\Delta^2\theta - \gamma\Delta\theta,$$

Duhamel's principle implies

$$\theta(2\tau_1^*) = \mathcal{S}_{t_1, 2\tau_1^*}^{u, 1}\theta(t_1) - \gamma \int_{t_1}^{2\tau_1^*} \mathcal{S}_{s, 2\tau_1^*}^{u, 1}(\Delta^2\theta(s) + \Delta\theta(s)) ds,$$

where \mathcal{S} is the solution operator from Definition 1.1. Since $2\tau_1^* - t_1 \geq \tau_1^*$, and \mathcal{S} is an L^2 contraction, this implies

$$(4.5) \quad \|\theta(2\tau_1^*)\|_{L^2} \leq \frac{\|\theta_0\|_{L^2}}{2} + 2\gamma \int_{t_1}^{2\tau_1^*} \|\Delta^2\theta\|_{L^2} ds.$$

To estimate the second term on the right, we multiply (1.3) by $\Delta^2\theta$ and integrate in space to obtain

$$\frac{1}{2}\partial_t\|\Delta\theta\|_{L^2}^2 + \gamma\|\Delta^2\theta\|_{L^2}^2 \leq C_1\|u\|_{C^2}\|\Delta\theta\|_{L^2}^2,$$

for some explicit dimensional constant C_1 . Integration in time together with (4.3) and (4.4) yields

$$2\gamma \int_{t_1}^{2\tau_1^*} \|\Delta^2\theta\|_{L^2}^2 ds \leq \lambda(4C_1\|u\|_{C^2}\tau_1^* + 2)\|\theta_0\|_{L^2}^2.$$

Using this in (4.5) we have

$$\|\theta(2\tau_1^*)\|_{L^2} \leq \left(\frac{1}{2} + 2(\gamma\tau_1^*\lambda(2 + 4C_1\|u\|_{C^2}\tau_1^*))^{1/2}\right)\|\theta_0\|_{L^2}.$$

By our choice of λ this implies

$$\|\theta(2\tau_1^*)\|_{L^2} \leq (1 - 2\lambda\gamma\tau_1^*)\|\theta_0\|_{L^2} \leq e^{-2\lambda\gamma\tau_1^*}\|\theta_0\|_{L^2},$$

finishing the proof. \square

The proof of Lemma 1.5 follows quickly from Lemmas 4.1 and 4.2.

Proof of Lemma 1.5. Iterating Lemmas 4.1 and 4.2 repeatedly we see that for any $t_0 \geq 0$ and $n \in \mathbb{N}$ we have

$$\|\theta(t_0 + 2n\tau_1^*)\|_{L^2} \leq e^{-2n\lambda\gamma\tau_1^*}\|\theta(t_0)\|_{L^2}.$$

Thus we must have $\tau_2^* \leq (\ln 2)/(\lambda\gamma)$, from which (1.10) follows. \square

Appendix A. Dissipation time bounds of mixing vector fields

In this section, we prove Theorem 3.2. As in Section 3, we assume here that M is a smooth, closed, Riemannian manifold with volume 1, and Δ is the Laplace–Beltrami operator on M . We also fix a divergence free vector field $u \in L^\infty([0, \infty); C^2(M))$, and let θ be the solution to the advection hyper-diffusion equation (1.3) with $\alpha = 2$ on the manifold M , with mean-zero initial data $\theta_0 \in \dot{L}^2(M)$.

The idea behind the proof of Theorem 3.2 is to divide the analysis into two cases. When $\|\Delta\theta\|_{L^2}/\|\theta\|_{L^2}$ is large, the energy inequality implies $\|\theta\|_{L^2}$ decays rapidly. On the other hand, when $\|\Delta\theta\|_{L^2}/\|\theta\|_{L^2}$ is small, we use the mixing assumption on u to show that $\|\theta\|_{L^2}$ still decays rapidly. The outline of the proof is the same as

that of Theorem 1.2; however, the proof of the second case is substantially different. We begin by stating two lemmas handling each of the above cases.

Lemma A.1. *The solution θ satisfies the energy inequality*

$$(A.1) \quad \partial_t \|\theta\|_{L^2}^2 = -2\gamma \|\Delta\theta\|_{L^2}^2.$$

Consequently, if for some $c_0 > 0$ we have

$$\|\Delta\theta(t)\|_{L^2}^2 \geq c_0 \|\theta(t)\|_{L^2}^2, \quad \text{for all } 0 \leq t \leq t_0,$$

then

$$(A.2) \quad \|\theta(t)\|_{L^2}^2 \leq e^{-2\gamma c_0 t} \|\theta_0\|_{L^2}^2, \quad \text{for all } 0 \leq t \leq t_0.$$

Lemma A.2. *Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. Suppose u is weakly mixing with rate function h . There exists positive, finite dimensional constants \tilde{C} , \tilde{c} such that the following holds: If λ_N is an eigenvalue of the Laplace–Beltrami operator such that*

$$(A.3) \quad h^{-1}\left(\frac{1}{\tilde{c}\lambda_N^{(d+4)/4}}\right) \leq \frac{1}{\tilde{C}\lambda_N\sqrt{\gamma}\|u\|_{C^2}^{1/2}},$$

and if

$$(A.4) \quad \|\Delta\theta_0\|_{L^2}^2 < \lambda_N^2 \|\theta_0\|_{L^2}^2$$

holds, then we have

$$(A.5) \quad \|\theta(t_0)\|_{L^2}^2 \leq \exp\left(-\frac{\gamma\lambda_N^2 t_0}{4}\right) \|\theta_0\|_{L^2}^2,$$

at a time t_0 given by

$$(A.6) \quad t_0 \stackrel{\text{def}}{=} h^{-1}\left(\frac{1}{\tilde{c}\lambda_N^{(d+4)/4}}\right).$$

If instead u is strongly mixing, then the analog of Lemma A.2 is as follows.

Lemma A.3. *Suppose u is strongly mixing with rate function h . There exists a finite dimensional $\tilde{C} > 0$ such that the following holds: If λ_N is an eigenvalue of the Laplace–Beltrami operator such that*

$$(A.7) \quad 2h^{-1}\left(\frac{1}{2\lambda_N}\right) \leq \frac{1}{\tilde{C}\lambda_N\sqrt{\gamma}\|u\|_{C^2}^{1/2}},$$

and if (A.4) holds, then (A.5) holds at a time t_0 given by

$$(A.8) \quad t_0 \stackrel{\text{def}}{=} 2h^{-1}\left(\frac{1}{2\lambda_N}\right).$$

Momentarily postponing the proof of Lemmas A.1–A.3, we prove Theorem 3.2.

Proof of Theorem 3.2. For the first assumption, we assume u is weakly mixing with rate function h . Let \tilde{c} , \tilde{C} be the constants from Lemma A.2. Note that the intermediate value theorem readily implies the existence of a unique $\lambda_* > 0$ such that

$$(A.9) \quad h^{-1}\left(\frac{1}{\tilde{c}\lambda_*^{(d+4)/4}}\right) = \frac{1}{\tilde{C}\lambda_*\sqrt{\gamma}\|u\|_{C^2}^{1/2}}.$$

Further, it is easy to see that $\lambda_* \rightarrow \infty$ as $\gamma \rightarrow 0$. Thus, for all sufficiently small γ , Weyl's lemma¹ guarantees the existence of $N = N(\gamma) \in \mathbb{N}$ such that

$$(A.11) \quad \frac{\lambda_*^2}{2} \leq \lambda_N^2 \leq \lambda_*^2.$$

Now choosing $c_0 = \lambda_N^2$ and repeatedly applying Lemmas A.1 and Lemma A.2, we obtain an increasing sequence of times (t'_k) , such that $t'_k \rightarrow \infty$, $t'_{k+1} - t'_k \leq t_0$, and

$$\|\theta_s(t'_k)\|_{L^2}^2 \leq \exp\left(-\frac{\gamma \lambda_N^2 t'_k}{4}\right) \|\theta_0\|_{L^2}^2.$$

This immediately implies

$$(A.12) \quad \tau_2^*(u, \gamma) \leq \frac{8 \ln 2}{\gamma \lambda_N^2} + t_0.$$

Choosing

$$t_* \stackrel{\text{def}}{=} \frac{\sqrt{2}}{\tilde{C} \lambda_* \sqrt{\gamma} \|u\|_{C^2}^{1/2}},$$

and using (A.9), (A.11), and (A.12) yields (3.2) as claimed.

The proof of the second assertion of Theorem 3.2 is almost identical to that of the first assertion. The only change required is to replace Lemma A.2 with A.3. \square

It remains to prove Lemmas A.1–A.3.

Proof of Lemma A.1. Multiplying (1.3) by θ , integrating over M and using the fact that u is divergence free immediately yields (A.1). The second assertion of Lemma A.1 follows from this and Gronwall's lemma. \square

For Lemmas A.2 and A.3 we will need a standard result estimating the difference between θ and solutions to the inviscid transport equation.

Lemma A.4. *Let ϕ be the solution of (3.1) with initial data θ_0 . There exists a dimensional constant C_d such that for all $t \geq 0$ we have*

$$(A.13) \quad \|\theta(t) - \phi(t)\|_{L^2}^2 \leq \sqrt{2\gamma t} \|\theta_0\|_{L^2} \left(C_d \|u\|_{C^2} \int_0^t \|\Delta \theta\|_{L^2}^2 ds + \|\Delta \theta_0\|_{L^2}^2 \right)^{1/2}.$$

Proof. Subtracting (1.3) and (3.1) shows

$$\partial_t(\theta - \phi) + u \cdot \nabla(\theta - \phi) + \gamma \Delta^2 \theta = 0.$$

Multiplying this by $\theta(t) - \phi(t)$ and integrating over space and time gives

$$(A.14) \quad \|\theta(t) - \phi(t)\|_{L^2}^2 = -2\gamma \int_0^t \int_M (\theta - \phi) \Delta^2 \theta dx ds \leq 2\gamma \|\theta_0\|_{L^2} \int_0^t \|\Delta^2 \theta\|_{L^2} ds.$$

On the other hand, multiplying (1.3) by $\Delta^2 \theta$ and integrating over M gives

$$\partial_t \|\Delta \theta\|_{L^2}^2 + 2\langle u \cdot \nabla \theta, \Delta^2 \theta \rangle + 2\gamma \|\Delta^2 \theta\|_{L^2}^2 = 0.$$

¹Recall that Weyl's lemma (see for instance [MP49]) says that

$$(A.10) \quad \lambda_j \approx \frac{4\pi \Gamma(\frac{d}{2} + 1)^{2/d}}{\text{vol}(M)^{2/d}} j^{2/d},$$

asymptotically as $j \rightarrow \infty$. This implies $\lambda_{j+1} - \lambda_j = o(\lambda_j)$ as $j \rightarrow \infty$. Hence, for all sufficiently large λ_* , one can always find N large enough such that (A.11) holds.

Integrating the middle term by parts, using the fact that u is divergence free, and integrating in time yields

$$2\gamma \int_0^t \|\Delta^2 \theta\|_{L^2}^2 ds \leq C_d \|u\|_{C^2} \int_0^t \|\Delta \theta\|_{L^2}^2 ds + \|\Delta \theta_0\|_{L^2}^2,$$

for some dimensional constant C_d . Substituting this in (A.14) and using the Cauchy–Schwartz inequality gives (A.13) as claimed. \square

We now prove Lemma A.2.

Proof of Lemma A.2. We claim that our choice of λ_N and t_0 will guarantee

$$(A.15) \quad \int_0^{t_0} \|\Delta \theta(s)\|_{L^2}^2 ds \geq \frac{\lambda_N^2 t_0 \|\theta_0\|_{L^2}^2}{8}.$$

Once this is established, integrating (A.1) in time immediately yields (A.5).

Thus, to prove Lemma A.2, we only need to prove (A.15). Suppose, for contradiction, the inequality (A.15) does not hold. Letting $P_N: \dot{L}^2(M) \rightarrow \dot{L}^2(M)$ denote the orthogonal projection onto the span of the first N eigenfunctions of the Laplace–Beltrami operator, we observe

$$\begin{aligned} (A.16) \quad & \frac{\lambda_N^2 t_0 \|\theta_0\|_{L^2}^2}{8} > \int_0^{t_0} \|\Delta \theta(s)\|_{L^2}^2 ds \geq \lambda_N^2 \int_{t_0/2}^{t_0} \|(I - P_N)\theta(s)\|_{L^2}^2 ds \\ & \geq \frac{\lambda_N^2}{2} \int_{t_0/2}^{t_0} \|(I - P_N)\phi(s)\|_{L^2}^2 ds - \lambda_N^2 \int_{t_0/2}^{t_0} \|(I - P_N)(\theta(s) - \phi(s))\|_{L^2}^2 ds \\ & \geq \frac{\lambda_N^2 t_0}{4} \|\theta_0\|_{L^2}^2 - \frac{\lambda_N^2}{2} \int_{t_0/2}^{t_0} \|P_N \phi(s)\|_{L^2}^2 ds - \lambda_N^2 \int_0^{t_0} \|\theta(s) - \phi(s)\|_{L^2}^2 ds. \end{aligned}$$

We will now bound the last two terms in (A.16).

For the last term in (A.16), we use Lemma A.4 to obtain

$$\begin{aligned} (A.17) \quad & \int_0^{t_0} \|\theta(s) - \phi(s)\|_{L^2}^2 ds \\ & \leq \int_{s=0}^{t_0} \sqrt{2\gamma s} \|\theta_0\|_{L^2} \left(C_d \|u\|_{C^2} \int_0^s \|\Delta \theta(t)\|_{L^2}^2 dt + \|\Delta \theta_0\|_{L^2}^2 \right)^{1/2} ds \\ & \leq C \sqrt{\gamma} t_0^{3/2} \|\theta_0\|_{L^2} \left(\|u\|_{C^2} \int_0^{t_0} \|\Delta \theta(t)\|_{L^2}^2 dt + \|\Delta \theta_0\|_{L^2}^2 \right)^{1/2} \\ & \leq C \sqrt{\gamma} t_0^{3/2} \lambda_N \|\theta_0\|_{L^2}^2 \left(\|u\|_{C^2} t_0 + 1 \right)^{1/2}. \end{aligned}$$

For the last inequality above, we used our assumption that the inequality (A.15) does not hold.

To estimate the second term on the right of (A.16), let e_j denote the eigenfunction of the Laplace–Beltrami operator corresponding to the eigenvalue λ_j . Now

$$\int_{t_0/2}^{t_0} \|P_N \phi(s)\|_{L^2}^2 ds \leq \sum_{j=1}^N \int_0^{t_0} |\langle \phi(s), e_j \rangle|^2 ds \leq t_0 h^2(t_0) \|\phi_0\|_{H^1}^2 \sum_{j=1}^N \lambda_j.$$

Using Weyl’s lemma (A.10) and the assumption (A.4), we see

$$(A.18) \quad \int_{t_0/2}^{t_0} \|P_N \phi(s)\|_{L^2}^2 ds \leq C t_0 h^2(t_0) \|\phi_0\|_{L^2}^2 \lambda_N^{(d+4)/2},$$

for some constant $C = C(M)$.

We now let C_1 be the larger of the constants appearing in (A.17) and (A.18). Using these two inequalities in (A.16) shows

$$(A.19) \quad \frac{1}{8} > \frac{1}{4} - C_1 \lambda_N \sqrt{\gamma t_0} (1 + t_0 \|u\|_{C^2})^{1/2} - C_1 \lambda_N^{(d+4)/2} h^2(t_0).$$

If we choose $\tilde{c} \geq \sqrt{16C_1}$, then by equation (A.6) the last term on the right is at most $1/16$. Next, when γ is sufficiently small we will have $t_0 \|u\|_{C^2} \geq 1$. Thus, if $\tilde{C} \geq 16\sqrt{2}C_1$ and λ_N is the largest eigenvalue for which (A.3) holds, then the second term above is also at most $1/16$. This implies $1/8 > 1/8$, which is the desired contradiction. \square

The proof of Lemma A.3 is very similar to that of Lemma A.2.

Proof of Lemma A.3. Follow the proof of Lemma A.2 until (A.18). Now, to estimate the second term on the right of (A.16), the strongly mixing property of u gives

$$(A.20) \quad \begin{aligned} \int_{t_0/2}^{t_0} \|P_N \phi(s)\|_{L^2}^2 ds &\leq \lambda_N \int_{t_0/2}^{t_0} \|\phi(s)\|_{H^{-1}}^2 ds \leq \lambda_N \int_{t_0/2}^{t_0} h^2(s) \|\theta_0\|_{H^1}^2 ds \\ &\leq \frac{t_0}{2} \lambda_N h^2(t_0/2) \|\Delta \theta_0\|_{L^2} \|\theta_0\|_{L^2} \leq \frac{t_0}{2} \lambda_N^2 h^2(t_0/2) \|\theta_0\|_{L^2}^2. \end{aligned}$$

Above, the last inequality followed from interpolation and the assumption (A.4).

Now let C_1 be the constant appearing in (A.17). Using (A.17) and (A.20) in (A.16) implies

$$\frac{1}{8} > \frac{1}{4} - C_1 \lambda_N \sqrt{\gamma t_0} (1 + t_0 \|u\|_{C^2})^{1/2} - \frac{1}{4} \lambda_N^2 h^2(t_0/2).$$

If t_0 is defined by (A.8), then the last term above is at most $1/16$. Moreover, if $\tilde{C} = 2^{9/2} C_1$ and λ_N is the largest eigenvalue of the Laplace–Beltrami operator satisfying (A.7), then the second term above is also at most $1/16$. This again forces $1/8 > 1/8$, which is our desired contradiction. \square

References

- [ACM19] G. Alberti, G. Crippa, and A. L. Mazzucato. Exponential self-similar mixing by incompressible flows. *J. Amer. Math. Soc.*, 32(2):445–490, 2019. doi:[10.1090/jams/913](https://doi.org/10.1090/jams/913).
- [BBPS19] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith. Almost-sure exponential mixing of passive scalars by the stochastic Navier-Stokes equations, 2019, [1905.03869](https://arxiv.org/abs/1905.03869).
- [BH17] J. Bedrossian and S. He. Suppression of blow-up in Patlak-Keller-Segel via shear flows. *SIAM J. Math. Anal.*, 49(6):4722–4766, 2017. doi:[10.1137/16M1093380](https://doi.org/10.1137/16M1093380).
- [BKNR10] H. Berestycki, A. Kiselev, A. Novikov, and L. Ryzhik. The explosion problem in a flow. *J. Anal. Math.*, 110:31–65, 2010. doi:[10.1007/s11854-010-0002-7](https://doi.org/10.1007/s11854-010-0002-7).
- [Cah61] J. W. Cahn. On spinodal decomposition. *Acta Metallurgica*, 9(9):795–801, 1961. doi:[10.1016/0001-6160\(61\)90182-1](https://doi.org/10.1016/0001-6160(61)90182-1).
- [CH58] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958. doi:[10.1063/1.1744102](https://doi.org/10.1063/1.1744102).
- [CKRZ08] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš. Diffusion and mixing in fluid flow. *Ann. of Math. (2)*, 168(2):643–674, 2008. doi:[10.4007/annals.2008.168.643](https://doi.org/10.4007/annals.2008.168.643).
- [CPB88] C. K. Chan, F. Perrot, and D. Beysens. Effects of hydrodynamics on growth: Spinodal decomposition under uniform shear flow. *Phys. Rev. Lett.*, 61:412–415, Jul 1988. doi:[10.1103/PhysRevLett.61.412](https://doi.org/10.1103/PhysRevLett.61.412).
- [CZDE18] M. Coti Zelati, M. G. Delgadino, and T. M. Elgindi. On the relation between enhanced dissipation time-scales and mixing rates. *ArXiv e-prints*, June 2018, [1806.03258](https://arxiv.org/abs/1806.03258).
- [Ell89] C. M. Elliott. The Cahn-Hilliard model for the kinetics of phase separation. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 35–73. Birkhäuser, Basel, 1989.

- [ES86] C. M. Elliott and Z. Songmu. On the Cahn-Hilliard equation. *Arch. Rational Mech. Anal.*, 96(4):339–357, 1986. doi:[10.1007/BF00251803](https://doi.org/10.1007/BF00251803).
- [EZ19] T. M. Elgindi and A. Zlatoš. Universal mixers in all dimensions. *Adv. Math.*, 356:106807, 33, 2019. doi:[10.1016/j.aim.2019.106807](https://doi.org/10.1016/j.aim.2019.106807).
- [Fen19] Y. Feng. *Dissipation enhancement by mixing*. Carnegie Mellon University, 2019. Ph.D. Thesis.
- [FI19] Y. Feng and G. Iyer. Dissipation enhancement by mixing. *Nonlinearity*, 32(5):1810–1851, 2019. doi:[10.1088/1361-6544/ab0e56](https://doi.org/10.1088/1361-6544/ab0e56).
- [FKR06] A. Fannjiang, A. Kiselev, and L. Ryzhik. Quenching of reaction by cellular flows. *Geom. Funct. Anal.*, 16(1):40–69, 2006. doi:[10.1007/s00039-006-0554-y](https://doi.org/10.1007/s00039-006-0554-y).
- [FW03] A. Fannjiang and L. Wołowski. Noise induced dissipation in Lebesgue-measure preserving maps on d -dimensional torus. *J. Statist. Phys.*, 113(1-2):335–378, 2003. doi:[10.1023/A:1025787124437](https://doi.org/10.1023/A:1025787124437).
- [HL09] T. Y. Hou and Z. Lei. On the stabilizing effect of convection in three-dimensional incompressible flows. *Comm. Pure Appl. Math.*, 62(4):501–564, 2009. doi:[10.1002/cpa.20254](https://doi.org/10.1002/cpa.20254).
- [IXZ19] G. Iyer, X. Xu, and A. Zlatoš. Convection-induced singularity suppression in the Keller-Segel and other non-linear PDEs. *arXiv e-prints*, Aug 2019, [1908.01941](https://arxiv.org/abs/1908.01941).
- [KSZ08] A. Kiselev, R. Shterenberg, and A. Zlatoš. Relaxation enhancement by time-periodic flows. *Indiana Univ. Math. J.*, 57(5):2137–2152, 2008. doi:[10.1512/iumj.2008.57.3349](https://doi.org/10.1512/iumj.2008.57.3349).
- [KX16] A. Kiselev and X. Xu. Suppression of chemotactic explosion by mixing. *Arch. Ration. Mech. Anal.*, 222(2):1077–1112, 2016. doi:[10.1007/s00205-016-1017-8](https://doi.org/10.1007/s00205-016-1017-8).
- [LDE⁺13] J. Liu, L. Dedè, J. A. Evans, M. J. Borden, and T. J. Hughes. Isogeometric analysis of the advective Cahn–Hilliard equation: Spinodal decomposition under shear flow. *Journal of Computational Physics*, 242:321 – 350, 2013. doi:[10.1016/j.jcp.2013.02.008](https://doi.org/10.1016/j.jcp.2013.02.008).
- [LLG95] J. Läger, C. Laubner, and W. Gronski. Correlation between shear viscosity and anisotropic domain growth during spinodal decomposition under shear flow. *Phys. Rev. Lett.*, 75:3576–3579, Nov 1995. doi:[10.1103/PhysRevLett.75.3576](https://doi.org/10.1103/PhysRevLett.75.3576).
- [LTD11] Z. Lin, J.-L. Thiffeault, and C. R. Doering. Optimal stirring strategies for passive scalar mixing. *J. Fluid Mech.*, 675:465–476, 2011. doi:[10.1017/S0022112011000292](https://doi.org/10.1017/S0022112011000292).
- [MD18] C. J. Miles and C. R. Doering. Diffusion-limited mixing by incompressible flows. *Nonlinearity*, 31(5):2346, 2018. doi:[10.1088/1361-6544/aab1c8](https://doi.org/10.1088/1361-6544/aab1c8).
- [MP49] S. Minakshisundaram and Å. Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math.*, 1:242–256, 1949. doi:[10.4153/CJM-1949-021-5](https://doi.org/10.4153/CJM-1949-021-5).
- [ONT07a] L. Ó Náraigh and J.-L. Thiffeault. Bubbles and filaments: Stirring a Cahn–Hilliard fluid. *Phys. Rev. E*, 75:016216, Jan 2007. doi:[10.1103/PhysRevE.75.016216](https://doi.org/10.1103/PhysRevE.75.016216).
- [ONT07b] L. Ó Náraigh and J.-L. Thiffeault. Dynamical effects and phase separation in cooled binary fluid films. *Phys. Rev. E*, 76:035303, Sept. 2007. doi:[10.1103/PhysRevE.76.035303](https://doi.org/10.1103/PhysRevE.76.035303).
- [ONT08] L. Ó Náraigh and J.-L. Thiffeault. Bounds on the mixing enhancement for a stirred binary fluid. *Physica D*, 237(21):2673–2684, Nov. 2008. doi:[10.1016/j.physd.2008.04.012](https://doi.org/10.1016/j.physd.2008.04.012).
- [Peg89] R. L. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London Ser. A*, 422(1863):261–278, 1989.
- [Pie94] R. Pierrehumbert. Tracer microstructure in the large-eddy dominated regime. *Chaos, Solitons & Fractals*, 4(6):1091–1110, 1994.
- [Poo96] C.-C. Poon. Unique continuation for parabolic equations. *Comm. Partial Differential Equations*, 21(3-4):521–539, 1996. doi:[10.1080/03605309608821195](https://doi.org/10.1080/03605309608821195).
- [Thi12] J.-L. Thiffeault. Using multiscale norms to quantify mixing and transport. *Nonlinearity*, 25(2):R1–R44, 2012. doi:[10.1088/0951-7715/25/2/R1](https://doi.org/10.1088/0951-7715/25/2/R1).
- [Wei18] D. Wei. Diffusion and mixing in fluid flow via the resolvent estimate. *arXiv e-prints*, Nov 2018, [1811.11904](https://arxiv.org/abs/1811.11904).
- [YZ17] Y. Yao and A. Zlatoš. Mixing and un-mixing by incompressible flows. *J. Eur. Math. Soc. (JEMS)*, 19(7):1911–1948, 2017. doi:[10.4171/JEMS/709](https://doi.org/10.4171/JEMS/709).
- [Zla10] A. Zlatoš. Diffusion in fluid flow: dissipation enhancement by flows in 2D. *Comm. Partial Differential Equations*, 35(3):496–534, 2010. doi:[10.1080/03605300903362546](https://doi.org/10.1080/03605300903362546).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN – MADISON, MADISON, WI 53706
Email address: `feng65@wisc.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA
15213
Email address: `yuanyuaf@math.cmu.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA
15213
Email address: `gautam@math.cmu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN – MADISON, MADISON, WI 53706
Email address: `jeanluc@math.wisc.edu`