

CONVECTION-INDUCED SINGULARITY SUPPRESSION IN THE KELLER-SEGEL AND OTHER NON-LINEAR PDES

GAUTAM IYER, XIAOQIAN XU, AND ANDREJ ZLATOŠ

ABSTRACT. In this paper we study the effect of the addition of a convective term, and of the resulting increased dissipation rate, on the growth of solutions to a general class of non-linear parabolic PDEs. In particular, we show that blow-up in these models can always be prevented if the added drift has a small enough dissipation time. We also prove a general result relating the dissipation time and the effective diffusivity of stationary cellular flows, which allows us to obtain examples of simple incompressible flows with arbitrarily small dissipation times.

As an application, we show that blow-up in the Keller-Segel model of chemotaxis can always be prevented if the velocity field of the ambient fluid has a sufficiently small dissipation time. We prove a similar result for a fluid dynamics model derived from the 3D Navier-Stokes equations by adding a convection term to their vorticity formulation. Finally we study reaction-diffusion equations with ignition-type nonlinearities, and show that the reaction can always be quenched by the addition of a convective term with a small enough dissipation time, provided the average initial temperature is initially below the ignition threshold.

1. Introduction and main results

The question of growth and blow-up of solutions to non-linear parabolic PDEs is of widespread interest, and arises in their applications to areas such as fluid dynamics, population dynamics, combustion, and cosmology. Convection is often included in the relevant models, and its presence may have two diametrically opposite effects on the solution dynamics. While, on the one hand, it does have the ability to promote formation of singular structures, in this paper we concentrate on the opposite, stabilizing effect of convection.

2010 *Mathematics Subject Classification*. Primary 76R05; Secondary 35B27, 35B44, 35Q35.

This work has been partially supported by the National Science Foundation under grants DMS-1652284 and DMS-1900943 to AZ, and DMS-1814147 to GI, as well as by the Center for Nonlinear Analysis.

The ability of strong (incompressible) drifts to slow down growth of solutions to non-linear PDEs, and even prevent their blow-up, has been studied by many authors (see for instance [FKR06, BKNR10, KX16, BH17, He18, HT19]). Many of the results focus on the study of special convective motions with strong mixing properties, and their ability to prevent singularity formation in specific models. While strongly mixing flows are good candidates for this purpose, there are various other (often much simpler) flows that can be used to control singularities.

Indeed, our first main result shows that addition of flows that simply enhance dissipation to a certain degree is sufficient to keep solutions regular. This, of course, includes many strong mixing flows, whose dissipation-enhancing properties have recently been studied extensively (see, e.g., [CKRZ08, CZDE18, FI18]). But our results apply to general flows, and the simplicity of the relevant hypotheses allows us to apply them to a wide class of non-linear parabolic models.

Suppression of blow-up in general non-linear PDEs. In order to keep the presentation simple, we restrict our attention to the spatially periodic setting. We hence study the PDE

$$(1.1) \quad \partial_t \theta + u \cdot \nabla \theta = \Delta \theta + N(\theta)$$

on the d -dimensional torus \mathbb{T}^d and with initial data $\theta_0 \in L_0^2(\mathbb{T}^d)$, where $L_0^p(\mathbb{T}^d)$ is the space of all mean-zero functions in $L^p(\mathbb{T}^d)$. The velocity u in the convection term is a prescribed (i.e. independent of θ) time-dependent Lipschitz divergence-free vector field. The non-linear operator $N: H^1(\mathbb{T}^d) \rightarrow L_0^2(\mathbb{T}^d)$ is measurable, and its target space being $L_0^2(\mathbb{T}^d)$ means that solutions to (1.1) remain mean-zero. Finally, N satisfies the following crucial hypotheses:

(H1) There exists $\varepsilon_0 \in (0, 1]$ and an increasing continuous function $F: [0, \infty) \rightarrow [0, \infty)$ such that for every $\varphi \in H^1(\mathbb{T}^d)$ we have

$$\left| \int_{\mathbb{T}^d} \varphi N(\varphi) dx \right| \leq (1 - \varepsilon_0) \|\nabla \varphi\|_{L^2}^2 + F(\|\varphi\|_{L^2}).$$

(H2) There exists $C_0 < \infty$ and an increasing continuous function $G: [0, \infty) \rightarrow [0, \infty)$ such that for every $\varphi \in H^1(\mathbb{T}^d)$ we have

$$\|N(\varphi)\|_{L^2} \leq C_0 \|\nabla \varphi\|_{L^2}^2 + G(\|\varphi\|_{L^2}).$$

Under hypotheses (H1)–(H2), our main result shows that for any mean-zero initial data θ_0 , the corresponding weak solution to (1.1) is uniformly bounded in $L^2(\mathbb{T}^d)$ on its time interval of existence, provided the *dissipation time* of u is sufficiently small. The latter is a measure of the dissipation rate enhancement provided by the convection term, defined as follows (see also [FW03, Zla10, FI18]).

Definition 1.1. Let $u \in L^\infty((0, \infty), W^{1,\infty}(\mathbb{T}^d))$ be a divergence-free vector field, and let $\mathcal{S}_{s,t}$ be the solution operator to the advection-diffusion equation

$$(1.2) \quad \partial_t \varphi + u \cdot \nabla \varphi = \Delta \varphi$$

on $(0, \infty) \times \mathbb{T}^d$. That is, $\varphi_t \stackrel{\text{def}}{=} \mathcal{S}_{s,t} f$ solves (1.2) with initial data $\varphi_s = f$. The *dissipation time* of u is

$$\tau_*(u) \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 \mid \|\mathcal{S}_{s,s+t}\|_{L_0^2 \rightarrow L_0^2} \leq \frac{1}{2} \text{ for all } s \geq 0 \right\}.$$

Here, and throughout this paper, $\varphi_t \stackrel{\text{def}}{=} \varphi(t, \cdot)$ denotes the slice of the function φ at time t . Note also that since the L^2 -norm of solutions to (1.2) is non-decreasing, $\|\mathcal{S}_{s,s+t}\|_{L_0^2 \rightarrow L_0^2} \leq \frac{1}{2}$ implies $\|\mathcal{S}_{s,s+t'}\|_{L_0^2 \rightarrow L_0^2} \leq \frac{1}{2}$ for all $t' \geq t$.

We can now state our first main result. We refer the reader to Section 2 below for the definition of weak solutions to (1.1) used here.

Theorem 1.2. Assume that N satisfies hypotheses (H1)–(H2) and

$$(1.3) \quad \theta \in L_{\text{loc}}^2((0, T), H^1(\mathbb{T}^d)) \cap C([0, T], L_0^2(\mathbb{T}^d))$$

is a weak solution to (1.1) with $u \in L^\infty((0, \infty), W^{1,\infty}(\mathbb{T}^d))$ a divergence-free vector field. There is $\tau_0 = \tau_0(\|\theta_0\|_{L^2}, N)$ such that if $\tau_*(u) \leq \tau_0$, then

$$\sup_{t \in [0, T]} \|\theta_t\|_{L^2} \leq 2\|\theta_0\|_{L^2} + 1.$$

If the functions F and G from hypotheses (H1) and (H2) also satisfy

$$(1.4) \quad \limsup_{y \rightarrow 0^+} \frac{\sqrt{F(y)} + G(y)}{y} < \infty$$

and $T = \infty$, then $\|\theta_t\|_{L^2} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

Remark. The time τ_0 can be computed explicitly in terms of F , G , C_0 , ε_0 and $\|\theta_0\|$. It is given by $T_0(\|\theta_0\|_{L^2})$ from Proposition 2.2 below for the first claim, and by $T_1(\|\theta_0\|_{L^2})$ from Proposition 2.3 below for the second claim. The exponential decay rate can also be computed explicitly from Proposition 2.3, and is discussed in Remark 2.4 below.

Cellular flows with small dissipation times. In order to apply Theorem 1.2, we need to construct flows with arbitrarily small dissipation times. One construction of such flows is through appropriate rescaling of mixing flows. Indeed, the action of mixing flows transfers energy from lower to higher frequencies; the faster this happens, the faster solutions to (1.2) dissipate energy and the smaller $\tau_*(u)$ becomes. This principle has previously been used to obtain rigorous bounds on

the dissipation time. In [CKRZ08] (see also [BHN05, KSZ08]) the authors show that for time-independent velocity fields $u = u(x)$ we have $\lim_{A \rightarrow \infty} \tau_*(Au) = 0$ if and only if the operator $u \cdot \nabla$ on \mathbb{T}^d has no eigenfunctions in $H^1(\mathbb{T}^d)$ other than constants. In particular, if the flow of u is weakly mixing (i.e., the spectrum of $u \cdot \nabla$ is continuous), then $\tau_*(Au)$ must vanish as the flow amplitude $A \rightarrow \infty$. Moreover, in [FI18, CZDE18] the authors obtain explicit bounds on $\tau_*(u)$ from the (appropriately defined) *mixing rate* of u . As a result, if u generates an exponentially mixing flow, then $\tau_*(Au(At, \cdot)) \leq \frac{c}{A} (\ln A)^2$ for some constant $c > 0$.

The disadvantage of constructing flows with small dissipation times in this manner is that known examples of strongly mixing flows are either quite complicated or not very regular (see for instance [CKRZ08, YZ17, EZ18, ACM19]). There are, however, many flows that are far from mixing in any sense but still have small dissipation times. While these times cannot vanish as the amplitude of the flow is increased, such flows are still sufficient for our purposes.

Here we construct flows with arbitrarily small dissipation times by rescaling a general class of smooth (time-independent) cellular flows. A prototypical example of a 2D cellular flow is given by

$$(1.5) \quad u(x) = \nabla^\perp \sin(2\pi x_1) \sin(2\pi x_2) = 2\pi \begin{pmatrix} -\sin(2\pi x_1) \cos(2\pi x_2) \\ \cos(2\pi x_1) \sin(2\pi x_2) \end{pmatrix}.$$

In two dimensions, all cellular flows have closed trajectories and are therefore not mixing.

Nevertheless, we will show that by rescaling both the cell size and the flow amplitude, the dissipation time of such flows can be made arbitrarily small. We will achieve this by establishing a relation between their dissipation time and their (direction-dependent) *effective diffusivity*. The latter, denoted $D_e(u)$ for any $e \in \mathbb{S}^{d-1}$, is the asymptotic long-time mean square displacement in direction e of the stochastic process on \mathbb{R}^d associated to the operator $\Delta - u \cdot \nabla$, normalized by the factor $\frac{1}{2t}$ (a precise definition can be found at beginning of Section 6 below). If we let $D(u) \stackrel{\text{def}}{=} \min\{D_{e_1}(u), \dots, D_{e_n}(u)\}$ be the minimum of the effective diffusivities of the flow u in each of the coordinate directions, then we have the following result.

Theorem 1.3. *For each $n \in \mathbb{N}$, let $u_n \in W^{1,\infty}(\mathbb{T}^d)$ be a mean-zero divergence-free velocity field that is symmetric in all coordinates (i.e., it satisfies (6.5) below). If $\lim_{n \rightarrow \infty} D(u_n) = \infty$, then there exists $\nu_n \in \mathbb{N}$ such that the rescaled velocity fields $v_n(x) \stackrel{\text{def}}{=} -\nu_n u_n(\nu_n x)$ on \mathbb{T}^d satisfy*

$$\lim_{n \rightarrow \infty} \tau_*(v_n) = 0.$$

In particular, for $d \in \{2, 3\}$ and any $T > 0$, there exists a smooth cellular flow u on \mathbb{T}^d such that $\tau_*(u) \leq T$.

We will in fact provide an upper bound on the required reciprocal cell size of ν_n , and further show that

$$\tau_*(v_n) \leq C D(u_n)^{-\alpha} \ln(1 + D(u_n)),$$

for some constant C and some explicit $\alpha > 0$. The precise details are in Remark 6.6 in Section 6 below.

We note that effective diffusivity of cellular flows has been extensively studied by many authors (see for instance [CS89, FP94, Kor04, RZ07]), particularly in two dimensions. In fact, typical 2D cellular flows (including the one in (1.5)) satisfy $D(Au) \sim A^{1/2}$ as $A \rightarrow \infty$. While the asymptotic behavior of $D(Au)$ is not as well understood in three dimensions, a large class of 3D cellular flows still have $\lim_{A \rightarrow \infty} D(Au) = \infty$ (see, e.g., [RZ07]). Thus, in both two and three dimensions, we can apply Theorem 1.3 by choosing $u_n(x) = nu(x)$ for some cellular flow u . We revisit this in Examples 6.7 and 6.8 below.

We end this introduction by discussing applications of the above results to specific non-linear models.

Suppression of blow-up in the Keller-Segel system. Chemotaxis is the movement of organisms in response to chemical stimuli, and arises in many contexts such as the movement of bacteria towards food sources and sperm towards eggs, as well as migration of neurons and leukocytes. The mathematical study of chemotaxis was initiated by Patlak [Pat53], and Keller and Segel [KS70, KS71] who modelled the process as a coupled parabolic system. Here, we study a simplified, parabolic-elliptic version of this system introduced by Jäger and Luckhaus [JL92] (see also [Hor03, Hor04, Per07]). If $\rho \geq 0$ represents the bacterial population density and $c \geq 0$ represents the concentration of a chemoattractant produced by the bacteria, then the evolution of ρ and c is governed by

$$(1.6a) \quad \partial_t \rho - \Delta \rho = -\nabla \cdot (\rho \chi \nabla c),$$

$$(1.6b) \quad -\Delta c = \rho - \bar{\rho},$$

$$(1.6c) \quad \bar{\rho} = \int_{\mathbb{T}^d} \rho \, dx,$$

which we consider on \mathbb{T}^d . Here $\chi > 0$ is a sensitivity parameter and we will assume that the dimension d is either 2 or 3. This model stipulates that bacterial diffusion is biased in the direction of the gradient of the concentration of a chemoattractant that is emitted by the bacteria themselves, and that the chemoattractant diffuses much faster than the bacteria do.

It is proved in [JL92] that if $d = 2$ and the initial data is below a certain critical threshold, solutions to (1.6a)–(1.6c) are regular for all positive time. Above this threshold, [JL92] constructs solutions that blow up in finite time by concentrating positive mass at a single point. In three dimensions, similar results were proved by Herrero et al. [HMY97, HMY98, Her00] (see also [Hor03, Per07]).

We now consider the Keller-Segel system in the presence of a drift, generated by the movement of the ambient fluid. If u is a divergence-free vector field representing the fluid velocity, then (1.6a) is replaced by

$$(1.6a') \quad \partial_t \rho + u \cdot \nabla \rho = \Delta \rho - \nabla \cdot (\rho \chi \nabla c).$$

This model was studied previously by Kiselev and Xu in [KX16], where they show that for any initial population distribution, there exists an ambient velocity field u that ensures the solution to (1.6a') remains regular for all positive time. They prove this showing that two families of velocity fields u , the relaxation-enhancing flows from [CKRZ08] and the (initial-data-dependent and time-dependent) exponentially mixing flows from [YZ17], can be used to prevent a blow-up in solutions to (1.6a').

Both these types of flows have fairly complicated geometries. In contrast, our results above allow us to prove a result similar to that in [KX16], using general flows that have sufficiently small dissipation times. This of course includes the flows considered in [KX16], but also the much simpler fast cellular flows with small cells.

Theorem 1.4. *Let $d \in \{2, 3\}$ and $\rho_0 \in C^\infty(\mathbb{T}^d)$ be any non-negative function. There exists $\tau_0 = \tau_0(\|\rho_0\|_{L^2}, \chi) > 0$, such that if for some divergence-free Lipschitz u we have $\tau_*(u) \leq \tau_0$, then the unique solution ρ to (1.6a'), (1.6b), (1.6c) with initial data ρ_0 is globally regular and*

$$(1.7) \quad \lim_{t \rightarrow \infty} \|\rho_t - \bar{\rho}\|_{L^2} = 0.$$

In particular, for each such ρ_0 , there is a time-independent smooth cellular flow u on \mathbb{T}^d that prevents singularity formation in (1.6a'), (1.6b), (1.6c).

It is also important to point out a difference between Theorem 1.4 and results on domains of infinite volume, where strong convection may easily increase dissipation by simply spreading L^1 initial data across large regions (see, e.g., [KZ06, HT19] for such results on the plane involving chemotaxis as well as quenching of reactions from the next subsection). This is clearly not possible on \mathbb{T}^d , the domain considered here.

Suppression of blow-up in the vorticity formulation of the 3D Navier-Stokes equations. The incompressible Navier-Stokes equations describe the evolution of the velocity field of an incompressible fluid. Global existence of their smooth solutions in 3D is a longstanding open problem that has been extensively studied since the seminal work of Leray [Ler34]. While there are currently many conditional global regularity results (see for instance [Pro59, Ser63, CF93, ISS03, SŠ02]), the full problem remains one of the six unresolved Clay Millennium problems [Con01, Fef06].

Using Theorem 1.2 and standard estimates, we show here that given any smooth initial data, one can always add a smooth time-independent convection term into the vorticity form of the 3D Navier-Stokes equations so that the solution stays globally regular. This is Theorem 4.1 in Section 4 below, and the only requirement on the convection term is that it has a small enough dissipation time. We remark, however, that while the vorticity form is an equivalent reformulation of the Navier-Stokes equations, adding a convection term to it is not the same as studying the Navier-Stokes equations with a background flow.

Quenching in models of combustion. Reaction-diffusion equations with ignition type nonlinearities are a well established model used to study the temperature of a combusting substance. It is well known that if, on a bounded domain, the average temperature is above the ignition threshold initially, then the fuel eventually burns completely. However, when the average temperature is below the ignition threshold, the fuel may or may not burn completely, and it is possible for the reaction to be quenched. We will show in Theorem 5.1 in Section 5 below that, in fact, the reaction can always be quenched in this case by the addition of any convective term with a sufficiently small dissipation time. The proof does not use Theorem 1.2 but instead involves showing that $\|\mathcal{S}_{0,T}\|_{L_0^1 \rightarrow L_0^\infty}$ (for any fixed $T > 0$) can be made arbitrarily small by making the dissipation time of the advecting velocity field small enough.

Organization of the paper. We prove Theorem 1.2 in Section 2, Theorem 1.4 in Section 3, and discuss applications to fluid dynamics and combustion models in Sections 4 and 5, respectively. Theorem 1.3 is proved in Section 6, along with some results concerning asymptotic mean displacement of stochastic processes corresponding to (1.2) on \mathbb{R}^d with periodic flows.

Acknowledgements. We thank Alexander Kiselev and Bruce Driver for helpful discussions.

2. Uniform energy bounds for general PDEs

In this section we prove Theorem 1.2. We begin by stating the definition of weak solutions to (1.1).

Definition 2.1. We say that θ satisfying (1.3) is a weak solution to (1.1) on $(0, T) \times \mathbb{T}^d$ if it is a distributional solution to the PDE, and for every $\psi \in H^1(\mathbb{T}^d)$ and almost every $t \in (0, T)$ we obtain equality if we multiply the PDE at time t by ψ and integrate over \mathbb{T}^d .

Note that once θ is a distributional solution to (1.1) satisfying (1.3), and $u \in L^\infty((0, \infty), W^{1, \infty}(\mathbb{T}^d))$ and N satisfies (H2), it follows that $\Delta\theta \in L^2_{\text{loc}}((0, T), H^{-1}(\mathbb{T}^d))$ and $u \cdot \nabla\theta, N(\theta) \in L^1_{\text{loc}}((0, T), L^2(\mathbb{T}^d))$, so

$$(2.1) \quad \partial_t \theta \in L^2_{\text{loc}}((0, T), H^{-1}(\mathbb{T}^d)) + L^1_{\text{loc}}((0, T), L^2(\mathbb{T}^d)).$$

This means that for any $\psi \in H^1(\mathbb{T}^d)$ and almost any $t \in (0, T)$, we can indeed multiply each term in (1.1) by ψ and integrate over \mathbb{T}^d .

Theorem 1.2 is now an immediate corollary of the following two results.

Proposition 2.2. *Suppose N satisfies the hypotheses (H1)–(H2), $u \in L^\infty((0, \infty), W^{1, \infty}(\mathbb{T}^d))$ is a divergence-free vector field, and θ is a weak solution to (1.1) such that (1.3) holds. For any $B \geq 0$, let*

$$T_0(B) \stackrel{\text{def}}{=} \min \left\{ \int_B^{2B+1} \frac{y}{F(y)} dy, \frac{B}{2C_0 \varepsilon_0^{-1} F(2B+1) + 2G(2B+1)} \right\}.$$

If for some $t_0 \geq 0$ we have $\tau_*(u) \leq T_0(\|\theta_{t_0}\|_{L^2})$, then

$$(2.2a) \quad \|\theta_{t_0+n\tau_*(u)}\|_{L^2} \leq \|\theta_{t_0}\|_{L^2} \quad \text{for every } n \in \mathbb{N},$$

$$(2.2b) \quad \|\theta_t\|_{L^2} \leq 2\|\theta_{t_0}\|_{L^2} + 1 \quad \text{for every } t > t_0.$$

Proposition 2.3. *Assume the hypotheses of Proposition 2.2 as well as (1.4). Let $T_1(0) \stackrel{\text{def}}{=} \infty$, and for any $B > 0$ let*

$$T_1(B) \stackrel{\text{def}}{=} \inf_{b \in (0, B]} \min \left\{ \int_b^{2b} \frac{y}{F(y)} dy, \frac{b}{4C_0 \varepsilon_0^{-1} F(2b) + 4G(2b)} \right\}.$$

If for some $t_0 \geq 0$ we have $\tau_*(u) \leq T_1(\|\theta_{t_0}\|_{L^2})$, then

$$(2.3) \quad \|\theta_t\|_{L^2} \leq 2\Psi^{\lfloor (t-t_0)/\tau_*(u) \rfloor} (\|\theta_{t_0}\|_{L^2}) \quad \text{for every } t \geq t_0,$$

where $\Psi(B) \stackrel{\text{def}}{=} B - \min\{\frac{B}{16}, \frac{\varepsilon_0}{8C_0}\}$ and $\Psi^n \stackrel{\text{def}}{=} \Psi \circ \Psi^{n-1}$ for $n \geq 2$.

Remark. Hypothesis (1.4) ensures that $T_1(B) > 0$ for every $B > 0$.

Remark 2.4. The time decay obtained in (2.3) is exponential. The exponential decay rate is $\tau_*(u)^{-1} |\ln(1-r)|$, where r is initially

$$\min \left\{ \frac{1}{16}, \frac{\varepsilon_0}{8C_0 \|\theta_{t_0}\|_{L^2}} \right\},$$

and once t becomes large enough, r changes to $\frac{1}{16}$. By optimizing the proof further, this number can be increased to any number smaller than

$$\frac{1}{2} - \lim_{B \rightarrow 0^+} \frac{\tau_*(u)}{4T_1(B)}.$$

We now prove both propositions.

Proof of Proposition 2.2. Without loss of generality, we assume $t_0 = 0$. For notational convenience let $B \stackrel{\text{def}}{=} \|\theta_0\|_{L^2}$ and $\tau_* = \tau_*(u)$.

Let us now assume that $\|\theta_s\|_{L^2} \leq B$ for some $s \geq 0$ (which holds for $s = 0$). Note that similarly to [Eva98, Theorem 5.9.3], one can use (2.1) and (1.3) to show that $\|\theta_t\|_{L^2}^2$ is a locally absolutely continuous function of t , whose derivative equals $2 \int_{\mathbb{T}^d} (\partial_t \theta)_t \theta_t dx$ at almost every $t \in (0, T)$. Hence, multiplying (1.2) by θ and integrating in space yields

$$(2.4) \quad \begin{aligned} \frac{1}{2} \partial_t \|\theta_t\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2 &\leq \left| \int_{\mathbb{T}^d} \theta_t N(\theta_t) dx \right| \\ &\leq (1 - \varepsilon_0) \|\nabla \theta_t\|_{L^2}^2 + F(\|\theta_t\|_{L^2}) \end{aligned}$$

for almost every $t \in (0, T)$, so for these t we have

$$(2.5) \quad \partial_t \|\theta_t\|_{L^2} \|\theta_t\|_{L^2} \leq F(\|\theta_t\|_{L^2}).$$

Thus, for almost every $t \in (0, T)$ we have

$$\partial_t \int_B^{|\theta_t|_{L^2}} \frac{y}{F(y)} dy = \frac{\partial_t \|\theta_t\|_{L^2} \|\theta_t\|_{L^2}}{F(\|\theta_t\|_{L^2})} \leq 1.$$

Since $\int_B^{|\theta_s|_{L^2}} \frac{y}{F(y)} dy \leq 0$, it follows that for all $t \in [s, s + T_0(B)]$ we have

$$(2.6) \quad \int_B^{|\theta_t|_{L^2}} \frac{y}{F(y)} dy \leq t - s \leq T_0(B) \leq \int_B^{2B+1} \frac{y}{F(y)} dy,$$

which forces $\|\theta_t\|_{L^2} \leq 2B + 1$ for all $t \in [s, s + T_0(B)]$.

Next, integrating (2.4) in time yields

$$(2.7) \quad \|\theta_{s+\tau_*}\|_{L^2}^2 - \|\theta_s\|_{L^2}^2 \leq 2 \int_s^{s+\tau_*} F(\|\theta_t\|_{L^2}) dt - 2\varepsilon_0 \int_s^{s+\tau_*} \|\nabla \theta_t\|_{L^2}^2 dt.$$

If

$$(2.8) \quad \int_s^{s+\tau_*} \|\nabla \theta_t\|_{L^2}^2 dt \geq \frac{1}{\varepsilon_0} \int_s^{s+\tau_*} F(\|\theta_t\|_{L^2}) dt,$$

then the right hand side of (2.7) is at most 0, and hence $\|\theta_{s+\tau_*}\|_{L^2} \leq B$.

If (2.8) fails, then Duhamel's formula shows that

$$\theta_{s+\tau_*} = \mathcal{S}_{s, s+\tau_*} \theta_s + \int_s^{s+\tau_*} \mathcal{S}_{t, s+\tau_*} N(\theta_t) dt,$$

and hence

$$\begin{aligned}
\|\theta_{s+\tau_*}\|_{L^2} &\leq \|\mathcal{S}_{s,s+\tau_*}\theta_s\|_{L^2} + \int_s^{s+\tau_*} \|\mathcal{S}_{t,s+\tau_*}N(\theta_t)\|_{L^2} dt \\
&\leq \frac{B}{2} + \int_s^{s+\tau_*} \|N(\theta_t)\|_{L^2} dt \\
&\leq \frac{B}{2} + \int_s^{s+\tau_*} \left(C_0\|\nabla\theta_t\|_{L^2}^2 + G(\|\theta_t\|_{L^2})\right) dt \\
(2.9) \quad &\leq \frac{B}{2} + \int_s^{s+\tau_*} \left(\frac{C_0}{\varepsilon_0}F(\|\theta_t\|_{L^2}) + G(\|\theta_t\|_{L^2})\right) dt.
\end{aligned}$$

Since $\tau_* \leq T_0(B)$ (so that $\|\theta_t\|_{L^2} \leq 2B + 1$ for all t in the integral), it follows that

$$\|\theta_{s+\tau_*}\|_{L^2} \leq \frac{B}{2} + \left(\frac{C_0}{\varepsilon_0}F(2B + 1) + G(2B + 1)\right)\tau_* \leq B.$$

From this and $\|\theta_t\|_{L^2} \leq 2B + 1$ for all $t \in [s, s + \tau_*]$ both holding whenever $\|\theta_s\|_{L^2} \leq B$ (in particular, when $s = 0$), the claim follows via induction on n . \square

Proof of Proposition 2.3. Without loss of generality, we again assume $t_0 = 0$. We again also denote $B \stackrel{\text{def}}{=} \|\theta_0\|_{L^2}$ and $\tau_* = \tau_*(u)$. By (1.4) and continuity of G we have $G(0) = 0$, therefore $N(0) = 0$ by (H2). This proves the claim when $\theta_0 = 0$, so let us now assume that $B > 0$.

As in the previous proof, from (2.5) we obtain

$$(2.10) \quad \|\theta_t\|_{L^2} \leq 2\|\theta_s\|_{L^2} \quad \text{whenever} \quad t \in \left[s, s + \int_{\|\theta_s\|_{L^2}}^{2\|\theta_s\|_{L^2}} \frac{y}{F(y)} dy\right].$$

Next, let $C_1 \stackrel{\text{def}}{=} \max\{C_0B, 2\varepsilon_0\}$. From (2.7) we see that if

$$(2.11) \quad \int_0^{\tau_*} \|\nabla\theta_t\|_{L^2}^2 dt \geq \frac{1}{\varepsilon_0} \int_0^{\tau_*} F(\|\theta_t\|_{L^2}) dt + \frac{C_1 - \varepsilon_0}{4C_1^2} B^2,$$

then we obtain (also using $\varepsilon_0 \leq C_1/2$ and the definition of C_1)

$$\begin{aligned}
\|\theta_{\tau_*}\|_{L^2} &\leq \left(1 - \frac{\varepsilon_0(C_1 - \varepsilon_0)}{2C_1^2}\right)^{1/2} B \\
(2.12) \quad &\leq \left(1 - \frac{\varepsilon_0}{8C_1}\right) B \leq \left(1 - \min\left\{\frac{1}{16}, \frac{\varepsilon_0}{8C_0B}\right\}\right) B.
\end{aligned}$$

And if (2.11) fails, then the argument proving (2.9) shows that

$$\|\theta_{\tau_*}\|_{L^2} \leq \frac{B}{2} + \int_0^{\tau_*} \left(\frac{C_0}{\varepsilon_0}F(\|\theta_t\|_{L^2}) + G(\|\theta_t\|_{L^2})\right) dt + \frac{C_0(C_1 - \varepsilon_0)}{4C_1^2} B^2.$$

From (2.10) and F, G being increasing, we see that if $\tau_* \leq T_1(B)$, then the integral is bounded above by $\frac{B}{4}$ and so

$$\|\theta_{\tau_*}\|_{L^2} \leq \left(\frac{3}{4} + \frac{C_0 C_1 - \varepsilon_0}{4C_1^2} B \right) B \leq \left(1 - \min \left\{ \frac{1}{8}, \frac{\varepsilon_0}{4C_0 B} \right\} \right) B.$$

Iterating this shows that

$$(2.13) \quad \|\theta_{n\tau_*}\|_{L^2} \leq \Psi^n(\|\theta_0\|_{L^2})$$

for all $n \in \mathbb{N}$. From (2.10) we know that $\|\theta_t\|_{L^2}$ at most doubles at times between integer multiples of τ_* . Combining this fact with (2.13) yields (2.3), as desired. \square

3. Preventing blow-up of Keller-Segel dynamics

In this section we prove Theorem 1.4. For the sake of convenience, we denote

$$\nabla^{-1} \stackrel{\text{def}}{=} \nabla \Delta^{-1}.$$

That is, we let $\nabla^{-1}\psi \stackrel{\text{def}}{=} \nabla\phi$, where ϕ solves $\Delta\phi = \psi$ on \mathbb{T}^d . With this notation, the first two equations of the Keller-Segel system (1.6a'), (1.6b), (1.6c) read

$$(3.1) \quad \partial_t \rho + u \cdot \nabla \rho = \Delta \rho + \chi \nabla \cdot (\rho \nabla^{-1}(\rho - \bar{\rho})).$$

The main idea here is to use the results from Section 2 to show that $\|\rho_t\|_{L^2}$ remains uniformly bounded in time. Once this is established, well known results can be used to obtain global regularity, and then (1.7) will be obtained from (2.3).

Lemma 3.1. *Let ρ be a smooth solution to (1.6a'), (1.6b), (1.6c) on the time interval $[0, T)$, and let $t_0 \in [0, T)$. There are $c = c(\chi \|\rho_{t_0} - \bar{\rho}\|_{L^2}) > 0$ and $\tau_1 = \tau_1(\|\rho_{t_0} - \bar{\rho}\|_{L^2}, \bar{\rho}, \chi) > 0$ such that if $\tau_*(u) \leq \tau_1$ for the incompressible drift u , then*

$$(3.2) \quad \|\rho_t - \bar{\rho}\|_{L^2} \leq 3e^{-c(t-t_0)/\tau_*(u)} \|\rho_{t_0} - \bar{\rho}\|_{L^2},$$

for all $t \in [t_0, T)$. Moreover, τ_1 and c can be chosen to be decreasing in each argument.

Proof. Notice that if we let

$$\begin{aligned} \theta &\stackrel{\text{def}}{=} \rho - \bar{\rho}, \\ N(\theta) &\stackrel{\text{def}}{=} \chi \nabla \cdot ((\theta + \bar{\rho}) \nabla^{-1} \theta), \end{aligned}$$

then θ satisfies (1.1) (here we think of $\bar{\rho}_0$ as a parameter) and $N(\theta)$ is mean-zero. Thus, in order to apply Proposition 2.3, we only need to verify hypotheses (H1)–(H2) and (1.4). For (H1), we compute

$$(3.3) \quad \begin{aligned} \chi^{-1} \int_{\mathbb{T}^d} \theta N(\theta) dx &= \int_{\mathbb{T}^d} (\theta + \bar{\rho}) \theta^2 dx + \int_{\mathbb{T}^d} \theta \nabla \theta \cdot \nabla^{-1} \theta dx \\ &= \frac{1}{2} \|\theta\|_{L^3}^3 + \bar{\rho} \|\theta\|_{L^2}^2 \end{aligned}$$

because

$$\int_{\mathbb{T}^d} \theta \nabla \theta \cdot \nabla^{-1} \theta dx = \frac{1}{2} \int_{\mathbb{T}^d} \nabla \theta^2 \cdot \nabla^{-1} \theta dx = -\frac{1}{2} \int_{\mathbb{T}^d} \theta^3 dx.$$

Using the Gagliardo-Nirenberg and Young inequalities (see for instance [Maz85]), we have

$$\|\theta\|_{L^3}^3 \leq C_1 \|\theta\|_{L^2}^{3-\frac{d}{2}} \|\nabla \theta\|_{L^2}^{\frac{d}{2}} \leq \chi^{-1} \|\nabla \theta\|_{L^2}^2 + C_2 \chi^{\frac{d}{4-d}} \|\theta\|_{L^2}^{\frac{12-2d}{4-d}},$$

for some universal constants C_1, C_2 . Using this in (3.3) shows that (H1) holds with $\varepsilon_0 = \frac{1}{2}$ and

$$F(y) = \left(C_1 \chi^{\frac{4}{4-d}} y^{\frac{4}{4-d}} + \chi \bar{\rho} \right) y^2.$$

For (H2), note that the Hardy-Littlewood-Sobolev inequality implies

$$\begin{aligned} \chi^{-1} \|N(\theta)\|_{L^2} &\leq \|\theta(\theta + \bar{\rho})\|_{L^2} + \|\nabla \theta \cdot \nabla^{-1} \theta\|_{L^2} \\ &\leq \bar{\rho} \|\theta\|_{L^2} + \|\theta\|_{L^4}^2 + \|\nabla \theta\|_{L^2} \|\nabla^{-1} \theta\|_{L^\infty} \\ &\leq \bar{\rho} \|\theta\|_{L^2} + C_3 \left(\|\theta\|_{\dot{H}^{d/4}}^2 + \|\nabla \theta\|_{L^2} \|\theta\|_{\dot{H}^{d/2-1+\varepsilon}} \right) \\ &\leq \bar{\rho} \|\theta\|_{L^2} + C_4 \|\nabla \theta\|_{L^2}^2 \end{aligned}$$

for any $\varepsilon > 0$ and constants C_3, C_4 depending only on ε . (Above we also used for $\psi = \Delta^{-1} \theta$ that for any $s > 0$ we have $\|D^2 \psi\|_{\dot{H}^s} \leq C \|\Delta \psi\|_{\dot{H}^s}$, with some constant $C = C(s, d)$.) Hence (H2) is satisfied with $C_0 = C_4 \chi$ (after fixing, e.g., $\varepsilon = \frac{1}{2}$) and $G(y) = \chi \bar{\rho} y$.

From the above formulae for F and G we can see that (1.4) also holds. Thus Proposition 2.3 applies, proving (3.2) as well as the last claim. \square

Next, we use the fact that a time-uniform bound on $\|\rho_t\|_{L^2}$ implies global existence for (3.1). This was shown in [KX16], and we present here a short proof for the sake of completeness.

Lemma 3.2 (Thm. 2.1 in [KX16]). *Let $d \in \{2, 3\}$ and $\rho_0 \in C^\infty(\mathbb{T}^d)$ be non-negative. If the maximal time T of existence of the unique smooth*

solution ρ to (3.1) with initial data ρ_0 and incompressible drift u is finite, then

$$(3.4) \quad \int_0^T \|\rho_t - \bar{\rho}\|_{L^2}^{\frac{4}{4-d}} dt = \infty.$$

Proof. Multiplying (3.1) by $-\Delta\rho$ and integrating in space yields

$$(3.5) \quad \frac{1}{2} \partial_t \|\nabla\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 \leq \left| \int_{\mathbb{T}^d} \nabla\rho \cdot \nabla^{-1}(\rho - \bar{\rho}) \Delta\rho dx \right| \\ + \left| \int_{\mathbb{T}^d} u \cdot \nabla\rho \Delta\rho dx \right| + \left| \int_{\mathbb{T}^d} \rho(\rho - \bar{\rho}) \Delta\rho dx \right| = I + II + III.$$

To estimate I , we integrate by parts to get

$$I \leq \frac{1}{2} \left| \int_{\mathbb{T}^d} \nabla|\nabla\rho|^2 \cdot \nabla^{-1}(\rho - \bar{\rho}) dx \right| + \sum_{i=1}^d \left| \int_{\mathbb{T}^d} (\nabla\rho) \cdot (\nabla^{-1}(\rho - \bar{\rho}))_{x_i} \rho_{x_i} dx \right|.$$

Another integration by parts in the first term shows that it is bounded above by

$$\int_{\mathbb{T}^d} |\nabla\rho|^2 |\rho - \bar{\rho}| dx \leq \|\rho - \bar{\rho}\|_{L^4} \|\nabla\rho\|_{L^{8/3}}^2$$

while L^4 -boundedness of double Riesz transforms shows that the second term is bounded above by $C\|\rho - \bar{\rho}\|_{L^4} \|\nabla\rho\|_{L^{8/3}}^2$ for some constant C . The Gagliardo-Nirenberg inequality applied to both norms in this product then yields

$$I \leq C \|\nabla\rho\|_{L^2}^{\frac{d}{4}} \|\rho - \bar{\rho}\|_{L^2}^{1-\frac{d}{4}} \|\Delta\rho\|_{L^2}^{\frac{d}{4}} \|\nabla\rho\|_{L^2}^{2-\frac{d}{4}} = C \|\nabla\rho\|_{L^2}^2 \|\rho - \bar{\rho}\|_{L^2}^{1-\frac{d}{4}} \|\Delta\rho\|_{L^2}^{\frac{d}{4}}$$

and therefore finally

$$I \leq C \|\nabla\rho\|_{L^2}^{2-\frac{d}{2}} \|\rho - \bar{\rho}\|_{L^2} \|\Delta\rho\|_{L^2}^{\frac{d}{2}} \leq \frac{1}{2} \|\Delta\rho\|_{L^2}^2 + C \|\rho - \bar{\rho}\|_{L^2}^{\frac{4}{4-d}} \|\nabla\rho\|_{L^2}^2$$

(with a new C in each inequality).

For II , we again integrate by parts and use $\nabla \cdot u = 0$ to get

$$II \leq \left| \int_{\mathbb{T}^d} u \cdot \nabla|\nabla\rho|^2 dx \right| + C \|u\|_{C^1} \|\nabla\rho\|_{L^2}^2 = C \|u\|_{C^1} \|\nabla\rho\|_{L^2}^2.$$

For III , we integrate by parts and use the Gagliardo-Nirenberg inequality exactly as we did for term I to obtain

$$III \leq 2 \int_{\mathbb{T}^d} |\rho - \bar{\rho}| |\nabla\rho|^2 dx + \bar{\rho} \int_{\mathbb{T}^d} |\nabla\rho|^2 dx \\ \leq \frac{1}{2} \|\Delta\rho\|_{L^2}^2 + \left(C \|\rho - \bar{\rho}\|_{L^2}^{\frac{4}{4-d}} + \bar{\rho} \right) \|\nabla\rho\|_{L^2}^2.$$

Using the above bounds in (3.5) yields

$$(3.6) \quad \partial_t \|\nabla\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 \leq C \left(\|u\|_{C^1} + \bar{\rho} + \|\rho - \bar{\rho}\|_{L^2}^{\frac{4}{4-d}} \right) \|\nabla\rho\|_{L^2}^2.$$

Gronwall's lemma now shows that if (3.4) does not hold, then $\|\nabla\rho\|_{L^2}$ and $\|\Delta\rho\|_{L^2}$ remain uniformly bounded on $[0, T)$. Then a standard Galerkin approximations argument shows that ρ can be smoothly extended beyond T , a contradiction. This completes the proof. \square

We can now prove Theorem 1.4.

Proof of Theorem 1.4. Let $\tau_0(y, \chi) \stackrel{\text{def}}{=} \tau_1(y, y, \chi)$, with τ_1 being from Lemma 3.1 (so τ_0 is also decreasing in both arguments). If T is the maximal time of existence of ρ , then (3.2) holds (with $t_0 = 0$) for all $t \in [0, T)$. Therefore we must have $T = \infty$ because otherwise (3.4) would not hold, contradicting Lemma 3.2. This also means that (1.7) follows from (3.2). The last claim follows from Theorem 1.3. \square

4. Preventing blow-up in the vorticity equation

The 3D Navier-Stokes equations in vorticity form are

$$(4.1a) \quad \partial_t \omega = \Delta \omega + (\omega \cdot \nabla) u - (u \cdot \nabla) \omega,$$

$$(4.1b) \quad \omega = \nabla \times u,$$

$$(4.1c) \quad \nabla \cdot u = 0.$$

Here ω represents the (incompressible) vorticity field of an incompressible 3D fluid and u is the corresponding velocity field. We impose periodic boundary conditions on (4.1a)–(4.1c), and always assume that u has mean zero (it is then uniquely determined from (4.1b), (4.1c) to be $u = -\Delta^{-1} \nabla \times \omega$).

Global existence of smooth solutions to (4.1a)–(4.1c) is a longstanding open problem and has been extensively studied (see for instance [Con01, Fef06]). In this section we show that for any given ω_0 , one can add a smooth, uniformly bounded drift to (4.1a) which will ensure that the solution remains globally regular. The drift, which we denote here by v , can be any velocity field with a small enough dissipation time. We remark, however, that the resulting model is not the same as when one adds the drift to the original Navier-Stokes system in the velocity form.

Theorem 4.1. *Consider the system*

$$(4.1a') \quad \partial_t \omega + (v \cdot \nabla) \omega = \Delta \omega + (\omega \cdot \nabla) u - (u \cdot \nabla) \omega,$$

along with (4.1b) and (4.1c), with mean-zero divergence-free initial data $\omega_0 \in L^2(\mathbb{T}^3)$ and a divergence-free drift $v \in C^\infty([0, \infty) \times \mathbb{T}^3)$. There is $\tau_0 = \tau_0(\|\omega_0\|_{L^2}) > 0$ such that if $\tau_(v) \leq \tau_0$, then the unique solution ω to the above system is globally regular and*

$$(4.2) \quad \lim_{t \rightarrow \infty} \|\omega_t\|_{L^2} = 0.$$

In particular, for each such ω_0 , there is a time-independent smooth cellular flow u on \mathbb{T}^d that prevents singularity formation in this system.

As in the previous section, we will use results from Section 2 to show that $\|\omega_t\|_{L^2}$ remains uniformly bounded in time, and then that it also decays to 0.

Lemma 4.2. *Let $\omega \in C^\infty((0, T) \times \mathbb{T}^3)$ be a smooth solution to (4.1a'), (4.1b), (4.1c), and let $t_0 \in [0, T)$. There exists $\tau_0 = \tau_0(\|\omega_{t_0}\|_{L^2}) > 0$ and a constant $c = c(\|\omega_{t_0}\|_{L^2}) > 0$ such that if $\tau_*(v) \leq \tau_0$ then*

$$(4.3) \quad \|\omega_t\|_{L^2} \leq 3e^{-c(t-t_0)/\tau_*} \|\omega_{t_0}\|_{L^2},$$

for all $t \in [t_0, T)$. Moreover, τ_0 and c can be chosen to be decreasing.

Proof. As with Lemma 3.1, we will prove Lemma 4.2 by applying Proposition 2.3. With $u = -\Delta^{-1}\nabla \times \omega$, the system becomes the single equation (4.1a'), and we define

$$N(\omega) \stackrel{\text{def}}{=} (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.$$

Incompressibility of u and ω shows that $N(\omega)$ is mean-zero.

Note that the proofs in Section 2 extend to the vector-valued case if we take the dot product of (1.1) with θ at the start of the proof of Proposition 2.2, and then hypothesis (H1) also involves a dot product. We now get from L^3 -boundedness of double Riesz transforms (note that $\nabla u = -\nabla(\nabla \times \Delta^{-1}\omega)$)

$$\left| \int_{\mathbb{T}^3} \omega \cdot N(\omega) dx \right| = \left| \int_{\mathbb{T}^3} \omega \cdot (\omega \cdot \nabla)u dx \right| \leq \|\omega\|_{L^3}^2 \|\nabla u\|_{L^3} \leq C\|\omega\|_{L^3}^3,$$

and then conclude via the Gagliardo-Nirenberg inequality (with a new constant C in each inequality)

$$\left| \int_{\mathbb{T}^3} \omega \cdot N(\omega) dx \right| \leq C\|\omega\|_{L^2}^{3/2} \|\nabla \omega\|_{L^2}^{3/2} \leq \frac{1}{2} \|\nabla \omega\|_{L^2}^2 + C\|\omega\|_{L^2}^6.$$

Thus, hypothesis (H1) holds with $\varepsilon_0 = \frac{1}{2}$ and $F(y) = Cy^6$.

For hypothesis (H2), via similar arguments we obtain

$$\begin{aligned} \|N(\omega)\|_{L^2} &\leq \|(\omega \cdot \nabla)u\|_{L^2} + \|(u \cdot \nabla)\omega\|_{L^2} \\ &\leq \|\omega\|_{L^4} \|\nabla u\|_{L^4} + \|u\|_{L^\infty} \|\nabla \omega\|_{L^2} \\ &\leq C\|\omega\|_{L^4}^2 + C\|\nabla u\|_{L^4} \|\nabla \omega\|_{L^2} \\ &\leq C\|\nabla \omega\|_{L^2}^2. \end{aligned}$$

Hence hypothesis (H2) is satisfied with some C_0 and $G(y) = 0$.

From the above formulae for F and G we see that (1.4) also holds. Thus Proposition 2.3 applies and immediately yields (4.3). \square

Proof of Theorem 4.1. Let τ_0 be from Lemma 4.2, and suppose if T is the maximal time of existence of ω . Then (4.3) holds (with $t_0 = 0$) for all $t \in [0, T)$. Recall, a standard result (see, e.g., [CF88, Tem01]), shows that if for some $T < \infty$, the maximal interval of existence for the Navier-Stokes equations (4.1a)–(4.1c) is $[0, T)$, then we must have $\limsup_{t \rightarrow T} \|\omega_t\|_{L^2} = \infty$. Its proof applies essentially without change to our system (4.1a'), (4.1b), (4.1c), and hence (4.3) (with $t_0 = 0$) forces $T = \infty$. This also means that (4.2) follows from (4.3). The last claim follows from Theorem 1.3. \square

5. Quenching of reactions by stirring

A well established model for the (normalized) temperature of a combusting substance (see, e.g., the reviews [Ber02, Xin00, Xin09]) is the reaction diffusion equation

$$(5.1) \quad \partial_t \theta = \Delta \theta + f(\theta).$$

Here, the function θ takes values in $[0, 1]$ and the non-linear Lipschitz *reaction function* f is of the *ignition type* (other types of reaction functions model other reactive processes, including chemical kinetics and population dynamics). That is, we assume that there is $\alpha_0 \in (0, 1)$ such that

$$f(\theta) = 0 \quad \text{for } \theta \in [0, \alpha_0] \cup \{1\} \quad \text{and} \quad f(\theta) > 0 \quad \text{for } \theta \in (\alpha_0, 1).$$

The number α_0 is *ignition temperature*, below which no burning occurs. Also note that since $\theta \equiv 0, 1$ are stationary solutions, comparison principle indeed yields $0 \leq \theta \leq 1$ whenever $0 \leq \theta_0 \leq 1$.

When the combusting process is also subject to mixing due to a prescribed incompressible drift u , equation (5.1) becomes

$$(5.1') \quad \partial_t \theta + u \cdot \nabla \theta = \Delta \theta + f(\theta).$$

With or without the drift u , one can show using

$$\partial_t \|\theta_t\|_{L^1} = \int_{\mathbb{T}^d} f(\theta_t) dx \geq 0$$

that the fuel burns completely (i.e., $\lim_{t \rightarrow \infty} \|1 - \theta_t\|_{L^\infty} = 0$) for all initial data $0 \leq \theta_0 \leq 1$ that satisfy $\bar{\theta}_0 \stackrel{\text{def}}{=} \int_{\mathbb{T}^d} \theta_0 dx \geq \alpha_0$ and $\theta_0 \not\equiv \alpha_0$.

When instead $\bar{\theta}_0 < \alpha_0$, the fuel may or may not burn completely, and it is possible for the reaction to be *quenched*, that is, $\|\theta_T\|_{L^\infty} \leq \alpha_0$ for some $T \geq 0$. Comparison principle shows that after such time T , evolution of the temperature will be governed by the linear equation (1.2), and hence we will have $\lim_{t \rightarrow \infty} \|\theta_t - \bar{\theta}_0\|_{L^\infty} = 0$.

The main result of this section shows that if $\bar{\theta}_0 < \alpha_0$, then one can always ensure quenching by choosing an incompressible drift u with a small enough dissipation time.

Theorem 5.1. *Let θ be the solution to (5.1') on \mathbb{T}^d with nonnegative initial data $\theta_0 \in L^\infty(\mathbb{T}^d)$ with $\bar{\theta}_0 < \alpha_0$. There is $\tau_0 = \tau_0(\alpha_0, \bar{\theta}_0)$ such that if $\tau_*(u) \leq \tau_0$ for the incompressible drift u , then the reaction is quenched. In particular, if $\bar{\theta}_0 < \alpha_0$ and $d \in \{2, 3\}$, then there is a time-independent smooth cellular flow u on \mathbb{T}^d that quenches the reaction.*

Remark. The last claim should also hold for $d \geq 4$, but we are not aware of a construction of such flows. Time-periodic flows with this property can be constructed by alternating flows from Example 6.7 below acting in different couples of coordinates.

Theorem 5.1 and its proof are closely related to Theorem 7.2 in [CKRZ08], which is a qualitative statement about a certain class of time-independent drifts (so-called relaxation-enhancing ones) on general compact manifolds. Theorem 5.1 is a more quantitative result that applies to general time-dependent drifts. It is an immediate consequence of the comparison principle and the following result about equation (1.2).

Proposition 5.2. *There is a constant C_d such that the solution operator $\mathcal{S}_{s,t}$ for (1.2) with any $u \in L^\infty((0, \infty), W^{1,\infty}(\mathbb{T}^d))$ satisfies*

$$(5.2) \quad \sup_{s \geq 0} \|\mathcal{S}_{s,s+t}\|_{L^1_0 \rightarrow L^1_0} \leq \varepsilon$$

for every $\varepsilon > 0$, provided

$$(5.3) \quad t \geq (C_d + d \ln^- \tau_*(u) + 2 \ln^- \varepsilon) \tau_*(u).$$

Proof. There is a constant $c_d \geq 1$ such that if φ is the solution to (1.2) with mean-zero initial data $\varphi_0 \in L^1(\mathbb{T}^d)$, then

$$(5.4) \quad \|\varphi_{s+t}\|_{L^\infty} \leq c_d t^{-d/4} \|\varphi_s\|_{L^2}$$

and

$$(5.5) \quad \|\varphi_{s+t}\|_{L^2} \leq c_d t^{-d/4} \|\varphi_s\|_{L^1}$$

for any $s \geq 0$ and $t \in [0, 1]$. The first claim is contained in Lemma 5.4 in [Zla10] (see also Lemma 5.6 in [CKRZ08] and Lemmas 3.1, 3.3 in [FKR06]), while the second follows from it and a simple duality argument (it is also contained in Lemma 5.4 in [Zla10] but with $\frac{d}{2}$ in place of $\frac{d}{4}$, which would also suffice here). Notice also that while [Zla10] only considers time-independent drifts, the proof of its Lemma 5.4 equally applies to the time-dependent case.

Write $\tau_* = \tau_*(u)$ and let $\sigma = \min\{\tau_*, 1\}$. Then for any $s \geq 0$, $n \in \mathbb{N}$, and $t \geq n\tau_* + 2\sigma$ we have

$$\begin{aligned} \|\varphi_{s+t}\|_{L^\infty} &\leq \|\varphi_{s+n\tau_*+2\sigma}\|_{L^\infty} \leq c_d \sigma^{-d/4} \|\varphi_{s+n\tau_*+\sigma}\|_{L^2} \\ &\leq c_d \sigma^{-d/4} 2^{-n} \|s + \varphi_\sigma\|_{L^2} \leq c_d^2 \sigma^{-d/2} 2^{-n} \|\varphi_s\|_{L^1}. \end{aligned}$$

The result now follows from this estimate with

$$n = \left\lceil \log_2(c_d^2 \sigma^{-d/2} \varepsilon^{-1}) \right\rceil$$

and from $\frac{1}{\ln 2} \leq 2$, with $C_d \stackrel{\text{def}}{=} 2 + 4 \ln c_d$. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Notice that

$$\lambda \stackrel{\text{def}}{=} \sup_{y \in (0,1)} \frac{f(y)}{y}$$

is finite, and if φ solves (1.2) on \mathbb{T}^d with initial data $\phi_0 \stackrel{\text{def}}{=} \theta_0$, then the comparison principle shows that $\theta_t \leq e^{\lambda t} \varphi_t$ for all $t \geq 0$. Let $t_0 \stackrel{\text{def}}{=} -\frac{1}{\lambda} \ln \frac{\alpha_0 + \bar{\theta}_0}{2\alpha_0} > 0$ and

$$\varepsilon \stackrel{\text{def}}{=} \alpha_0 e^{-\lambda t_0} - \bar{\theta}_0 = \frac{\alpha_0 - \bar{\theta}_0}{2} > 0.$$

If now $\tau_0 > 0$ is such that for any $\tau_*(u) \leq \tau_0$ we have that the right-hand side of (5.3) is at most t_0 , then Proposition 5.2 shows that

$$\|\varphi_{t_0} - \bar{\theta}_0\|_{L^\infty} \leq \varepsilon \|\theta_0 - \bar{\theta}_0\|_{L^1} \leq \varepsilon \leq \alpha_0 e^{-\lambda t_0} - \bar{\theta}_0.$$

Therefore $\|\theta_{t_0}\|_{L^\infty} \leq \|\varphi_{t_0}\|_{L^\infty} e^{\lambda t_0} \leq \alpha_0$, completing the proof of the first claim. The last claim follows from Theorem 1.3. □

6. Dissipation times of periodic and cellular flows

In this section we will prove Theorem 1.3 and also provide examples of cellular flows in 2D and 3D satisfying its hypotheses.

Consider now the advection-diffusion equation (1.2) on \mathbb{R}^d , with initial data φ_0 and a time-independent mean-zero divergence-free Lipschitz vector field u . Consider also the stochastic process $X_t^x = X_t^x(\omega)$ satisfying the SDE

$$(6.1) \quad dX_t^x = \sqrt{2} dB_t - u(X_t^x) dt, \quad X_0^x = x.$$

Here $B_t = B_t(\omega)$ is a normalized Brownian motion on \mathbb{R}^d with $B_0 = 0$, defined on some probability space $(\Omega, \mathcal{B}_\infty, \mathbb{P}_\Omega)$. Lemma 7.8 in [Øks03] shows that if $k_t(x, y)$ is the fundamental solution for (1.2) (i.e., $k_t(x, \cdot)$

is the density for X_t^x) and \mathbb{E}_Ω the expectation with respect to $\omega \in \Omega$, then solutions to (1.2) satisfy

$$(6.2) \quad \varphi(t, x) = \int_{\mathbb{R}^d} k_t(x, y) \varphi_0(y) dy = \mathbb{E}_\Omega(\varphi_0(X_t^x)).$$

Finally, for each vector $e \in \mathbb{R}^d$, the *effective diffusivity* of u in direction e is the number

$$(6.3) \quad D_e(u) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \mathbb{E}_\Omega \left(\frac{|(X_t^x - x) \cdot e|^2}{2t} \right) \quad (\geq 1),$$

with the limit being independent of $x \in \mathbb{R}^d$. It will be also convenient to denote

$$D(u) \stackrel{\text{def}}{=} \min\{D_{e_1}(u), \dots, D_{e_d}(u)\} \geq 1$$

the minimum of effective diffusivities in all the coordinate directions. We refer the reader to the discussion in Sections 1 and 2 of [Zla11] for more details.

It follows from (6.3) that the stochastic process travels far (relative to \sqrt{t} , for large t) with large probability when the effective diffusivity is large, which may of course aid mixing when such flows are scaled down and acting on a torus. In fact, cellular flows in two dimensions have their effective diffusivities growing proportionally to the square root of their amplitudes [FP94, Kor04], so this suggests that large amplitude cellular flows (with small cells) should be good short time mixers on \mathbb{T}^2 . To show that, we need to use the following result from [Zla11], which is a short time large probability one-sided analogue of (6.3) for 1-periodic flows, with u -independent constants.

Lemma 6.1 (Theorem 2.1 in [Zla11]). *There is $C \geq 1$ such that for any 1-periodic incompressible mean-zero Lipschitz flow u on \mathbb{R}^d , any $e \in \mathbb{R}^d$, any $\alpha > 0$, and any $\tau \geq 1$, there are $t \in [1, \tau + 1]$ and $x \in \mathbb{R}^d$ such that*

$$(6.4) \quad \mathbb{P}_\Omega \left(|(X_t^x - x) \cdot e| \geq \alpha \sqrt{\tau D_e(u)} \right) \geq 1 - C\alpha.$$

Remark. The result in [Zla11] has $t \in [0, \tau]$, but the proof can be easily modified to obtain this version.

Because we want to consider general periods, we now extend this result to that case.

Lemma 6.2. *There is $C \geq 1$ such that for any l -periodic incompressible mean-zero Lipschitz flow u , any $e \in \mathbb{R}^d$, any $\alpha > 0$, and any $\tau \geq l^2$, there are $t \in [l^2, \tau + l^2]$ and $x \in \mathbb{R}^d$ such that (6.4) holds.*

Proof. Let $v(x) \stackrel{\text{def}}{=} l u(lx)$, let X_t^x be from (6.1), and let $Y_t^x \stackrel{\text{def}}{=} \frac{1}{l} X_{l^2 t}^{lx}$. Then Y_t^x satisfies

$$dY_t^x = \sqrt{2} \frac{1}{l} dB_{l^2 t} - v(Y_t^x) dt, \quad Y_0^x = x.$$

Since $\frac{1}{l} B_{l^2 t}$ equals B_t in law, it follows that Y_t^x is a stochastic process corresponding to the 1-periodic flow v via (6.1). From (6.3) we immediately see that $D_e(v) = D_e(u)$ for all $e \in \mathbb{R}^d$, and Lemma 6.1 applied to v then finishes the proof. \square

Next we extend the claim to all x , in an appropriate sense. Let $\Psi(x) \stackrel{\text{def}}{=} \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$.

Theorem 6.3. *There is $C \geq 1$ such that for any l -periodic incompressible mean-zero Lipschitz flow u on \mathbb{R}^d , any $e \in \mathbb{S}^d$, any $\alpha \in (0, 1)$, and any $\tau \geq l^2$, there is $t \in [l^2, \tau + l^2]$ such that for any $x \in \mathbb{R}^d$ we have*

$$\mathbb{P}_\Omega \left(|(X_t^x - x) \cdot e| \geq \alpha \sqrt{\tau D_e(u)} - 3l\Psi^{-1}(\alpha) - 2l^2\|u\|_{L^\infty} \right) \geq 1 - C\sqrt{\alpha}.$$

Also, $D_e(u) = D_e(u^L)$ for any $L > 0$, where $u^L(x) \stackrel{\text{def}}{=} \frac{1}{L} u(\frac{x}{L})$.

Remark. Note that for any 1-periodic flow u we now have

$$\lim_{L \rightarrow 0} \left(3L\Psi^{-1}(C\alpha) - 2L^2\|u^L\|_{L^\infty} \right) = 0,$$

with u^L being L -periodic. So if u has a large effective diffusivity, X_t^x will travel far for all t as small as one needs and all $x \in \mathbb{R}^d$, provided we scale the flow down sufficiently (and multiply by the same scaling factor). This will result in good mixing properties of such flows on \mathbb{T}^d .

Proof. The last claim was established in the previous proof.

It is well known that there is $c > 0$ such that for any l -periodic flow u , the probability density function $h_t(x, y) \stackrel{\text{def}}{=} \sum_{k \in (l\mathbb{Z})^d} k_t(x, y + k)$ of the process $X_t^x \bmod l \in (l\mathbb{T})^d$ takes values in $[cl^{-d}, c^{-1}l^{-d}]$ when $t = l^2$ (for each $x \in \mathbb{T}^d$). Given $\tau \geq l^2$, take (t, x) from Lemma 6.2. Then for any $\alpha > 0$ and C from Lemma 6.2 we have

$$\begin{aligned} & 1 - C\alpha - \mathbb{P}_\Omega \left(|\sqrt{2}B_1 \cdot e| > a \right) \\ & \leq \int_{(l\mathbb{T})^d} h_{l^2}(x, y) \mathbb{P}_\Omega \left(|(X_{t-l^2}^y - y) \cdot e| \geq \alpha \sqrt{\tau D_e(u)} - la - l^2\|u\|_{L^\infty} \right) dy. \end{aligned}$$

This is due to the Markov property of X_t^x as well as the fact that if $|\sqrt{2}B_{l^2} \cdot e| \leq la$ (which has the same probability as $|\sqrt{2}B_1 \cdot e| \leq a$), then $|X_{l^2}^x - x| \leq la + l^2\|u\|_{L^\infty}$. Let us now pick $a \stackrel{\text{def}}{=} \sqrt{2}\Psi^{-1}(\alpha)$, so we have

$\mathbb{P}_\Omega(|\sqrt{2}B_1 \cdot e| > a) \leq 2C\alpha$. Using $h_{l^2} \geq cl^{-d}$ and $\int_{(l\mathbb{T})^d} h_{l^2}(x, y) dy = 1$, we find that the measure of the set of all $y \in (l\mathbb{T})^d$ such that

$$\mathbb{P}_\Omega\left(\left|(X_{t-l^2}^y - y) \cdot e\right| \geq \alpha\sqrt{\tau D_e(u)} - la - l^2\|u\|_{L^\infty}\right) < 1 - 3c^{-1}\sqrt{C\alpha}$$

is at most $\sqrt{C\alpha}l^d$.

Let now $z \in \mathbb{R}^d$ be arbitrary. Then Markov property again yields

$$\begin{aligned} & \mathbb{P}_\Omega\left(\left|(X_t^z - z) \cdot e\right| \geq \alpha\sqrt{\tau D_e(u)} - 2la - 2l^2\|u\|_{L^\infty}\right) \\ & \quad + \mathbb{P}_\Omega\left(|\sqrt{2}B_1 \cdot e| > a\right) \\ & \geq \int_{(l\mathbb{T})^d} h_{l^2}(z, y) \cdot \\ & \quad \cdot \mathbb{P}_\Omega\left(\left|(X_{t-l^2}^y - y) \cdot e\right| \geq \alpha\sqrt{\tau D_e(u)} - la - l^2\|u\|_{L^\infty}\right) dy \\ & \geq 1 - 3c^{-1}\sqrt{C\alpha} - c^{-1}l^{-d}\sqrt{C\alpha}l^d. \end{aligned}$$

So $C\alpha \leq 1 \leq c^{-1}$ shows for the above t and all $z \in \mathbb{R}^d$,

$$\mathbb{P}_\Omega\left(\left|(X_t^z - z) \cdot e\right| \geq \alpha\sqrt{\tau D_e(u)} - 2la - 2l^2\|u\|_{L^\infty}\right) \geq 1 - 5c^{-1}\sqrt{C\alpha}.$$

If we now change C to be $5 \max\{C, c^{-2}\}$, the result follows. \square

From now on we will consider flows on \mathbb{T}^d with period $\frac{1}{\nu}$ ($\nu \in \mathbb{N}$) that are also symmetric. We say that a flow u is symmetric in x_n if we have

$$(6.5) \quad u(R_n x) = R_n u(x) \quad \text{for all } x \in \mathbb{T}^d,$$

where $R_n v \stackrel{\text{def}}{=} v - 2v_n e_n$ for $v = \sum_{n=1}^d v_n e_n \in \mathbb{R}^d$ is the reflection in the n^{th} coordinate. Note that a periodic flow that is symmetric in all d coordinates has a cellular structure.

We let X_t^x be the process above, when u is considered on \mathbb{R}^d (extended periodically), and note that $X_t^x \bmod 1$ is the corresponding process on \mathbb{T}^d . Finally, for any divisor μ of ν , we denote by $\mathcal{C}_k^\mu \stackrel{\text{def}}{=} \prod_{n=1}^d \left[\frac{k_n}{\mu}, \frac{k_n+1}{\mu}\right)$ ($k = (k_1, \dots, k_d) \in \{0, \dots, \mu-1\}^d$) the ‘‘cells’’ of u on \mathbb{T}^d of size $\frac{1}{\mu}$ (each of which is an invariant set for the flow when u is symmetric).

Lemma 6.4. *There is $C \geq 1$ such that for any $\frac{1}{\nu}$ -periodic incompressible mean-zero Lipschitz flow u that is symmetric in x_n we have*

$$(6.6) \quad |\mathbb{P}_\Omega(X_\tau^x \in \mathcal{C}_k^\mu) - \mathbb{P}_\Omega(X_\tau^x \in \mathcal{C}_m^\mu)| \leq C\sqrt{\alpha}$$

for any $x \in \mathbb{R}^d$, any $\alpha \in (0, 1)$, any divisor μ of ν , and any k, m as above such that $k - m$ is a multiple of e_n , provided

$$(6.7) \quad \tau \geq \frac{(6\nu\Psi^{-1}(\alpha) + 4\|u\|_{L^\infty} + \nu^2)^2}{4\nu^4\alpha^2 D_{e_n}(u)} + \frac{2}{\nu^2}.$$

In particular, if u is symmetric in all coordinates, then we have

$$(6.8) \quad \left| \mathbb{P}_\Omega (X_\tau^x \in \mathcal{C}_k^\mu) - \frac{1}{\mu^d} \right| \leq C d \sqrt{\alpha}$$

for any $x \in \mathbb{R}^d$, any α, μ as above, and any $k, m \in \{0, \dots, \mu - 1\}^d$, provided

$$(6.9) \quad \tau \geq \frac{(6\nu\Psi^{-1}(\alpha) + 4\|u\|_{L^\infty} + \nu^2)^2}{4\nu^4\alpha^2 D(u)} + \frac{2}{\nu^2}.$$

Proof. The second claim follows from the first applied successively in all d coordinates, and from $\sum_m \mathbb{P}_\Omega (X_\tau^x \in \mathcal{C}_m^\mu) = 1$.

As for the first claim, notice that (6.6) implies $\tau \geq \frac{2}{\nu^2}$ and

$$\alpha \sqrt{(\tau - \nu^{-2}) D_{e_j}(u)} - 3\nu^{-1} \Psi^{-1}(\alpha) - 2\nu^{-2} \|u\|_{L^\infty} \geq \frac{1}{2}.$$

Theorem 6.3 shows that there is $s \in [\frac{1}{\nu^2}, \tau]$ such that

$$\mathbb{P}_\Omega \left(|(X_s^x - x)_n| \geq \frac{1}{2} \right) \geq 1 - C\sqrt{\alpha},$$

when X_t^x is considered on \mathbb{R}^d . This means that there is $\Omega' \subseteq \Omega$ with $\mathbb{P}_\Omega(\Omega') \geq 1 - C\sqrt{\alpha}$ such that for each $\omega \in \Omega'$, the process $X_t^x(\omega) \bmod 1$ hits at least one of the two hyperplanes on \mathbb{T}^d with $x_n \in \{\frac{k_n+m_n+1}{2\mu}, \frac{k_n+m_n+1}{2\mu} + \frac{1}{2}\}$ by time s .

But symmetry in x_n and $\frac{1}{\nu}$ -periodicity show that u is symmetric across any hyperplane with $x_n = \frac{j}{2\nu}$ ($j = 0, \dots, 2\nu - 1$), that is,

$$u \left(R_n \left(x - \frac{j}{2\nu} e_n \right) + \frac{j}{2\nu} e_n \right) = u \left(R_n x + \frac{j}{\nu} e_n \right) = u(R_n x) = R_n u(x)$$

for all $x \in \mathbb{T}^d$. Since also \mathcal{C}_k^μ and \mathcal{C}_m^μ are mapped into each other by reflections across the hyperplanes with $x_n \in \{\frac{k_n+m_n+1}{2\mu}, \frac{k_n+m_n+1}{2\mu} + \frac{1}{2}\}$, the law of B_t is invariant under the reflection R_n , and X_t^x satisfies the strong Markov property, it follows that

$$\mathbb{P}_\Omega(\omega \in \Omega' \text{ and } X_\tau^x(\omega) \in \mathcal{C}_k^\mu) = \mathbb{P}_\Omega(\omega \in \Omega' \text{ and } X_\tau^x(\omega) \in \mathcal{C}_m^\mu)$$

because $\tau \geq s$. This finishes the proof. \square

If h_t is as above and h'_t is the corresponding probability density kernel for the flow $-u$ on \mathbb{T}^d , then we have $h_t(x, y) = h'_t(y, x)$ for all $x, y \in \mathbb{T}^d$ because $\nabla \cdot u \equiv 0$ means that the adjoint of the operator $e^{t(\Delta - u \cdot \nabla)}$ (whose kernel is h_t) is $e^{t(\Delta + u \cdot \nabla)}$ (whose kernel is h'_t). Therefore, the solution to (1.2) on \mathbb{T}^d satisfies

$$\int_{\mathcal{C}_k^\mu} \varphi(t, x) dx = \int_{\mathcal{C}_k^\mu} \int_{\mathbb{T}^d} h_t(x, y) \varphi_0(y) dy dx$$

$$\begin{aligned}
 &= \int_{\mathbb{T}^d} \int_{\mathcal{C}_k^\mu} h'_t(y, x) dx \varphi_0(y) dy \\
 &= \int_{\mathbb{T}^d} \mathbb{P}_\Omega(X_\tau^y \in \mathcal{C}_k^\mu) \varphi_0(y) dy,
 \end{aligned}$$

with X_t^y the process corresponding to $-u$. If (6.8) holds and φ_0 is mean-zero, then we obtain

$$\left| \int_{\mathcal{C}_k^\mu} \varphi(\tau, x) dx \right| \leq C d \sqrt{\alpha} \|\varphi_0\|_{L^1}$$

for each k . Poincaré inequality now shows that

$$\int_{\mathcal{C}_k^\mu} |\nabla \varphi(\tau, x)|^2 dx \geq C_d^{-2} \mu^2 \inf_{|a| \leq C_d \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^1}} \int_{\mathcal{C}_k^\mu} |\varphi(\tau, x) - a|^2 dx$$

for each k and some $C_d \geq 1$ (only dependent on d). But then

$$\begin{aligned}
 \mu^{-1} \|\nabla \varphi_\tau\|_{L^2} &\geq C_d^{-1} \left\| \left(|\varphi_\tau| - C_d \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^1} \right)^+ \right\|_{L^2} \\
 &\geq C_d^{-1} \|\varphi_\tau\|_{L^2} - \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^1},
 \end{aligned}$$

which now implies that

$$(6.10) \quad \frac{d}{d\tau} \|\varphi_\tau\|_{L^2}^2 = -2 \|\nabla \varphi_\tau\|_{L^2}^2 \leq -C_d^{-1} \mu \|\varphi_\tau\|_{L^2}$$

as long as $\|\varphi_\tau\|_{L^2} \geq 2C_d \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^2}$ (because $\|\varphi_0\|_{L^2} \geq \|\varphi_0\|_{L^1}$). From this and Lemma 6.4 we obtain the following result.

Theorem 6.5. *For any $d \in \mathbb{N}$, there is $C_d \geq 1$ such that for any $\frac{1}{\nu}$ -periodic incompressible mean-zero Lipschitz flow u on \mathbb{T}^d that is symmetric in all coordinates and any mean-zero $\varphi_0 \in L^2(\mathbb{T}^d)$, the solution to (1.2) on \mathbb{T}^d satisfies*

$$(6.11) \quad \|\varphi_t\|_{L^2(\mathbb{T}^d)} \leq 2C_d \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^2(\mathbb{T}^d)}$$

for each $\alpha \in (0, 1)$ and each divisor μ of ν , provided

$$(6.12) \quad t \geq \frac{(6\nu\Psi^{-1}(\alpha) + 4\|u\|_{L^\infty} + \nu^2)^2}{4\nu^4\alpha^2 D(-u)} + \frac{2}{\nu^2} + \frac{2C_d}{\mu} \ln^+ \frac{1}{2C_d \sqrt{\alpha} \mu^d}$$

Proof. This follows from (6.10) for all times τ satisfying (6.9) such that $\|\varphi_\tau\|_{L^2} \geq 2C_d \sqrt{\alpha} \mu^d \|\varphi_0\|_{L^1}$, noting also that $\|\varphi_\tau\|_{L^2} \leq \|\varphi_0\|_{L^2}$. \square

We can now prove Theorem 1.3.

Proof. Let $\alpha_n \stackrel{\text{def}}{=} D(u_n)^{-1/4}$ and $\mu_n \stackrel{\text{def}}{=} \lfloor \alpha_n^{-1/4d} \rfloor$. Using these values for α, μ in Theorem 6.5, as well as $u(x) = -\nu_n u_n(\nu_n x)$ with a sufficiently large $\nu = \nu_n$ (a multiple of μ_n), together with the last claim in Theorem 6.3, yields that the sum of the first two fractions in (6.12) is less than $D(u_n)^{-1/2}$, while $2C_d \sqrt{\alpha} \mu^d$ in (6.11) is bounded above by $2C_d D(u_n)^{-1/16}$

and the last term in (6.12) is bounded above by $\frac{C}{\mu_n} \ln 2\mu_n$ for some $C \geq 1$ and all n . Since all three bounds converge to 0 as $n \rightarrow \infty$, the proof of the first claim is finished.

The last claim follows from this and Examples 6.7 and 6.8 below. \square

Remark 6.6. We can in fact pick ν_n to be the smallest multiple of $\mu_n = \lfloor D(u_n)^{1/16d} \rfloor$ greater than $\|u_n\|_{L^\infty} D(u_n)^{(1-8d)/32d}$ and still have the right-hand side of (6.12) bounded above by $\frac{C}{\mu_n} \ln 2\mu_n$ for some $C \geq 1$. That is, we have $\tau_*(v_n) \leq \frac{C}{\mu_n} \ln 2\mu_n$. In particular, if u_n are the 2D flows from Example 6.7 below, we obtain $\nu_n \sim A_n^{113/128}$ and $\tau_*(v_n) \leq CA_n^{-1/64} \ln A_n$.

Example 6.7. When $d = 2$, hypotheses of Theorem 1.3 are satisfied by $u_n(x, y) \stackrel{\text{def}}{=} A_n \nabla^\perp \psi(x, y)$ with $\lim_{n \rightarrow \infty} A_n = \infty$ and any stream function $\psi \in C^{2,\delta}(\mathbb{T}^d)$ that has only non-degenerate critical points and vanishes when $x = 0$ or $y = 0$, for instance $\psi(x, y) = \sin(2\pi x) \sin(2\pi y)$. (When periodically extended onto \mathbb{R}^2 , these flows are called cellular, with their cells being $(k, k + 1) \times (m, m + 1)$ for any $k, m \in \mathbb{Z}$.) This is because then $D(u_n) \sim A_n^{1/2}$ by [FP94, Kor04].

Example 6.8. When $d = 3$, hypotheses of Theorem 1.3 are satisfied by

$$u_n(x, y, z) \stackrel{\text{def}}{=} A_n \left(\Phi_x(x, y) W'(z), \Phi_y(x, y) W'(z), 8\pi^2 \Phi(x, y) W(z) \right)$$

with $\lim_{n \rightarrow \infty} A_n = \infty$, and the functions $\Phi(x, y) \stackrel{\text{def}}{=} \cos(2\pi x) \cos(2\pi y)$ and $W(z) \stackrel{\text{def}}{=} \sin(2\pi z)$ (this again extends periodically to a cellular flow on \mathbb{R}^3). We now have $\lim_{n \rightarrow \infty} D(u_n) = \infty$ by [RZ07].

References

- [ACM19] G. Alberti, G. Crippa, and A. L. Mazzucato. Exponential self-similar mixing by incompressible flows. *J. Amer. Math. Soc.*, 32(2):445–490, 2019. doi:[10.1090/jams/913](https://doi.org/10.1090/jams/913).
- [Ber02] H. Berestycki. *The Influence of Advection on the Propagation of Fronts in Reaction-Diffusion Equations*, pages 11–48. Springer Netherlands, Dordrecht, 2002. doi:[10.1007/978-94-010-0307-0_2](https://doi.org/10.1007/978-94-010-0307-0_2).
- [BH17] J. Bedrossian and S. He. Suppression of blow-up in Patlak-Keller-Segel via shear flows. *SIAM J. Math. Anal.*, 49(6):4722–4766, 2017. doi:[10.1137/16M1093380](https://doi.org/10.1137/16M1093380).
- [BHN05] H. Berestycki, F. Hamel, and N. Nadirashvili. Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Comm. Math. Phys.*, 253(2):451–480, 2005. doi:[10.1007/s00220-004-1201-9](https://doi.org/10.1007/s00220-004-1201-9).
- [BKNR10] H. Berestycki, A. Kiselev, A. Novikov, and L. Ryzhik. The explosion problem in a flow. *J. Anal. Math.*, 110:31–65, 2010. doi:[10.1007/s11854-010-0002-7](https://doi.org/10.1007/s11854-010-0002-7).

- [CF88] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
- [CF93] P. Constantin and C. Fefferman. Direction of vorticity and the problem of global regularity for the Navier-Stokes equations. *Indiana Univ. Math. J.*, 42(3):775–789, 1993. doi:[10.1512/iumj.1993.42.42034](https://doi.org/10.1512/iumj.1993.42.42034).
- [CKRZ08] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš. Diffusion and mixing in fluid flow. *Ann. of Math. (2)*, 168(2):643–674, 2008. doi:[10.4007/annals.2008.168.643](https://doi.org/10.4007/annals.2008.168.643).
- [Con01] P. Constantin. Some open problems and research directions in the mathematical study of fluid dynamics. In *Mathematics unlimited—2001 and beyond*, pages 353–360. Springer, Berlin, 2001.
- [CS89] S. Childress and A. M. Soward. Scalar transport and alpha-effect for a family of cat’s-eye flows. *J. Fluid Mech.*, 205:99–133, 1989. doi:[10.1017/S0022112089001965](https://doi.org/10.1017/S0022112089001965).
- [CZDE18] M. Coti Zelati, M. G. Delgadino, and T. M. Elgindi. On the relation between enhanced dissipation time-scales and mixing rates. *ArXiv e-prints*, June 2018, [1806.03258](https://arxiv.org/abs/1806.03258).
- [Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [EZ18] T. M. Elgindi and A. Zlatoš. Universal mixers in all dimensions. *arXiv e-prints*, page arXiv:1809.09614, Sep 2018, [1809.09614](https://arxiv.org/abs/1809.09614).
- [Fef06] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.
- [FI18] Y. Feng and G. Iyer. Dissipation enhancement by mixing. 2018.
- [FKR06] A. Fannjiang, A. Kiselev, and L. Ryzhik. Quenching of reaction by cellular flows. *Geom. Funct. Anal.*, 16(1):40–69, 2006. doi:[10.1007/s00039-006-0554-y](https://doi.org/10.1007/s00039-006-0554-y).
- [FP94] A. Fannjiang and G. Papanicolaou. Convection enhanced diffusion for periodic flows. *SIAM J. Appl. Math.*, 54(2):333–408, 1994. doi:[10.1137/S0036139992236785](https://doi.org/10.1137/S0036139992236785).
- [FW03] A. Fannjiang and L. Wołowski. Noise induced dissipation in Lebesgue-measure preserving maps on d -dimensional torus. *J. Statist. Phys.*, 113(1-2):335–378, 2003. doi:[10.1023/A:1025787124437](https://doi.org/10.1023/A:1025787124437).
- [He18] S. He. Suppression of blow-up in parabolic-parabolic Patlak-Keller-Segel via strictly monotone shear flows. *Nonlinearity*, 31(8):3651–3688, 2018. doi:[10.1088/1361-6544/aac1ce](https://doi.org/10.1088/1361-6544/aac1ce).
- [Her00] M. A. Herrero. Asymptotic properties of reaction-diffusion systems modeling chemotaxis. In *Applied and industrial mathematics, Venice–2, 1998*, pages 89–108. Kluwer Acad. Publ., Dordrecht, 2000.
- [HMV97] M. A. Herrero, E. Medina, and J. J. L. Velázquez. Finite-time aggregation into a single point in a reaction-diffusion system. *Nonlinearity*, 10(6):1739–1754, 1997. doi:[10.1088/0951-7715/10/6/016](https://doi.org/10.1088/0951-7715/10/6/016).
- [HMV98] M. A. Herrero, E. Medina, and J. J. L. Velázquez. Self-similar blow-up for a reaction-diffusion system. *J. Comput. Appl. Math.*, 97(1-2):99–119, 1998. doi:[10.1016/S0377-0427\(98\)00104-6](https://doi.org/10.1016/S0377-0427(98)00104-6).
- [Hor03] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*,

- 105(3):103–165, 2003. URL http://www.mis.mpg.de/preprints/2003/preprint2003_3.pdf.
- [Hor04] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein.*, 106(2):51–69, 2004.
- [HT19] S. He and E. Tadmor. Suppressing chemotactic blow-up through a fast splitting scenario on the plane. *Arch. Ration. Mech. Anal.*, 232(2):951–986, 2019. doi:[10.1007/s00205-018-01336-7](https://doi.org/10.1007/s00205-018-01336-7).
- [ISS03] L. Iskauriaza, G. A. Serëgin, and V. Shverak. $L_{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2(350)):3–44, 2003. doi:[10.1070/RM2003v058n02ABEH000609](https://doi.org/10.1070/RM2003v058n02ABEH000609).
- [JL92] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.*, 329(2):819–824, 1992. doi:[10.2307/2153966](https://doi.org/10.2307/2153966).
- [Kor04] L. Korálov. Random perturbations of 2-dimensional Hamiltonian flows. *Probab. Theory Related Fields*, 129(1):37–62, 2004. doi:[10.1007/s00440-003-0320-0](https://doi.org/10.1007/s00440-003-0320-0).
- [KS70] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26(3):399 – 415, 1970. doi:[10.1016/0022-5193\(70\)90092-5](https://doi.org/10.1016/0022-5193(70)90092-5).
- [KS71] E. F. Keller and L. A. Segel. Model for chemotaxis. *J. Theor. Biol.*, 30(2):225 – 234, 1971. doi:[10.1016/0022-5193\(71\)90050-6](https://doi.org/10.1016/0022-5193(71)90050-6).
- [KSZ08] A. Kiselev, R. Shterenberg, and A. Zlatoš. Relaxation enhancement by time-periodic flows. *Indiana Univ. Math. J.*, 57(5):2137–2152, 2008. doi:[10.1512/iumj.2008.57.3349](https://doi.org/10.1512/iumj.2008.57.3349).
- [KX16] A. Kiselev and X. Xu. Suppression of chemotactic explosion by mixing. *Arch. Ration. Mech. Anal.*, 222(2):1077–1112, 2016. doi:[10.1007/s00205-016-1017-8](https://doi.org/10.1007/s00205-016-1017-8).
- [KZ06] A. Kiselev and A. Zlatoš. Quenching of combustion by shear flows. *Duke Math. J.*, 132(1):49–72, 2006. doi:[10.1215/S0012-7094-06-13212-X](https://doi.org/10.1215/S0012-7094-06-13212-X).
- [Ler34] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934. doi:[10.1007/BF02547354](https://doi.org/10.1007/BF02547354).
- [Maz85] V. G. Maz’ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. doi:[10.1007/978-3-662-09922-3](https://doi.org/10.1007/978-3-662-09922-3). Translated from the Russian by T. O. Shaposhnikova.
- [Øks03] B. Øksendal. *Stochastic differential equations*. Universitext. Springer-Verlag, Berlin, sixth edition, 2003. doi:[10.1007/978-3-642-14394-6](https://doi.org/10.1007/978-3-642-14394-6). An introduction with applications.
- [Pat53] C. S. Patlak. Random walk with persistence and external bias. *Bull. Math. Biophys.*, 15(3):311–338, Sep 1953. doi:[10.1007/BF02476407](https://doi.org/10.1007/BF02476407).
- [Per07] B. Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007. doi:[10.1007/978-3-7643-7842-4](https://doi.org/10.1007/978-3-7643-7842-4).
- [Pro59] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.
- [RZ07] L. Ryzhik and A. Zlatoš. KPP pulsating front speed-up by flows. *Commun. Math. Sci.*, 5(3):575–593, 2007. URL <http://projecteuclid.org/euclid.cms/1188405669>.

- [Ser63] J. Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 69–98. Univ. of Wisconsin Press, Madison, Wis., 1963.
- [SŠ02] G. Seregin and V. Šverák. Navier-Stokes equations with lower bounds on the pressure. *Arch. Ration. Mech. Anal.*, 163(1):65–86, 2002. doi:[10.1007/s002050200199](https://doi.org/10.1007/s002050200199).
- [Tem01] R. Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.
- [Xin00] J. Xin. Front propagation in heterogeneous media. *SIAM Rev.*, 42(2):161–230, 2000. doi:[10.1137/S0036144599364296](https://doi.org/10.1137/S0036144599364296).
- [Xin09] J. Xin. *An introduction to fronts in random media*, volume 5 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, New York, 2009. doi:[10.1007/978-0-387-87683-2](https://doi.org/10.1007/978-0-387-87683-2).
- [YZ17] Y. Yao and A. Zlatoš. Mixing and un-mixing by incompressible flows. *J. Eur. Math. Soc. (JEMS)*, 19(7):1911–1948, 2017. doi:[10.4171/JEMS/709](https://doi.org/10.4171/JEMS/709).
- [Zla10] A. Zlatoš. Diffusion in fluid flow: dissipation enhancement by flows in 2D. *Comm. Partial Differential Equations*, 35(3):496–534, 2010. doi:[10.1080/03605300903362546](https://doi.org/10.1080/03605300903362546).
- [Zla11] A. Zlatoš. Reaction-diffusion front speed enhancement by flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(5):711–726, 2011. doi:[10.1016/j.anihpc.2011.05.004](https://doi.org/10.1016/j.anihpc.2011.05.004).

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY,
PITTSBURGH, PA 15213, USA

Email address: gautam@math.cmu.edu

DEPARTMENT OF MATHEMATICS, DUKE KUNSHAN UNIVERSITY, KUNSHAN,
CHINA

Email address: xiaoqian.xu@dukekunshan.edu.cn

DEPARTMENT OF MATHEMATICS, UC SAN DIEGO, LA JOLLA, CA 92130, USA

Email address: zlatos@ucsd.edu