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Dynamic cubic instability in a 2D Q-tensor model for liquid crystals

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We consider a four-elastic-constant Landau–de Gennes energy characterizing nematic liquid crystal configurations described using the Q-tensor formalism. The energy contains a cubic term and is unbounded from below. We study dynamical effects produced by the presence of this cubic term by considering an L^2 gradient flow generated by this energy. We work in two dimensions and concentrate on understanding the relations between the physicality of the initial data and the global well-posedness of the system.

Keywords: Gradient system; liquid crystals; Q-tensor.

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1. Introduction

This paper studies the dynamics of an important instability phenomenon that arises in the Landau–de Gennes theory of nematic liquid crystals.[2–4] Mathematically our

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results address global well-posedness of the L^2 gradient flow generated by an energy functional that is unbounded from below in its natural energy space. This turns out to be related to quantifying how the flow affects the convex hull of the initial data.

We consider a Landau–de Gennes energy functional

$$\mathcal{E}[Q] = \int_\Omega \mathcal{F}(Q(x))dx,$$

where $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and Q is a matrix-valued function defined on Ω that takes values into the space of Q-tensors, namely $\mathcal{S}^{(d)} \stackrel{\text{def}}{=} \{M \in \mathbb{R}^{d\times d}, M = M^T, \operatorname{tr}(M) = 0\}$. The matrix $Q(x)$ is a measure of the local preferred orientation of the nematic molecules at the point $x \in \Omega$, see for instance Refs. 4 and 17.

The energy density $\mathcal{F}(Q)$ can be decomposed as:

$$\mathcal{F}(Q) = \mathcal{F}_{\text{el}} + \mathcal{F}_{\text{bulk}},$$

where $\mathcal{F}_{\text{el}}$ is the "elastic part" which depends on gradients of Q, and $\mathcal{F}_{\text{bulk}}$ is the "bulk part" that contains no gradients.

Invariances under physical symmetries impose certain restrictions on the form of the elastic and bulk parts. The simplest and most common form that is invariant under physical symmetries and still captures the essential features[2,17] assumes that $\mathcal{F}_{\text{el}}$ and $\mathcal{F}_{\text{bulk}}$ are given by:

$$\mathcal{F}_{\text{el}}(Q) \stackrel{\text{def}}{=} L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik} + L_4 Q_{lk}\partial_k Q_{ij}\partial_l Q_{ij}, \tag{1.1}$$

$$\mathcal{F}_{\text{bulk}}(Q) \stackrel{\text{def}}{=} \frac{a}{2}\operatorname{tr}(Q^2) - \frac{b}{3}\operatorname{tr}(Q^3) + \frac{c}{4}\operatorname{tr}^2(Q^2). \tag{1.2}$$

Here and in the following we assume the Einstein summation convention by which repeated indices $i, j, k = 1, \ldots, d$ are implicitly summed.

The coefficients a, b, c and $L_k, k = 1, 2, 3, 4$, are assumed to be non-dimensional (see Ref. 16). For spatially homogeneous systems the term $\mathcal{F}_{\text{bulk}}$ is bounded from below only if $c > 0$ (see Ref. 18). Physical considerations impose that $b \geq 0$ (see Ref. 14) and a is a temperature-dependent parameter that can be taken to be either positive or negative. The most physically relevant case is when a is small. This corresponds to a temperature near the supercooling point, below which the isotropic phase becomes unstable. Thus we make the assumptions

$$b \geq 0 \quad \text{and} \quad c > 0. \tag{1.3}$$

In two dimensions observe that $Q \in \mathcal{S}^{(2)}$ implies $\operatorname{tr}(Q^3) = 0$. Hence we may, without loss of generality, assume $b = 0$.

For the elastic part we note that the first three terms are quadratic, while the fourth one (with coefficient L_4) is cubic. The presence of a cubic term is rather unusual in most physical systems. The retention of this term in our situation is motivated by the fact that it allows reduction of the elastic energy $\mathcal{F}[Q]$ to the classical Oseen–Frank energy of liquid crystals (with four elastic terms). This is done by formally taking

$$Q(x) = s_+ \left(n(x) \times n(x) - \frac{1}{d}\mathbb{I} \right) \quad \text{where } s_+ > 0, \quad n : \Omega \to \mathbb{S}^{d-1},$$

and substituting it in the definition of $\mathcal{E}[Q]$ (see Appendix B or Ref. 4). Here $\mathbb{I}$ denotes the identity matrix.

The cubic term, however, also comes with a price: the energy $\mathcal{E}[Q]$ now has the "unpleasant" feature of being unbounded from below.[2,3] On the other hand, if $L_4 = 0$ the elastic part of $\mathcal{E}[Q]$,

$$\mathcal{E}_{\text{el}}[Q] \stackrel{\text{def}}{=} \int_\Omega \mathcal{F}_{\text{el}}(Q(x))dx$$

is bounded from below (and coercive) if and only if L_1, L_2 and L_3 satisfy certain conditions. For $Q \in \mathcal{S}^{(3)}$ and three-dimensional domains these conditions are developed in Ref. 13 (see also Ref. 8). For $Q \in \mathcal{S}^{(2)}$ and two-dimensional domains the conditions

$$L_1 + L_2 > 0 \quad \text{and} \quad L_1 + L_3 > 0 \tag{1.4}$$

are equivalent to coercivity. (We prove this in Lemma C.1 in Appendix C.)

One way to deal with the unboundedness and lack of coercivity caused by the (necessary) presence of L_4 is to replace the bulk potential defined in (1.2) with a potential $\psi(Q)$, which is finite if and only if Q is *physical*[a] (see for instance Ref. 3 for $d = 3$). In this paper we aim to directly study the physical relevance of the energy $\mathcal{E}[Q]$ keeping the more common potential (1.2), instead of the singular potential as in Ref. 3 (see Refs. 10 and 22 for works in the dynamical context). Of course the static theory will not provide anything meaningful when the energy $\mathcal{E}[Q]$ is unbounded. Consequently, we focus our attention on the dynamical aspect.

We study a gradient flow in the "simplest setting": namely an L^2 gradient flow in $\mathbb{R}^2$ corresponding to the energy functional $\mathcal{E}[Q]$ where Q takes values in $\mathcal{S}^{(2)}$. Explicitly, this is

$$\frac{\partial Q_{ij}}{\partial t} = -\left(\frac{\delta \mathcal{E}}{\delta Q} \right)_{ij} + \lambda \delta_{ij} + \mu_{ij} - \mu_{ji}, \tag{1.5}$$

where λ is a Lagrange multiplier corresponding to the constraint $\text{tr}(Q) = 0$ and for $i, j \in \{1, 2\}$ the μ_{ij}'s are the Lagrange multipliers corresponding to the constraints $Q_{ij} = Q_{ji}$. Here $\frac{\delta \mathcal{E}}{\delta Q}$ denotes the variational derivative of $\mathcal{E}$ with respect to Q, defined by

$$\frac{\delta \mathcal{E}}{\delta Q}(\varphi) = \frac{d}{dt}\mathcal{E}(Q + t\varphi)\bigg|_{t=0},$$

for $\varphi \in C_c^\infty(\Omega, M^{d\times d}(\mathbb{R}))$. Integrating by parts as necessary we can identify the linear operator $\frac{\delta \mathcal{E}}{\delta Q}$ with a matrix-valued function.

[a]We recall[3,14] that Q is *physical* if $Q \in \mathcal{S}^{(d)}$ and after suitable non-dimensionalizations its eigenvalues are between $-\frac{1}{d}$ and $1 - \frac{1}{d}$.

After some lengthy but straightforward calculations (which we carry out in Appendix A) Eq. (1.5) reduces to

$$\begin{aligned} \frac{\partial Q_{ij}}{\partial t} &= 2L_1 \Delta Q_{ij} - aQ_{ij} - c\,\text{tr}(Q^2)Q_{ij} + (L_2 + L_3)(\partial_j \partial_k Q_{ik} + \partial_i \partial_k Q_{jk}) \\ &\quad - (L_2 + L_3)\partial_l \partial_k Q_{lk}\delta_{ij} + 2L_4 \partial_l Q_{ij} \partial_k Q_{lk} + 2L_4 \partial_l \partial_k Q_{ij} Q_{lk} \\ &\quad - L_4 \partial_i Q_{kl} \partial_j Q_{kl} + \frac{L_4}{2}|\nabla Q|^2 \delta_{ij}. \end{aligned} \tag{1.6}$$

We study this system of equations on a bounded domain $\Omega \subset \mathbb{R}^2$ with initial data and boundary conditions given by

$$Q(x, 0) = Q_0(x), \quad \text{and} \quad Q(x, t)|_{\partial\Omega} = \tilde{Q}(x), \quad Q_0|_{\partial\Omega} = \tilde{Q}. \tag{1.7}$$

The main results in this paper are to show:

(1) Global existence of weak solutions to (1.6) and (1.7) in two dimensions, for $H^1 \cap L^\infty$ initial data that is small in L^∞ (Theorem 2.1, below).
(2) Finite time blow-up (in L^2) of solutions to (1.6) and (1.7) in two dimensions, for specially constructed (large) initial data (Theorem 2.2, below).
(3) The "preservation of physicality" of the initial data in two or three dimensions and a simple version of the flow (Proposition 2.2, below).

We defer the precise statements (and proofs) of these results to subsequent sections, and momentarily pause to briefly outline the ideas involved in the proofs and the problems encountered.

The main difficulty in proving global existence stems from the fact that the energy is *a priori* unbounded from below. However, from Eq. (1.6) we see that if $\|Q\|_{L^\infty}$ is small enough, then the cubic term can be absorbed into the other terms, which are positive definite under the assumption (1.4). Here:

$$\|Q\|_{L^\infty} = \sup_{x\in\Omega} |Q(x)|, \quad \text{where} \quad |Q(x)|^2 = \text{tr}(Q(x)Q(x)^t) = \text{tr}(Q^2(x)).$$

Thus the usual H^1-level information provided by the energy in such gradient flows can be effectively utilized, provided we *a priori* guarantee a smallness condition on the L^∞-norm. Our main tool (Proposition 2.1) does precisely this: namely, Proposition 2.1 shows smallness of $\|Q\|_{L^\infty}$ globally in time, provided it is small enough initially. We use this to prove global existence of weak solutions in Theorem 2.1. Global existence of strong solutions should now follow using relative standard methods, provided the initial data is regular, small and is compatible with the boundary conditions (see for instance Ref. 9).

We complement Theorem 2.1 with Theorem 2.2 which shows the existence of a finite time blow-up using large, specially constructed initial data. The proof amounts to finding a nonlinear differential inequality for a quantity that blows up in finite time. The main difficulty in this context is again the high-order nonlinearity. We use the energy inequality for control of this, even though the sign of the energy is not *a priori* controlled.

Theorems 2.1 and 2.2 give a dichotomy common to many nonlinear PDE's: long time existence if the initial data is small enough, and examples of finite time blow-up for large data. This leads to an interesting question about the maximal size of initial data for which solutions exist globally in time. This is a very subtle one and we only provide a modest contribution in this direction. We think that an important factor affecting global existence is the *physicality* of the initial data — namely the requirement that after a particular normalization the eigenvalues of the initial data are within the interval $(-\frac{1}{d}, 1-\frac{1}{d})$ (see more about physicality in Refs. 2 and 3).

There exists a direct and delicate relation between the smallness of $\|Q\|_{L^\infty}$ and the aforementioned notion of "physicality". Specifically, the physicality of a Q-tensor imposes an upper bound on the size of $\|Q\|_{L^\infty}$ but in general the contrary is false. Namely having an upper bound for $\|Q\|_{L^\infty}$ implies physicality in 2D, but not necessarily in higher dimensions.

More precisely, if $Q \in \mathcal{S}^{(d)}$ is physical, i.e. its eigenvalues $\lambda_i, i = 1, \ldots, d$ are in the interval $(-\frac{1}{d}, 1-\frac{1}{d})$, hence $\mathrm{tr}(Q^2(x)) = \sum_{i=1}^d \lambda_i^2 \le d(1-\frac{1}{d})^2$. On the other hand, the condition $\mathrm{tr}(Q^2(x)) = \sum_{i=1}^d \lambda_i^2 \le d(1-\frac{1}{d})^2$ for $Q \in \mathcal{S}^{(d)}$ implies that the eigenvalues of Q are between $(-\frac{1}{d}, 1-\frac{1}{d})$ only for $d = 2$, but not for $d = 3$! For $d = 3$, the notion of physicality is related to Q belonging to a convex set (not just a ball as for $d = 2$). Proposition 2.2 explores how the gradient flow preserves the convex hull of the initial data in a simple setting, for both $d = 2$ and $d = 3$.

***Plan of this paper*.** This paper is organized as follows. In Sec. 2 we precisely state the main results of this paper and state our notational conventions. In Sec. 3 we prove the small data global existence result (Theorem 2.1). In Sec. 4 we exhibit an example of a finite time blow-up with large initial data. In Sec. 5 we prove the preservation of physicality (Proposition 2.2).

There are numerous technical calculations involved in this paper, which for clarity of presentation have been relegated to Appendices. Appendix A shows that the gradient flow defined by (1.5) satisfies (1.6). Appendix B shows how the Landau–de Gennes energy functional can be reduced to the Oseen–Frank energy functional in two dimensions, and the necessity of the cubic term for this purpose. Appendix C shows that the coercivity assumption (1.4) is equivalent to coercivity in two dimensions. Finally Appendix D reduces the evolution for Q into a one-dimensional problem when the initial data is of the type used to prove the blow-up in Theorem 2.2.

2. Main Results and Notational Conventions

Our first main result in this paper is global well-posedness of (1.6) for small initial data. The crucial step in the proof is the preservation of L^∞-smallness, and we begin by stating this.

Proposition 2.1. *Consider the* 2D *evolution problem* (1.6) *and* (1.7) *on a bounded smooth domain* $\Omega \subset \mathbb{R}^2$. *Suppose the coercivity condition* (1.4) *holds together with the structural assumptions* (1.3). *For smooth solutions* Q *there exists an explicitly*

computable constant η_1 *(depending on* $L_i, i = 1, \ldots, 4$*) so that if*

$$\|\tilde{Q}\|_{L^\infty(\partial\Omega)} \le \|Q_0\|_{L^\infty(\Omega)} < \sqrt{2\eta_1} \tag{2.1}$$

and

$$|a| \le 2c\eta_1, \tag{2.2}$$

then for any $T > 0$, *we have*

$$\|Q\|_{L^\infty((0,T)\times\Omega)} \le \sqrt{2\eta_1}. \tag{2.3}$$

Remark 2.1. As mentioned earlier, the physically relevant regime is when the parameter a has small magnitude. This is consistent with the assumption (2.2).

Furthermore a careful check of the proof of Proposition 2.1 shows that (2.3) still holds for weak solutions that satisfy (2.1) and (2.2).

Theorem 2.1. *Suppose the coefficients* a, b, c *and* $L_1, \ldots, L_4$ *satisfy the coercivity condition* (1.4) *together with the structural assumptions* (1.3), *and let* $\Omega \subset \mathbb{R}^2$ *be a smooth, bounded domain. There exists an explicitly computable constant* η_2 *(depending on* $L_i, i = 1, \ldots, 4$ *and* Ω*) so that if* $Q_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $\tilde{Q} \in H^{\frac{3}{2}}(\partial\Omega)$, *and the smallness conditions* (2.1) *and* (2.2) *hold with* η_1 *replaced by* η_2, *then the system* (1.6) *and* (1.7) *has a unique global weak solution.*[b] *Further the initial smallness* (2.1) *is preserved for all time.*

We prove Proposition 2.1 and Theorem 2.1 in Sec. 3. The smallness assumption on the initial data is essential; we complement Theorem 2.1 with a result showing that certain solutions exhibit a finite time blow-up.

Theorem 2.2. *Suppose the coefficients* a, b, c *and* $L_1, \ldots, L_4$ *satisfy the coercivity condition* (1.4) *together with the structural assumptions* (1.3). *There exists a smooth domain* Ω, *smooth initial data* Q_0, *and a smooth (time-independent) function* $\tilde{Q} : \partial\Omega \to \mathbb{R}$ *such that the system* (1.6) *with Dirichlet boundary conditions* $\tilde{Q}$ *does not admit a global smooth solution.*

Remark 2.2. Our proof (Sec. 4) chooses Ω to be the annulus $B_{R_1}(0)\backslash B_{R_0}(0) \subset \mathbb{R}^2$ where $0 < R_0 < R_1$, and "hedgehog" type initial data. Namely, we choose Q_0 of the form

$$Q(0) = \theta_0(|x|)\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \mathbb{I}_2\right),$$

where $\theta_0 : [R_0, R_1] \to \mathbb{R}$ is smooth. If θ_0 is large enough, and R_0, R_1 are such that

$$\frac{R_0^2\pi^2}{9(R_1 - R_0)^2} > 1,$$

we show $\|Q(t)\|_{L^2(\Omega)} \to \infty$ in finite time, for any smooth solution.

[b] See Definition 3.1 for the precise definition of a weak solution.

Finally in Sec. 5 we study how the flow distorts the convex hull of eigenvalues, in an attempt to understand what is the maximal size of initial data that would give global well-posedness. The situation is more interesting in 3D than in 2D as in 3D the convex set of physical Q-tensors cannot be described just in terms of the Fröbenius norm of the matrix. We restrict ourselves to a simple setting (with specific assumptions on the elastic constants L_i's, $i = 1, 2, 3, 4$ and work in the whole space). Our main result in this section is the following.

Proposition 2.2. *Let $Q(t,x) \in C([0,T]; H^k(\mathbb{R}^d))$ with $k > \frac{d}{2}$, $d = 2, 3$ and arbitrary $T > 0$ be a solution of the system* (1.6) *and* (1.7), *under assumptions* (1.3). *Assume further*:

- $L_1 \neq 0$, $L_4 = 0$ *and* (1.4) *holds if* $d = 2$,
- *or* $L_1 \neq 0$ *and* $L_2 + L_3 = L_4 = 0$ *if* $d = 3$.

Suppose the initial data $Q_0 \in H^k(\mathbb{R}^d)$ is such that for any $x \in \mathbb{R}^d$, the eigenvalues of $Q_0(x)$ are in the interval:

$$\left[-\sqrt{\frac{|a|}{2c}}, \sqrt{\frac{|a|}{2c}}\right] \quad \textit{when } d = 2, \quad \text{or}$$

$$\left[-\frac{b+\sqrt{b^2-24ac}}{12c}, \frac{b+\sqrt{b^2-24ac}}{6c}\right] \quad \textit{when } d = 3.$$

If $d = 3$, we further assume

$$|a| < \frac{b^2}{3c}. \tag{2.4}$$

Then, for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, the eigenvalues of $Q(t,x)$ stay in the same interval.

The "usual" energy methods do not seem to yield Proposition 2.2 in dimension $d = 3$. Instead we use a Trotter product formula and provide a somewhat atypical proof in Sec. 5.

Notational convention. We define $A : B \stackrel{\text{def}}{=} \operatorname{tr}(A^t B)$ when A, B are $d \times d$ matrices, and let $|Q|$ denote the Fröbenius norm of the matrix Q (i.e. $|Q| \stackrel{\text{def}}{=} \sqrt{\operatorname{tr}(Q^t Q)} = \sqrt{\operatorname{tr}(Q^2)}$). We denote the space of Q-tensors by $\mathcal{S}^{(d)}$, where

$$\mathcal{S}^{(d)} \stackrel{\text{def}}{=} \{M \in \mathbb{R}^{d\times d}, M = M^t, \operatorname{tr}(M) = 0\},$$

and define the matrix-valued L^p space by

$$L^p(\Omega, \mathcal{S}^{(d)}) \stackrel{\text{def}}{=} \{Q : \Omega \to \mathcal{S}^{(d)}, |Q| \in L^p(\Omega, \mathbb{R})\}, \quad \text{when } 1 \le p \le \infty.$$

For the sake of simplicity, we let $\|\cdot\|$ (with no subscripts) to denote $\|\cdot\|_{L^2(\Omega)}$. We denote the partial derivative with respect to x_k of the ij component of Q, by either $Q_{ij,k}$ or $\partial_k Q_{ij}$. Throughout the paper, we assume the Einstein summation convention over the repeated indices.

3. Global Well-Posedness for Small Initial Data

Using standard techniques the gradient flow structure of the equation should provide *a priori* estimates for (1.5) for smooth enough solutions. Taking the (matrix) inner product of Eq. (1.5) with $\frac{\delta\mathcal{E}}{\delta Q} - \lambda I + \mu - \mu^T$ and integrating yields

$$\frac{d}{dt}\mathcal{E}[Q] = -\int_\Omega \left|\frac{\delta\mathcal{E}}{\delta Q} - \lambda I + \mu - \mu^T\right|^2 dx.$$

This gives the energy equality

$$\mathcal{E}[Q(t)] + \int_0^t \int_\Omega \left|\frac{\delta\mathcal{E}}{\delta Q} - \lambda I + \mu - \mu^T\right|^2 dx ds = \mathcal{E}[Q(0)], \quad \forall\, t > 0. \tag{3.1}$$

The main defect of the energy $\mathcal{E}[Q]$ is that it is unbounded from below as $L_4 \neq 0$. Thus, unlike in the usual contexts, it does not provide *a priori* control over the H^1-norm of Q. On the other hand, if $\|Q\|_{L^\infty}$ is small enough, then we can absorb the cubic term into the three quadratic terms and force the elastic part of the energy to be positive. The idea behind our proof is to first prove *preservation of smallness*: namely, if $\|Q\|_{L^\infty}$ is small enough initially, then it does not increase with time. Now coercivity of the quadratic terms, and smallness of the cubic term force the energy $\mathcal{E}[Q]$ to stay positive, from which (3.1) will provide an *a priori* H^1 bound for Q. This will be enough to prove well-posedness of (1.5) (or equivalently Eq. (1.6)).

3.1. *Preservation of smallness in L^∞*

The goal of this section is to prove Proposition 2.1 showing that L^∞ smallness of the initial data is preserved in time. This in turn implies that the energy is positive definite and will allow us to obtain *a priori* estimates on higher norms.

We begin by recalling a few well-known results that come directly from Gagliardo–Nirenberg inequalities and elliptic PDE theory.

Lemma 3.1. *Suppose Ω is a smooth, bounded domain in $\mathbb{R}^2$. There exists a positive constant $C_1 = C_1(\Omega)$, such that for any $f \in H^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$, with $f|_{\partial\Omega} = g$, we have*:

$$\|f\|_{L^\infty(\Omega)} \le C_1 \|f\|^{\frac{1}{2}} \left(\|\Delta f\|^{\frac{1}{2}} + \|f\|^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}_{H^{\frac{3}{2}}}\right), \tag{3.2}$$

$$\|D^2 f\| \le C_1 (\|\Delta f\| + \|f\| + \|g\|_{H^{\frac{3}{2}}(\partial\Omega)}). \tag{3.3}$$

Moreover, for any $f \in H^2(\Omega)$, we have the interpolation estimate

$$\|\nabla f\|^2_{L^4} \le C_1 \|f\|_{L^\infty} \left(\|\Delta f\|_{L^2} + \|f\| + \|g\|^{\frac{1}{2}}_{H^{\frac{3}{2}}}\right). \tag{3.4}$$

Finally, for $f \in H^1_0(\Omega)$, we have the Ladyzhenskaya inequality[12]:

$$\|f\|^2_{L^4(\Omega)} \le C\|\nabla f\| \|f\|. \tag{3.5}$$

Remark 3.1. Further, for $f \in H^2(\Omega) \cap H_0^1(\Omega)$ the terms $\|f\|$ and $\|g\|_{H^{\frac{3}{2}}(\partial\Omega)}$ are not required in (3.2), (3.3) and (3.4). This follows from elliptic regularity (see for instance Ref. 9 (Sec. 6.3.2, Theorem 4 and remark (i) afterwards)).

The proofs of (3.2) and (3.3) follow from interpolation inequalities (see for instance Theorems 5.2 and 5.8 in Ref. 1) combined with the elliptic regularity (see Theorem 6.3.2.4 in Ref. 9). The estimate (3.4) is a consequence of Gagliardo–Nirenberg inequality (see for instance p. 313 in Ref. 6) combined with the elliptic regularity result previously mentioned.

We can now provide the proof of Proposition 2.1.

Proof of Proposition 2.1. Due to the special structure of Q in 2D, we expand Q as

$$Q(x,t) = \begin{pmatrix} p(x_1,x_2,t) & q(x_1,x_2,t) \\ q(x_1,x_2,t) & -p(x_1,x_2,t) \end{pmatrix}, \tag{3.6}$$

where p, q are two scalar functions. Inserting (3.6) into (1.6), we obtain the following evolution equations for p and q:

$$\begin{aligned} \frac{\partial p}{\partial t} &= \zeta\Delta p + L_4[(\partial_1 p)^2 - (\partial_1 q)^2 - (\partial_2 p)^2 + (\partial_2 q)^2 + 2\partial_1 p\partial_2 q + 2\partial_2 p\partial_1 q] \\ &\quad + 2L_4(p\partial_1\partial_1 p + 2q\partial_1\partial_2 p - p\partial_2\partial_2 p) - ap - 2c(p^2+q^2)p, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial q}{\partial t} &= \zeta\Delta q + 2L_4[\partial_1 q\partial_2 q - \partial_1 p\partial_2 p + \partial_1 p\partial_1 q - \partial_2 p\partial_2 q] \\ &\quad + 2L_4(p\partial_1\partial_1 q + 2q\partial_1\partial_2 q - p\partial_2\partial_2 q) - aq - 2c(p^2+q^2)q, \end{aligned} \tag{3.8}$$

$$p(x,0) = p_0(x), \quad q(x) = q_0(x), \quad p(x,t)|_{\partial\Omega} = \tilde{p}(x), \quad q(x,t)|_{\partial\Omega} = \tilde{q}(x). \tag{3.9}$$

Here, $\tilde{p}$ and $\tilde{q}$ are the corresponding components associated to $\tilde{Q}$ and

$$\zeta \stackrel{\text{def}}{=} 2L_1 + L_2 + L_3 > 0. \tag{3.10}$$

Note that positivity of ζ is a consequence of assumption (1.4).

Define

$$\eta_1 \stackrel{\text{def}}{=} \frac{\zeta^2}{(1+4\sqrt{2})^2 L_4^2} > 0. \tag{3.11}$$

Multiplying (3.7) with p, (3.8) with q, and adding gives:

$$\begin{aligned} \frac{1}{2}\frac{\partial h^2}{\partial t} &= \frac{\zeta}{2}\Delta h^2 - \zeta(|\nabla p|^2 + |\nabla q|^2) - ah^2 - 2ch^4 \\ &\quad - L_4(p\partial_2\partial_2 h^2 - p\partial_1\partial_1 h^2 - 2q\partial_1\partial_2 h^2) \\ &\quad + L_4 p[(\partial_2 p)^2 - (\partial_1 p)^2 - 3(\partial_1 q)^2 + 3(\partial_2 q)^2 + 2\partial_1 p\partial_2 q + 2\partial_2 p\partial_1 q] \\ &\quad + 2L_4 q[\partial_1 p\partial_1 q - 3\partial_1 p\partial_2 p - \partial_2 p\partial_2 q - \partial_1 q\partial_2 q], \end{aligned} \tag{3.12}$$

where

$$h(x_1,x_2,t) \stackrel{\text{def}}{=} \sqrt{p^2+q^2}. \tag{3.13}$$

Multiplying (3.12) by $(h^2-\eta_1)^+$ and integrating gives:

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int_\Omega |(h^2-\eta_1)^+|^2 dx \\ &= -\frac{\zeta}{2}\int_\Omega |\nabla(h^2-\eta_1)^+|^2 dx - \zeta\int_\Omega (h^2-\eta_1)^+(|\nabla p|^2 + |\nabla q|^2)dx \\ &\quad + L_4\int_\Omega p|\partial_2(h^2-\eta_1)^+|^2 dx \\ &\quad + L_4\int_\Omega \partial_2 p\partial_2(h^2-\eta_1)^+(h^2-\eta_1)^+ dx \\ &\quad - L_4\int_\Omega p|\partial_1(h^2-\eta_1)^+|^2 dx \\ &\quad - L_4\int_\Omega \partial_1 p\partial_1(h^2-\eta_1)^+(h^2-\eta_1)^+ dx \\ &\quad - 2L_4\int_\Omega q\partial_2(h^2-\eta_1)^+\partial_1(h^2-\eta_1)^+ dx \\ &\quad - 2L_4\int_\Omega \partial_1 q\partial_2(h^2-\eta_1)^+(h^2-\eta_1)^+ dx \\ &\quad - \int_\Omega 2ch^2\left(\frac{a}{2c}+h^2\right)(h^2-\eta_1)^+ dx \\ &\quad + L_4\int_\Omega p[(\partial_2 p)^2 - (\partial_1 p)^2 - 3(\partial_1 q)^2 + 3(\partial_2 q)^2 \\ &\quad + 2\partial_1 p\partial_2 q + 2\partial_2 p\partial_1 q](h^2-\eta_1)^+ dx \\ &\quad + 2L_4\int_\Omega q[\partial_1 p\partial_1 q - 3\partial_1 p\partial_2 p - \partial_2 p\partial_2 q - \partial_1 q\partial_2 q](h^2-\eta_1)^+ dx \\ &= -\frac{\zeta}{2}\int_\Omega |\nabla(h^2-\eta_1)^+|^2 dx - \zeta\int_\Omega (h^2-\eta_1)^+(|\nabla p|^2 + |\nabla q|^2)dx \\ &\quad + I_1 + \cdots + I_9. \end{aligned} \tag{3.14}$$

Above we used (3.9), (2.1) and integration by parts.

We estimate the terms I_1 through I_9 individually. Using the Schwarz inequality and the fact $|p|+|q| \le \sqrt{2p^2+2q^2}$, we obtain:

$$\begin{aligned} I_1 + I_3 + I_5 &\le |L_4|\int_\Omega (|p|+|q|)|\nabla(h^2-\eta_1)^+|^2 dx \\ &\le \sqrt{2}|L_4|\int_\Omega h|\nabla(h^2-\eta_1)^+|^2 dx. \end{aligned}$$

Also,

$$\begin{aligned} I_2 + I_4 &\le |L_4| \int_\Omega |\nabla p||\nabla (h^2-\eta_1)^+|(h^2-\eta_1)^+ dx \\ &\le \frac{|L_4|}{2}\int_\Omega |(h^2-\eta_1)^+|^{\frac{3}{2}}|\nabla p|^2 dx + \frac{|L_4|}{2}\int_\Omega |(h^2-\eta_1)^+|^{\frac{1}{2}}|\nabla (h^2-\eta_1)^+|^2 dx \\ &\le \frac{|L_4|}{2}\int_\Omega h(h^2-\eta_1)^+|\nabla p|^2 dx + \frac{|L_4|}{2}\int_\Omega h|\nabla (h^2-\eta_1)^+|^2 dx. \end{aligned} \tag{3.15}$$

Similarly,

$$I_6 \le |L_4| \int_\Omega h(h^2-\eta_1)^+|\nabla q|^2 dx + |L_4|\int_\Omega h|\nabla (h^2-\eta_1)^+|^2 dx. \tag{3.16}$$

Furthermore, assumption (2.2) implies

$$I_7 \le \int_\Omega 2ch^2\left(\frac{|a|}{2c} - h^2\right)(h^2-\eta_1)^+ dx \le 0. \tag{3.17}$$

Finally, using the Cauchy–Schwarz inequality and $|p|+|q| \le \sqrt{2p^2+2q^2}$ again, we get

$$I_8 + I_9 \le 4\sqrt{2}|L_4|\int_\Omega h(|\nabla p|^2 + |\nabla q|^2)(h^2-\eta_1)^+ dx. \tag{3.18}$$

Combining the above we get:

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int_\Omega |(h^2-\eta_1)^+|^2 dx \\ &\quad\le \frac{1}{2}\int_\Omega [(3+2\sqrt{2})|L_4|h - \zeta]|\nabla (h^2-\eta_1)^+|^2 dx \\ &\qquad + \int_\Omega [(1+4\sqrt{2})|L_4|h-\zeta](|\nabla p|^2+|\nabla q|^2)(h^2-\eta_1)^+ dx. \end{aligned} \tag{3.19}$$

Note that $3+2\sqrt{2} < 1+4\sqrt{2}$, hence if we assume at initial time

$$|Q_0| = \sqrt{2(p_0^2+q_0^2)} = \sqrt{2}h_0 < \frac{\sqrt{2}\zeta}{(1+4\sqrt{2})|L_4|}, \tag{3.20}$$

then it follows from (3.19) that

$$\frac{d}{dt}\int_\Omega |(h^2-\eta_1)^+|^2 dx \le 0, \quad \forall t \in (0,T),$$

which concludes the proof. □

Remark 3.2. For $L_4 = 0$ the previous result is to be expected, as the energy is just the usual Dirichlet type energy, up to a null-Lagrangian (see (C.4) in Appendix C). The unexpected aspect captured by the lemma is that through the gradient type evolution, the coercive part of the energy manages to control the size of the badly behaved cubic term that is present for $L_4 \neq 0$.

3.2. A priori *estimates for higher norms*

For small data, Proposition 2.1 shows that the L^∞ smallness is preserved. Consequently, this will imply coercivity of the second-order terms and positivity of the energy $\mathcal{E}$. The main result of this section uses this and the dissipative energy law (3.1) to *a priori* control higher-order norms of Q.

Proposition 3.1. *For $\Omega \subset \mathbb{R}^2$ smooth and bounded, there exists an $\eta_2 > 0$ depending on $L_i, i = 1,2,3,4$ and Ω so that if:*

$$\begin{aligned} &Q_0 \in H^1(\Omega)\cap L^\infty(\Omega), \quad \tilde{Q} \in H^{\frac{3}{2}}(\partial\Omega), \\ &\|Q_0\|_{L^\infty(\Omega)} \le \sqrt{2\eta_2}, \quad \textit{and} \quad \tfrac{|a|}{2c} \le \eta_2, \end{aligned} \tag{3.21}$$

then under the coercivity condition (1.4) *and structural assumptions* (1.3), *for any $T > 0$, and any smooth solution Q of* (1.6) *and* (1.7) *we have*

$$\|Q\|_{L^\infty(0,T;H^1(\Omega))} \le C \quad \textit{and} \quad \|Q\|_{L^2(0,T;H^2(\Omega))} \le C,$$

for some constant C depending on T, η_2, $\|Q_0\|_{H^1}$ and $\|\tilde{Q}\|_{H^{\frac{3}{2}}(\partial\Omega)}$.

Proof. As mentioned earlier, the assumption (1.4) guarantees coercivity of the linear terms in 2D and quantitatively gives

$$(L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3 \partial_j Q_{ij}\partial_k Q_{ik})(x) \ge \nu |\nabla Q|^2(x), \tag{3.22}$$

where

$$\nu \overset{\text{def}}{=} \min\{L_1+L_2, L_1+L_3\} > 0.$$

For continuity we prove this in Lemma C.1 in Appendix C below, and refer the reader to Refs. 8 and 13 for the three-dimensional analog.

Now define:

$$\eta_1 \overset{\text{def}}{=} \frac{\zeta^2}{(1+4\sqrt{2})^2L_4^2}, \quad \text{and} \quad \eta_2 \overset{\text{def}}{=} \frac{1}{60}\min\left\{\frac{\nu^2}{8L_4^2}, \frac{\zeta^2}{144L_4^2C_1^2}, \eta_1\right\}, \tag{3.23}$$

where C_1 is the constant appearing in Lemma 3.1.

To begin, an argument analogous to the proof of Proposition 2.1 gives

$$\|Q(t)\|_{L^\infty(\Omega)} \le \sqrt{2\eta_2}, \quad \forall t \in [0,T].$$

Next, we infer from the basic energy law (3.1), Lemma C.1, that there exists $\tilde{\eta} = \nu - 2|L_4|\sqrt{2\eta_2} > 0$, such that:

$$\begin{aligned} \mathcal{E}(Q_0) \ge \mathcal{E}(Q(t)) &\ge \int_\Omega \nu|\nabla Q|^2 + L_4 Q_{lk}\partial_l Q_{ij}\partial_k Q_{ij} + \frac{a}{2}\operatorname{tr}(Q^2) + \frac{c}{4}\operatorname{tr}^2(Q^2) dx \\ &\ge \int_\Omega \nu|\nabla Q|^2 dx - |L_4|\|Q\|_{L^\infty(\Omega)}\|\nabla Q\|^2 + \frac{c}{4}\int_\Omega\left\{\left[\operatorname{tr}(Q^2) + \frac{a}{c}\right]^2 - \frac{a^2}{c^2}\right\}dx \\ &\ge \tilde{\eta}\|\nabla Q(t)\|^2 - \frac{a^2}{4c}|\Omega|. \end{aligned}$$

Hence $Q \in L^\infty(0,T;H^1(\Omega))$. Furthermore, it follows from the basic energy law (3.1) and Eq. (1.6) that

$$Q_t \in L^2(0,T;L^2(\Omega)).$$

By Lemma 3.1, Proposition 2.1 and Cauchy–Schwarz inequality, we deduce from (3.7) and (3.8) that:

$$\begin{aligned}
\zeta\|\Delta p(t)\| &\le \|p_t\| + |L_4|\|(\partial_1 p)^2 - (\partial_1 q)^2 - (\partial_2 p)^2 + (\partial_2 q)^2 + 2\partial_1 p\partial_2 q + 2\partial_2 p\partial_1 q\| \\
&\quad + 2|L_4|\|p\partial_1\partial_1 p + 2q\partial_1\partial_2 p - p\partial_2\partial_2 p\| + \|ap + 2c(p^2+q^2)p\| \\
&\le \|p_t\| + 2|L_4|\|(\partial_1 p)^2 + (\partial_2 p)^2 + (\partial_1 q)^2 + (\partial_2 q)^2\| \\
&\quad + 2|L_4|\|p\|_{L^\infty(\Omega)}\|\partial_1\partial_1 p\| \\
&\quad + 4|L_4|\|q\|_{L^\infty(\Omega)}\|\partial_1\partial_2 p\| + 2|L_4|\|p\|_{L^\infty(\Omega)}\|\partial_2\partial_2 p\| + C \\
&\le \|p_t\| + 2|L_4|C_1\Big[\|p\|_{L^\infty}\Big(\|\Delta p\| + \|p\| + \|\tilde{p}\|_{H^{\frac{3}{2}}(\partial\Omega)}\Big) \\
&\quad + \|q\|_{L^\infty}\Big(\|\Delta q\| + \|q\| + \|\tilde{q}\|_{H^{\frac{3}{2}}(\partial\Omega)}\Big)\Big] \\
&\quad + 8|L_4|C_1\|h\|_{L^\infty}\Big(\|\Delta p\| + \|p\| + \|\tilde{p}\|_{H^{\frac{3}{2}}(\partial\Omega)}\Big) + C \\
&\le \|p_t\| + |L_4|C_1\|h\|_{L^\infty}(10\|\Delta p\| + 2\|\Delta q\|) + C.
\end{aligned} \tag{3.24}$$

Here C depends on Ω, Q_0, $\tilde{Q}$, and the coefficients of the system. Analogously, we know

$$\zeta\|\Delta q(t)\| \le \|q_t\| + |L_4|C_1\|h\|_{L^\infty}(10\|\Delta q\| + 2\|\Delta p\|) + C. \tag{3.25}$$

After summing up, we get

$$\zeta(\|\Delta p(t)\| + \|\Delta q(t)\|) \le \|Q_t\| + 12|L_4|C_1\|h\|_{L^\infty}(\|\Delta q\| + \|\Delta p\|) + C, \tag{3.26}$$

which yields the bound of $\|\Delta Q\|$ in $L^2(0,T)$, due to the choice of η_2 and the fact $\|h\|_{L^\infty(0,T;\Omega)} \le \sqrt{\eta_2}$. □

Remark 3.3. The factor $\frac{1}{60}$ in (3.23) is not used the proof above. However, it will be necessary in the proof of Theorem 2.1, Part (i), as described in the discussion before (3.46).

3.3. *Weak solutions*

The purpose of this section is to show that the *a priori* estimates previously established are enough to show global existence and uniqueness of weak solutions for small initial data. While this is usually standard, the nonlinearity appearing in the higher-order terms makes things complicated in our situation. Specifically, we crucially need $\|Q\|_{L^\infty}$ to be small in order to obtain coercivity of the second-order terms. Thus any approximating scheme devised to prove the existence of weak solutions must preserve L^∞ smallness of the initial data. Since Q is a 2×2 matrix we

do not have the luxury of a maximum principle that *a priori* preserves $\|Q\|_{L^\infty}$, and the approximating scheme must be constructed carefully. We carry out this construction below.

We begin by recalling the definition of weak solutions in our context.

Definition 3.1. For any $T \in (0,+\infty)$, a function Q satisfying:

$$\begin{aligned}
&Q \in L^\infty(0,T;H^1\cap L^\infty)\cap L^2(0,T;H^2),\\
&\partial_t Q \in L^2(0,T;L^2), \quad \text{and}\\
&Q \in S^{(2)}, \quad \text{a.e. in } \Omega\times(0,T)
\end{aligned}$$

is called a weak solution of the problem (3.7) and (3.8), if it satisfies the initial and boundary conditions (3.9), and we have:

$$\begin{aligned}
&-\int_{\Omega\times[0,T]} Q : \partial_t R\,dxdt \\
&\quad = -2L_1\int_{\Omega\times[0,T]} \partial_k Q : \partial_k R\,dxdt - \int_{\Omega\times[0,T]}[a + c\,\mathrm{tr}(Q^2)]Q : R\,dxdt \\
&\quad\quad - 2(L_2+L_3)\int_{\Omega\times[0,T]} \partial_k Q_{ik}\partial_j R_{ij}\,dxdt \\
&\quad\quad + (L_2+L_3)\int_{\Omega\times[0,T]} \partial_k Q_{lk}\partial_l R_{ii}\,dxdt \\
&\quad\quad - 2L_4\int_{\Omega\times[0,T]} Q_{lk}\partial_k Q_{ij}\partial_l R_{ij}\,dxdt - L_4\int_{\Omega\times[0,T]} \partial_i Q_{kl}\partial_j Q_{kl} R_{ij}\,dxdt \\
&\quad\quad + \frac{L_4}{2}\int_{\Omega\times[0,T]} |\nabla Q|^2 R_{ii}\,dxdt - \int_\Omega Q_0 : R(0)dx.
\end{aligned}$$

Here $R \in C_c^\infty([0,T)\times\Omega, M^{2\times 2}(\mathbb{R}))$ is arbitrary.

Remark 3.4. The notion of weak solution above is similar to the one considered in Definition 3.2 and Remark 4 of Ref. 20 for a related system. The main difference in our situation is the regularity requirement on R. The more standard requirement would be that $R \in H_0^1(\Omega)$, however, because of the presence of the cubic term we need a stronger assumption. A similar situation occurs in Ref. 7, where test functions are taken in a smaller space than $H_0^1(\Omega)$ because of the presence of similar terms. For simplicity we take here only smooth functions although a larger class of test function should still work for obtaining existence of solutions.

Proof of Theorem 2.1, Part (i). For simplicity we only consider the homogeneous boundary problem. The analysis of the corresponding inhomogeneous boundary condition is similar but involves many lengthy computations which obscure the heart of the matter. We start by augmenting (3.7) and (3.8), (1.6) with a singular potential. Explicitly, consider the system:

$$\frac{\partial p}{\partial t} = \zeta \Delta p - \varepsilon \frac{\partial f}{\partial p}(p,q) + L_4[(\partial_1 p)^2 - (\partial_1 q)^2 - (\partial_2 p)^2 + (\partial_2 q)^2 + 2\partial_1 p \partial_2 q]$$
$$+ 2L_4(\partial_2 p \partial_1 q + p\partial_1\partial_1 p + 2q\partial_1\partial_2 p - p\partial_2\partial_2 p) - ap - 2c(p^2+q^2)p, \quad (3.27)$$

$$\frac{\partial q}{\partial t} = \zeta \Delta q - \varepsilon \frac{\partial f}{\partial q}(p,q) + 2L_4[\partial_1 q \partial_2 q - \partial_1 p \partial_2 p + \partial_1 p \partial_1 q - \partial_2 p \partial_2 q]$$
$$+ 2L_4(p\partial_1\partial_1 q + 2q\partial_1\partial_2 q - p\partial_2\partial_2 q) - aq - 2c(p^2+q^2)q, \quad (3.28)$$

with initial data

$$p(x,0) = p_0^\varepsilon(x), \quad q(x) = q_0^\varepsilon(x), \quad \forall x \in \Omega, \quad (3.29)$$

and boundary conditions

$$p(x,t) = 0, \quad q(x,t) = 0, \quad \forall x \in \partial\Omega. \quad (3.30)$$

Here $p_0^\varepsilon, q_0^\varepsilon \in C^\infty(\Omega) \cap H_0^1(\Omega)$ are such that

$$p_0^\varepsilon \to p_0 \quad \text{and} \quad q_0^\varepsilon \to q_0 \quad \text{in } H^1(\Omega) \cap L^\infty(\Omega),$$

and $f(p,q)$ is the singular potential[c]:

$$f(p,q) = \begin{cases} -\ln(4\eta_2 - p^2 - q^2) & \text{if } p^2+q^2 < 4\eta_2, \\ \infty & \text{if } p^2+q^2 \geq 4\eta_2 \end{cases}$$

(where η_2 is as defined in (3.23)). The advantage of this approximating system is that it has a singular potential term in its energy:

$$\mathcal{E}^\varepsilon[p,q] \stackrel{\text{def}}{=} \int_\Omega \zeta(|\nabla p|^2 + |\nabla q|^2) + \varepsilon f(p,q)dx$$
$$+ 2L_4 \int_\Omega p(|\partial_1 p|^2 - |\partial_2 p|^2)dx$$
$$+ 4L_4 \int_\Omega q(\partial_1 p \partial_2 p + \partial_1 q \partial_2 q)dx$$
$$+ \int_\Omega a(p^2+q^2) + c(p^2+q^2)^2 dx. \quad (3.31)$$

Hence finite energy will imply $p^2 + q^2 \leq 4\eta_2$ almost everywhere. We will then prove an additional preservation of smallness principle for the approximate system (3.27)–(3.30). Namely, we will show the stronger L^∞ bound $p^2 + q^2 \leq \eta_2$ almost everywhere in time and space, provided this is true initially, at $t = 0$. Thus we will be able to conclude that the terms coming from the singular potential become uniformly small and disappear in the limit $\varepsilon \to 0$.

[c]Let us note that this choice of the singular potential ensures that the system thus obtained satisfies the symmetry and tracelessness constraints. The partial derivatives will only make sense for solutions of finite energy, hence such that $p^2 + q^2 < 4\eta_2$ a.e. so that we are in the effective domain of the convex potential f.

In order to obtain the existence of the approximate system (3.27)–(3.30), we regularize the singular potential f and construct an approximating sequence using the Galerkin method. To regularize the singular potential we use an approximating sequence of functions $f_N : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the following properties:

(1) $f_N : \mathbb{R}^2 \to \mathbb{R}$ is C^∞ and convex,
(2) there exists a constant $\alpha \in \mathbb{R}$ such that

$$-\alpha^2 \leq f_N(p,q), \quad \forall p,q \in \mathbb{R} \quad \text{and} \quad \forall N \geq 1, \quad (3.32)$$

(3) $f_N \leq f_{N+1} \leq f$ on $\mathbb{R}^2$ for all $N \in \mathbb{N}$,
(4) $f_N \to f$ in $C^k(D(f))$ as $N \to \infty$ (where $D(f)$ is the domain of f, namely $D(f) := \{(p,q) \in \mathbb{R}^2; p^2+q^2 < 4\eta_2\}$).

A similar construction was carried out in Ref. 22 using Moreau–Yosida approximation and a suitable smoothing, and we refer the reader to Ref. 22 for the details.

For the Galerkin approximation, let $\{\varphi_1, \ldots, \varphi_n, \ldots\}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of the Laplacian (with zero Dirichlet boundary conditions). Let $\mathcal{P}_m : L^2 \to H_m$ where $H_m \stackrel{\text{def}}{=} \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$. Consider the finite-dimensional system:

$$\frac{\partial p_m}{\partial t} = \zeta \Delta p_m - \varepsilon \mathcal{P}_m \left\{ \frac{\partial f_N}{\partial p}(p_m, q_m) \right\} + L_4 \mathcal{P}_m\{(\partial_1 p_m)^2 - (\partial_1 q_m)^2 - (\partial_2 p_m)^2\}$$
$$+ L_4 \mathcal{P}_m\{(\partial_2 q_m)^2 + 2\partial_1 p_m \partial_2 q_m + 2\partial_2 p_m \partial_1 q_m\} + 2L_4 \mathcal{P}_m\{p_m \partial_1\partial_1 p_m$$
$$+ 2q_m \partial_1\partial_2 p_m - p_m \partial_2\partial_2 p_m - ap_m - 2c(p_m^2 + q_m^2)p_m\}, \quad (3.33)$$

$$\frac{\partial q_m}{\partial t} = \zeta \Delta q_m - \varepsilon \mathcal{P}_m \left\{ \frac{\partial f_N}{\partial q}(p_m, q_m) \right\} + 2L_4 \mathcal{P}_m\{\partial_1 q_m \partial_2 q_m - \partial_1 p_m \partial_2 p_m\}$$
$$+ 2L_4 \mathcal{P}_m\{\partial_1 p_m \partial_1 q_m - \partial_2 p_m \partial_2 q_m\} + 2L_4 \mathcal{P}_m\{p_m \partial_1\partial_1 q_m$$
$$+ 2q_m \partial_1\partial_2 q_m - p_m \partial_2\partial_2 q_m - aq_m - 2c(p_m^2 + q_m^2)q_m\}, \quad (3.34)$$

with initial conditions

$$p_m(x,0) = (\mathcal{P}_m p_0^\varepsilon)(x), \quad q(x) = (\mathcal{P}_m q_0^\varepsilon)(x), \quad \forall x \in \Omega. \quad (3.35)$$

The above system depends on three parameters: ε, m and N. For simplicity we drop the explicit dependence on ε and N from the notation, and only keep the subscript m in the solutions p_m, q_m. We will first send $N \to \infty$ and then $m \to \infty$ to obtain solutions to the approximate continuous system (3.27) and (3.28). Finally we will pass to the limit $\varepsilon \to 0$. We divide the remainder of the proof into three steps.

Step 1: *Sending $N \to \infty$*. We look for solutions of the form

$$p_m(t,x) \stackrel{\text{def}}{=} \sum_{i=1}^m a_m^i(t)\varphi_i(x), \quad q_m(t,x) = \sum_{i=1}^m b_m^i(t)\varphi_i(x).$$

The existence of solutions for short time is a consequence of the standard Cauchy–Peano local existence theory for systems of ordinary differential equations. The bounds (3.39) obtained below will suffice for showing that the existence of the system holds for arbitrary intervals of time.

Note that for $\varepsilon > 0$ small enough we have $(p_0^\varepsilon)^2 + (q_0^\varepsilon)^2 < 2\eta_2$ almost everywhere. Since for $m \to \infty$ we have $\mathcal{P}_m p_0^\varepsilon \to p_0^\varepsilon$ in $H^2 \hookrightarrow L^\infty$ we can arrange

$$\|\mathcal{P}_m p_0^\varepsilon\|_{L^\infty}^2 + \|\mathcal{P}_m q_0^\varepsilon\|_{L^\infty}^2 < 2\eta_2, \tag{3.36}$$

for $m = m(\varepsilon)$ large enough, and ε sufficiently small.

Multiplying Eq. (3.33) by $\partial_t p_m$, and Eq. (3.34) by $\partial_t q_m$, adding and integrating over Ω gives

$$\begin{aligned}\mathcal{E}[p_m(t), q_m(t)] &+ \varepsilon \int_\Omega f_N(p_m, q_m)dx + \|\partial_t p_m\|_{L^2((0,T)\times\Omega)} \\ &+ \|\partial_t q_m\|_{L^2((0,T)\times\Omega)} \le \mathcal{E}^\varepsilon[p_m(0), q_m(0)].\end{aligned} \tag{3.37}$$

Here $\mathcal{E}[v,w] \stackrel{\text{def}}{=} \mathcal{E}^\varepsilon[v,w] - \varepsilon \int_\Omega f_N(v,w)dx$ with $\mathcal{E}^\varepsilon$ as defined in (3.31). To obtain (3.37) we integrated by parts and used the fact that $\mathcal{P}_m$ is a self-adjoint operator on L^2.

We now focus on understanding what *a priori* bounds are provided by (3.37). We claim that the finite dimensionality of H_m allows us to find a large enough constant $C(m)$, which depends on m but not on N or ε, such that

$$\int_\Omega \frac{\zeta}{2}(|\nabla p_m(t)|^2 + |\nabla q_m(t)|^2)dx \le \mathcal{E}[p_m(t), q_m(t)] + C(m). \tag{3.38}$$

To see this, observe that there exists a constant $\tilde{C}(m)$ depending only on m and Ω, such that:

$$\begin{aligned}&\int_\Omega 2L_4[p_m(|\partial_1 p_m|^2 - |\partial_2 p_m|^2) + 2q_m(\partial_1 p_m \partial_2 p_m + \partial_1 q_m \partial_2 q_m)]dx \\ &\quad + \int_\Omega a(p_m^2 + q_m^2) + c(p_m^2 + q_m^2)^2 dx \\ &\le C \int_\Omega \left[\frac{2}{3}L_4(p_m^3 + q_m^3) + \frac{4}{3}L_4|\nabla p_m|^3 \right. \\ &\quad \left. + \frac{4}{3}L_4|\nabla q_m|^3 + a(p_m^2 + q_m^2) + c(p_m^2 + q_m^2)^2\right] dx \\ &\le C \int_\Omega L_4(p_m^3 + q_m^3) + a(p_m^2 + q_m^2) + c(p_m^2 + q_m^2)^2 dx \\ &\quad + L_4 \tilde{C}(m) \int_\Omega (p_m^3 + q_m^3)dx\end{aligned}$$

(where for the first inequality we used Young's inequality $ab \le \frac{a^3}{3} + \frac{2b^{\frac{3}{2}}}{3}$ and for the second the finite dimensionality of H_m). This immediately implies (3.38) as claimed.

For the rest of this step, for the sake of clarity we will specify the hidden dependence on N, namely denote $p_m^N \stackrel{\text{def}}{=} p_m$, $q_m^N \stackrel{\text{def}}{=} q_m$.

Thus using (3.32), adding $C(m) + \varepsilon\alpha^2|\Omega|$ to both sides of (3.37) and taking into account (3.38), we have the *a priori* bounds:

$$\begin{aligned}\|p_m^N\|_{L^\infty(0,T;H^1)} \le C, \quad \|\partial_t p_m^N\|_{L^2(0,T;L^2)} \le C, \\ \|q_m^N\|_{L^\infty(0,T;H^1)} \le C, \quad \|\partial_t q_m^N\|_{L^2(0,T;L^2)} \le C,\end{aligned}$$

where the constant C is independent of N but depending on m. Further, since $p_m^N, q_m^N \in H_m$ and H_m is a finite-dimensional space with a C^∞ basis the above implies that for all $k \in \mathbb{N}$,

$$\begin{aligned}\sup_{N\in\mathbb{N}} \|p_m^N\|_{L^\infty(0,T;H^k)} &+ \|q_m^N\|_{L^\infty(0,T;H^k)} + \|\partial_t p_m^N\|_{L^2(0,T;H^k)} \\ &+ \|\partial_t q_m^N\|_{L^2(0,T;H^k)} < \infty.\end{aligned} \tag{3.39}$$

The above estimates show that as $N \to \infty$, the limit of p_m^N and q_m^N exist (along a subsequence) and in suitable spaces, to be denoted p_m, respectively q_m. Further, using the above *a priori* estimates in (3.37) we obtain

$$\varepsilon \int_\Omega f_N(p_m^N, q_m^N)dx \le C,$$

where the constant C is independent of N. In particular, using the monotonicity of $f_N(\cdot,\cdot)$ with respect to N, we have that for any $N_0 \ge 1$:

$$\varepsilon \int_\Omega f_{N_0}(p_m^N, q_m^N)dx \le C, \quad \forall N \ge N_0;$$

hence using the pointwise convergence of p_m^N, q_m^N respectively to p_m, q_m we get:

$$\varepsilon \int_\Omega f_{N_0}(p_m, q_m)dx \le C.$$

Since N_0 was chosen arbitrarily the monotone convergence theorem now implies

$$\varepsilon \int_\Omega f(p_m, q_m)dx \le C, \tag{3.40}$$

in the limit $N \to \infty$, along a subsequence. Thus, as $N \to \infty$ along a subsequence, we obtain a solution to (3.33) and (3.34) with f_N replaced by f. Further (3.40) shows that for all $t > 0$ the limiting functions p_m and q_m are in the effective domain of the convex potential f.

Step 2: *Sending* $m \to \infty$. Since (3.40) implies $p_m^2 + q_m^2 < 4\eta_2$ for all $m \in \mathbb{N}$, almost everywhere in $(0,T) \times \Omega$, the same argument as in Proposition 3.1 now shows:

$$2\tilde{\eta} \int_\Omega |\nabla p_m(x,t)|^2 + |\nabla q_m(x,t)|^2 dx - \frac{a^2}{4c}|\Omega| \le \mathcal{E}[p_m(t), q_m(t)], \quad \forall t > 0, \ m \in \mathbb{N}.$$

Using (3.37) (with f_N replaced by f) shows the existence of a constant $C(\varepsilon)$ such that

$$\int_0^T \int_\Omega (|\partial_t p_m|^2 + |\partial_t q_m|^2) dxdt < C(\varepsilon). \tag{3.41}$$

Since we work on a domain where f is finite almost everywhere, Eqs. (3.33) and (3.34) (with f_N replaced by f) show:

$$\int_0^T \int_\Omega \left(\zeta\Delta p_m - \varepsilon\mathcal{P}_m\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} + \mathcal{G}_m\right)^2 dxdt + \int_0^T \int_\Omega \left(\zeta\Delta q_m - \varepsilon\mathcal{P}_m\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} + \mathcal{H}_m\right)^2 dxdt < C(\varepsilon). \tag{3.42}$$

The quantities $\mathcal{G}_m$ and $\mathcal{H}_m$ above are defined by:

$$\mathcal{G}_m \stackrel{\text{def}}{=} \frac{\partial p_m}{\partial t} - \zeta\Delta p_m + \varepsilon\mathcal{P}_m\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} \quad \text{and} \tag{3.43}$$

$$\mathcal{H}_m \stackrel{\text{def}}{=} \frac{\partial q_m}{\partial t} - \zeta\Delta q_m + \varepsilon\mathcal{P}_m\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\}. \tag{3.44}$$

Expanding the left-hand side of (3.41), we have:

$$\begin{aligned}
C(\varepsilon) > & \int_0^T \int_\Omega (|\zeta\Delta p_m|^2 + |\zeta\Delta q_m|^2) dxdt + \int_0^T \int_\Omega (|\mathcal{G}_m|^2 + |\mathcal{H}_m|^2) dxdt \\
& + \int_0^T \int_\Omega \left|\varepsilon\mathcal{P}_m\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\}\right|^2 + \left|\varepsilon\mathcal{P}_m\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\}\right|^2 dxdt \\
& - \int_0^T \int_\Omega 2\zeta\varepsilon\Delta p_m \mathcal{P}_m\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} dxdt \\
& - \int_0^T \int_\Omega 2\zeta\varepsilon\Delta q_m \mathcal{P}_m\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} dxdt \\
& - \int_0^T \int_\Omega 2\varepsilon\mathcal{P}_m\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} \mathcal{G}_m dxdt \\
& - \int_0^T \int_\Omega 2\varepsilon\mathcal{P}_m\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} \mathcal{H}_m dxdt \\
& + \int_0^T \int_\Omega 2\zeta(\Delta p_m \mathcal{G}_m + \Delta q_m \mathcal{H}_m) dxdt \\
\stackrel{\text{def}}{=} & I_1 + \cdots + I_8.
\end{aligned} \tag{3.45}$$

Clearly I_1, I_2 and I_3 are positive. For $I_4 + I_5$, we integrate by parts to obtain:

$$\begin{aligned}
I_4 + I_5 &= 4\zeta\varepsilon \int_0^T \int_\Omega \nabla p_m \cdot \nabla\left\{\frac{2p_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} dxdt \\
&\quad + 4\zeta\varepsilon \int_0^T \int_\Omega \nabla q_m \cdot \nabla\left\{\frac{2q_m}{4\eta_2 - (p_m^2+q_m^2)}\right\} dxdt \\
&= 4\zeta\varepsilon \int_0^T \int_\Omega \frac{4\eta_2(|\nabla p_m|^2 + |\nabla q_m|^2)}{4\eta_2 - (p_m^2+q_m^2)} dxdt \\
&\quad + 4\zeta\varepsilon \int_0^T \int_\Omega \frac{p_m^2|\nabla p_m|^2 + q_m^2|\nabla q_m|^2}{4\eta_2 - (p_m^2+q_m^2)} dxdt \\
&\quad - 4\zeta\varepsilon \int_0^T \int_\Omega \frac{p_m^2|\nabla q_m|^2 + q_m^2|\nabla p_m|^2 - 4p_m q_m \nabla p_m \nabla q_m}{4\eta_2 - (p_m^2+q_m^2)} dxdt \\
&\geq 4\zeta\varepsilon \int_0^T \int_\Omega \frac{2p_m^2|\nabla p_m|^2 + 2q_m^2|\nabla q_m|^2 + 4p_m q_m \nabla p_m \nabla q_m}{4\eta_2 - (p_m^2+q_m^2)} dxdt \\
&\geq 0.
\end{aligned}$$

Here we used the fact that $p_m^2 + q_m^2 < 4\eta_2$ a.e. in $(0,T)\times\Omega$.

By Young's inequality we see

$$I_6 + I_7 + I_8 \geq -\frac{1}{2}(I_1 + I_3) - 16I_2.$$

Consequently

$$C(\varepsilon) > \frac{1}{2}(I_1 + I_3) - 15I_2.$$

We claim that due to our choice of η_2, the $15I_2$ term can be hidden in $I_1/2$. Indeed, using (3.33), (3.34), (3.43) and (3.44) we see that $\mathcal{G}_m$ and $\mathcal{H}_m$ are respectively all the terms in (3.33) and (3.34) that have L_4 as a coefficient. Of these, the second-order terms are all multiplied by p_m or q_m, both of which are uniformly bounded by η_2. The first-order terms can be handled by interpolation. Consequently when η_2 is sufficiently small we can arrange $|15I_2| \leq I_1/4$ (see also Remark 3.3).

The above shows

$$\int_0^T \int_\Omega \delta_0|\Delta p_m|^2 + \delta_0|\Delta q_m|^2 dxdt \leq C(\varepsilon), \tag{3.46}$$

for some small constant $\delta_0 > 0$ independent of m. This allows us to pass to the limit $m \to \infty$ and obtain weak solutions of (3.27) and (3.28). Moreover, these solutions are such that the limits $p^\varepsilon, q^\varepsilon$ belong to $L^\infty(0,T;H^1 \cap L^\infty) \cap L^2(0,T;H^2)$. Since $H^1 \hookrightarrow L^6$ we now have $p^\varepsilon, q^\varepsilon \in L^4(0,T;W^{1,3})$. Consequently using the definition of weak solutions we see that (3.27) and (3.28) hold pointwise with all the terms interpreted as elements of $L^2(0,T;L^{\frac{3}{2}})$.

Step 3: *Sending* $\varepsilon \to 0$. We recall that for clarity of presentation we have suppressed the ε superscript, and p, q are solutions of the ε-dependent system (3.27) and (3.28).

Since all terms in Eqs. (3.27) and (3.28) are $L^2(0,T;L^{\frac{3}{2}})$ we can use the same argument we used in the proof of Proposition 2.1. Namely letting $h^2 = p^2 + q^2$, multiplying (3.27) by $p(h^2 - \eta_2)^+$, (3.28) by $q(h^2 - \eta_2)^+$, adding and integrating over Ω leads to the analogue of (3.19):

$$\begin{aligned}\frac{1}{4}\partial_t \int_\Omega |(h^2-\eta_2)^+|^2(t)dx &\le \frac{1}{2}\int_\Omega [(3+2\sqrt{2})|L_4|h-\zeta]|\nabla(h^2-\eta_2)^+|^2 dx\\ &\quad + \int_\Omega [(1+4\sqrt{2})|L_4|h-\zeta](|\nabla p|^2+|\nabla q|^2)(h^2-\eta_2)^+ dx\\ &\quad - \int_\Omega \varepsilon \frac{2h^2(h^2-\eta_2)^+}{4\eta_2-(p^2+q^2)}dx. \qquad (3.47)\end{aligned}$$

Recall that we chose the initial data such that for the $\varepsilon > 0$ small enough we have

$$\|h(0,\cdot)\|_{L^\infty}^2 < \eta_2 < \eta_1 = \frac{\zeta^2}{(1+4\sqrt{2})^2 L_4^2}. \qquad (3.48)$$

Inequality (3.47) shows that

$$\partial_t \|(h(t)^2-\eta_2)^+\|_{L^2}^2 \le 0 \quad \text{provided } \|h(t)\|_{L^\infty}^2 \le \eta_1.$$

This immediately shows that if $\|(h(t)^2-\eta_2)^+\|_{L^2}^2 = 0$ at time 0, it must remain 0 for all $t \ge 0$. Consequently $p^2 + q^2 < \eta_2$ for all $t \ge 0$.

This immediately shows that $|p\partial_p f(p,q)| \le C(\eta_2)$, and the extra ε-terms appearing in (3.27) converge to 0 uniformly as $\varepsilon \to 0$. Following the proof of Proposition 3.1 this will now give (3.24) with additional ε terms that are uniformly converging to 0. This gives uniform in ε estimates for p, q in $L^2(0,T;H^2)$ and for $\partial_t p, \partial_t q$ in $L^2(0,T;L^2)$, which is enough to pass to the limit $\varepsilon \to 0$. □

Lemma 3.2. *Suppose*

$$Q_i = \begin{pmatrix} p_i & q_i \\ q_i & -p_i \end{pmatrix} \in L^\infty(0,\infty;H^1(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty;H^2(\Omega)) \quad (i=1,2)$$

are two global weak solutions to the problem (3.7)–(3.9) *on* $(0,T)$, *which satisfy*

$$\|Q_i\|_{L^\infty((0,\infty)\times\Omega)} \le \sqrt{2\eta_2} \quad (i=1,2),$$

with η_2 *as in Theorem* 2.1.

Then for any $t \in (0,T)$, *we have*

$$\|(Q_1-Q_2)(t)\| \le Ce^{Ct}\|Q_{01}-Q_{02}\|, \qquad (3.49)$$

where $C > 0$ *is a constant that depends on* Ω, Q_{0i} $(i=1,2)$, $\tilde{Q}$ *and the coefficients of the system, but not* t.

Proof. Let $\bar p = p_1 - p_2$, $\bar q = q_1 - q_2$. We see

$$\begin{aligned}\bar p_t &= \zeta\Delta\bar p - a\bar p - 2c(p_1^2+p_1p_2+p_2^2+q_1^2)\bar p - 2cp_2(q_1+q_2)\bar q\\ &\quad + L_4[\partial_1\bar p\partial_1(p_1+p_2) - \partial_1\bar q\partial_1(q_1+q_2) - \partial_2\bar p\partial_2(p_1+p_2)]\\ &\quad + L_4[\partial_2\bar q\partial_2(q_1+q_2) + 2\partial_1\bar p\partial_2 q_1 + 2\partial_1 p_2\partial_2\bar q + 2\partial_1 q_1\partial_2\bar p + 2\partial_2 p_2\partial_1\bar q]\\ &\quad + 2L_4(\bar p\partial_1\partial_1 p_1 + p_2\partial_1\partial_1\bar p + 2\bar q\partial_1\partial_2 p_1 + 2q_2\partial_1\partial_2\bar p - \bar p\partial_2\partial_2 p_1 - p_2\partial_2\partial_2\bar p),\end{aligned} \qquad (3.50)$$

and

$$\begin{aligned}\bar q_t &= \zeta\Delta\bar q - a\bar q - 2c(q_1^2+q_1q_2+q_2^2+p_2^2)\bar q - 2cq_1(p_1+p_2)\bar p\\ &\quad + 2L_4[\partial_1\bar q\partial_2 q_1 + \partial_1 q_2\partial_2\bar q - \partial_1\bar p\partial_2 p_1 - \partial_1 p_2\partial_2\bar p]\\ &\quad + 2L_4[\partial_1\bar p\partial_1 q_1 + \partial_1 p_2\partial_1\bar q - \partial_2\bar p\partial_2 q_1 - \partial_2 p_2\partial_2\bar q]\\ &\quad + 2L_4(\bar p\partial_1\partial_1 q_1 + p_2\partial_1\partial_1\bar q + 2\bar q\partial_1\partial_2 q_1\\ &\quad + 2q_2\partial_1\partial_2\bar q - \bar p\partial_2\partial_2 q_1 - p_2\partial_2\partial_2\bar q),\end{aligned} \qquad (3.51)$$

$$\bar p(0,x) = \bar q(0,x) = 0, \quad \forall x \in \Omega, \quad \bar p|_{\partial\Omega} = \bar q|_{\partial\Omega} = 0. \qquad (3.52)$$

Multiplying Eq. (3.50) with $\bar p$, Eq. (3.51) with $\bar q$, integrating over Ω and using the boundary condition (3.52) gives:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\bar p\|^2 + \|\bar q\|^2) + \zeta\|\nabla\bar p\|^2 + \zeta\|\nabla\bar q\|^2\\ &= L_4\int[\partial_1\bar p\partial_1(p_1+p_2) - \partial_1\bar q\partial_1(q_1+q_2) - \partial_2\bar p\partial_2(p_1+p_2) + \partial_2\bar q\partial_2(q_1+q_2)\\ &\quad + 2\partial_1\bar p\partial_2 q_1 + 2\partial_1 p_2\partial_2\bar q + 2\partial_1 q_1\partial_2\bar p + 2\partial_2 p_2\partial_1\bar q]\bar p dx\\ &\quad - \int a\bar p^2 + 2c(p_1^2+p_1p_2+p_2^2+q_1^2)\bar p^2 + 2cp_2[(q_1+q_2)\bar q]\bar p dx\\ &\quad + 2L_4\int(\bar p\partial_1\partial_1 p_1 + p_2\partial_1\partial_1\bar p + 2\bar q\partial_1\partial_2 p_1 + 2q_2\partial_1\partial_2\bar p\\ &\quad - \bar p\partial_2\partial_2 p_1 - p_2\partial_2\partial_2\bar p)\bar p dx\\ &\quad + 2L_4\int[\partial_1\bar q\partial_2 q_1 + \partial_1 q_2\partial_2\bar q - \partial_1\bar p\partial_2 p_1 - \partial_1 p_2\partial_2\bar p + \partial_1\bar p\partial_1 q_1 + \partial_1 p_2\partial_1\bar q\\ &\quad - \partial_2\bar p\partial_2 q_1 - \partial_2 p_2\partial_2\bar q]\bar q dx\\ &\quad - \int a\bar q^2 + 2c(q_1^2+q_1q_2+q_2^2+p_2^2)\bar q^2 + 2cq_1[(p_1+p_2)\bar p]\bar q dx\\ &\quad + 2L_4\int(\bar p\partial_1\partial_1 q_1 + p_2\partial_1\partial_1\bar q + 2\bar q\partial_1\partial_2 q_1 + 2q_2\partial_1\partial_2\bar q\\ &\quad - \bar p\partial_2\partial_2 q_1 - p_2\partial_2\partial_2\bar q)\bar q dx\\ &= I_1 + \cdots + I_6.\end{aligned} \qquad (3.53)$$

Note that $p_1, p_2, q_1, q_2, \bar{p}, \bar{q} \in L^\infty(0,\infty; H^1(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty; H^2(\Omega)) \cap L^\infty((0,\infty)\times\Omega)$, hence we know by Lemma 3.1 that:

$$\begin{aligned}
I_1 + I_4 &\le C(\|\bar{p}\|_{L^4(\Omega)} + \|\bar{q}\|_{L^4(\Omega)})(\|\nabla\bar{p}\| + \|\nabla\bar{q}\|)(\|\nabla Q_1\|_{L^4(\Omega)} + \|\nabla Q_2\|_{L^4(\Omega)})\\
&\le C(\|\Delta Q_1\|^{\frac12}\|\nabla Q_1\|^{\frac12} + \|\nabla Q_1\| + \|\Delta Q_2\|^{\frac12}\|\nabla Q_2\|^{\frac12} + \|\nabla Q_2\|)\\
&\quad \times (\|\bar{p}\|^{\frac12} + \|\bar{q}\|^{\frac12})(\|\nabla\bar{p}\|^{\frac32} + \|\nabla\bar{q}\|^{\frac32})\\
&\le \frac{\zeta}{9}(\|\nabla\bar{p}\|^2 + \|\nabla\bar{q}\|^2) + C(\|\Delta Q_1\|^2 + \|\Delta Q_2\|^2)(\|\bar{p}\|^2 + \|\bar{q}\|^2), \qquad (3.54)\\
I_2 + I_5 &\le C(\|\bar{p}\|^2 + \|\bar{q}\|^2). \qquad (3.55)
\end{aligned}$$

For I_3, integrating by parts gives:

$$\begin{aligned}
I_3 &= -2L_4\left\{2\int \bar{p}\partial_1\bar{p}\partial_1 p_1 dx + \int \partial_1 p_2\bar{p}\partial_1\bar{p}dx + \int p_2(\partial_1\bar{p})^2 dx + 2\int \bar{p}\partial_1\bar{q}\partial_2 p_1 dx\right.\\
&\quad + 2\int \bar{q}\partial_1\bar{p}\partial_2 p_1 dx + 2\int \bar{p}\partial_1 q_2\partial_2\bar{p}dx + 2\int q_2\partial_1\bar{p}\partial_2\bar{p}dx\\
&\quad \left. - 2\int \bar{p}\partial_2\bar{p}\partial_2 p_1 dx - \int \partial_2 p_2\bar{p}\partial_2\bar{p}dx - \int p_2(\partial_2\bar{p})^2 dx\right\}\\
&= I_{3a} + \cdots + I_{3j}.
\end{aligned}$$

Among all these $I_{3a}, \dots, I_{3j}$, we may estimate separately. First, by the assumption

$$\|Q_1\|_{L^\infty((0,\infty)\times\Omega)} \le \sqrt{2\eta_2}, \quad \|Q_2\|_{L^\infty((0,\infty)\times\Omega)} \le \sqrt{2\eta_2},$$

we see

$$\begin{aligned}
I_{3c} + I_{3g} + I_{3j} &= 2L_4\left\{\int p_2(\partial_1\bar{p})^2 dx + 2\int q_2\partial_1\bar{p}\partial_2\bar{p}dx - \int p_2(\partial_2\bar{p})^2 dx\right\}\\
&\le 2|L_4|\{\|p_2\|_{L^\infty}\|\partial_1\bar{p}\|^2 + \|q_2\|_{L^\infty}(\|\partial_1\bar{p}\|^2 + \|\partial_2\bar{p}\|^2)\\
&\quad + \|p_2\|_{L^\infty}\|\partial_2\bar{p}\|^2\}\\
&\le 2|L_4|\|h_2\|_{L^\infty}\{\|\partial_1\bar{p}\|^2 + \|\partial_1\bar{p}\|^2 + \|\partial_2\bar{p}\|^2 + \|\partial_2\bar{p}\|^2\}\\
&\le 4|L_4|\|h_2\|_{L^\infty}\|\nabla\bar{p}\|^2\\
&\le \frac{4\zeta}{1+4\sqrt{2}}\|\nabla\bar{p}\|^2\\
&\le \frac{2\zeta}{3}\|\nabla\bar{p}\|^2.
\end{aligned}$$

Here $h_2 = \sqrt{p_2^2 + q_2^2}$ is defined in the same way as (3.13), and we know from (3.23) and Proposition 3.1 that $\|h\|_{L^\infty(\Omega)} \le \sqrt{\eta_2} \le \frac{\zeta}{(1+4\sqrt{2})|L_4|}$. Next, similar to

the estimates for I_1 and I_4, we have

$$\begin{aligned}
&I_{3a} + I_{3b} + I_{3d} + I_{3e} + I_{3f} + I_{3h} + I_{3i}\\
&\quad \le \frac{\zeta}{9}(\|\nabla\bar{p}\|^2 + \|\nabla\bar{q}\|^2) + C(\|\Delta Q_1\|^2 + \|\Delta Q_2\|^2\| + 1)(\|\bar{p}\|^2 + \|\bar{q}\|^2).
\end{aligned}$$

Therefore,

$$I_3 \le \frac{7\zeta}{9}\|\nabla\bar{p}\|^2 + \frac{\zeta}{9}\|\nabla\bar{q}\|^2 + C(\|\Delta Q_1\|^2 + \|\Delta Q_2\|^2 + 1)(\|\bar{p}\|^2 + \|\bar{q}\|^2). \qquad (3.56)$$

We control I_6 in a manner similar to I_3:

$$I_6 \le \frac{\zeta}{9}\|\nabla\bar{p}\|^2 + \frac{7\zeta}{9}\|\nabla\bar{q}\|^2 + C(\|\Delta Q_1\|^2 + \|\Delta Q_2\|^2 + 1)(\|\bar{p}\|^2 + \|\bar{q}\|^2). \qquad (3.57)$$

Combining our estimates we have

$$\frac{1}{2}\frac{d}{dt}(\|\bar{p}\|^2 + \|\bar{q}\|^2) \le C(\|\Delta Q_1\|^2 + \|\Delta Q_2\|^2 + 1)(\|\bar{p}\|^2 + \|\bar{q}\|^2), \quad \forall t > 0. \qquad (3.58)$$

Here C is a positive constant that depends on Q_0, $\tilde{Q}$, and the coefficients of the system. Using Proposition 3.1 and (3.1), then a direct application of Gronwall's inequality leads to (3.49) □

4. Blow-Up for Large Initial Data

In this section we aim to prove Theorem 2.2 by constructing (large enough) initial data for which the solution of (1.6) exhibits a finite time blow-up of the L^2-norm. For this purpose we use a hedgehog type ansatz:

$$Q_{ij}(t,x) = \theta(t,|x|)S_{ij}, \quad \text{where} \quad S_{ij} = \left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right), \quad i,j = 1,2, \qquad (4.1)$$

on the spherical domain $B_{R_1}(0)\backslash B_{R_0}(0)$. Using the rotational symmetry of the ansatz and domain, we reduce the evolution of Q to a scalar one-dimensional scalar PDE for θ. For this it suffices to only take boundary conditions for θ. It turns out that boundary conditions of the form

$$\theta(t,R_0) = \theta(t,R_1) \ge 0, \quad \forall t > 0 \qquad (4.2)$$

are enough for our purposes. The main result of this section shows that any solution to (1.6) of the form (4.1) with boundary conditions (4.2) and large enough initial data blows up in finite time.

We begin with an evolution equation for θ.

Lemma 4.1. *Let Q be of the form* (4.1). *Then Q is a smooth solution of* (1.6) *if and only if θ is a smooth solution of*

$$\partial_t\theta = L_4\left(\frac{(\theta')^2}{2} + \frac{\theta\theta'}{r} + \theta\theta'' + \frac{6\theta^2}{r^2}\right) + \zeta\theta'' + \frac{\zeta\theta'}{r} - \frac{4\zeta\theta}{r^2} - a\theta - \frac{c\theta^3}{2}, \qquad (4.3)$$

where ζ is defined in (3.10).

Remark 4.1. By the coercivity condition (1.4) we know $\zeta > 0$.

Postponing the proof of Lemma 4.1 to Appendix D, we prove Theorem 2.2.

Proof of Theorem 2.2. Let $\theta_- = -\min\{\theta, 0\}$. Multiplying Eq. (4.3) by $-\theta_- r$, integrating over $[R_0, R_1]$ and integrating by parts gives:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{R_0}^{R_1} &\theta_-^2 r dr \\
&= -L_4 \int_{R_0}^{R_1} \left[\frac{(\theta_-')^2\theta_-}{2} r + \theta_-^2\theta_-' + \frac{6\theta_-^3}{r}\right] dr - \int_{R_0}^{R_1} L_4\theta_-^2\theta_-'' r + \zeta(\theta_-')^2 r dr \\
&\quad - \zeta\int_{R_0}^{R_1} \theta_-'\theta_- dr + \zeta\int_{R_0}^{R_1}\theta_-'\theta_- dr - \int_{R_0}^{R_1} 4\zeta\frac{\theta_-^2}{r} + (a\theta_-^2 + \frac{c}{2}\theta_-^4) r dr \\
&= -L_4 \int_{R_0}^{R_1} \left[\frac{(\theta_-')^2\theta_-}{2} r + \theta_-^2\theta_-' + \frac{6\theta_-^3}{r}\right] dr + L_4\int_{R_0}^{R_1} [\theta_-^2\theta_-' + 2(\theta_-')^2\theta_- r] dr \\
&\quad - \zeta\int_{R_0}^{R_1} (\theta_-')^2 r dr - 4\zeta\int_{R_0}^{R_1}\frac{\theta_-^2}{r} dr - \int_{R_0}^{R_1}\left(a\theta_-^2 + \frac{c}{2}\theta_-^4\right) r dr \\
&= \frac{3L_4}{2}\int_{R_0}^{R_1} (\theta_-')^2\theta_- r dr - 6L_4\int_{R_0}^{R_1}\frac{\theta_-^3}{r} dr - \zeta\int_{R_0}^{R_1}(\theta_-')^2 r dr - 4\zeta\int_{R_0}^{R_1}\frac{\theta_-^2}{r} dr \\
&\quad - \int_{R_0}^{R_1}\left(a\theta_-^2 + \frac{c}{2}\theta_-^4\right) r dr.
\end{aligned} \tag{4.4}$$

Next multiplying (4.3) by $-\partial_t\theta_- r$ and integrating over $[R_0, R_1]$, and integrating by parts wherever necessary gives:

$$\begin{aligned}
0 &\le \int_{R_0}^{R_1} (\partial_t\theta_-\partial_t\theta_-) r dr \\
&= -L_4\int_{R_0}^{R_1}\partial_t\theta_-\left[\frac{(\theta_-')^2}{2} + \frac{\theta_-\theta_-'}{r} + \frac{6\theta_-^2}{r^2}\right] r dr + L_4\int_{R_0}^{R_1}\partial_t\theta_-'\theta_-\theta_-' r dr \\
&\quad + L_4\int_{R_0}^{R_1}\partial_t\theta_-(\theta_-')^2 r dr + L_4\int_{R_0}^{R_1}\partial_t\theta_-\theta_-\theta_-' dr - \zeta\int_{R_0}^{R_1}\partial_t\theta_-'\theta_-' r dr \\
&\quad - \zeta\int_{R_0}^{R_1}\partial_t\theta_-\theta_-' dr + \zeta\int_{R_0}^{R_1}\partial_t\theta_-\theta_-' dr - \frac{d}{dt}\int_{R_0}^{R_1} 2\zeta\frac{\theta_-^2}{r} + \left(\frac{a\theta_-^2}{2} + \frac{c\theta_-^4}{8}\right) r dr \\
&= \frac{d}{dt}\int_{R_0}^{R_1}\left\{L_4\theta_-\left[\frac{(\theta_-')^2}{2} - \frac{2\theta_-^2}{r^2}\right] - \zeta\left[\frac{(\theta_-')^2}{2} + \frac{2\theta_-^2}{r^2}\right] - \left(\frac{a}{2}\theta_-^2 + \frac{c}{8}\theta_-^4\right)\right\} r dr.
\end{aligned}$$

Hence if we denote by

$$\mathcal{F}(t) \stackrel{\text{def}}{=} \int_{R_0}^{R_1}\left\{L_4\theta_-\left[\frac{(\theta_-')^2}{2} - \frac{2\theta_-^2}{r^2}\right] - \zeta\left[\frac{(\theta_-')^2}{2} + \frac{2\theta_-^2}{r^2}\right] - \left(\frac{a}{2}\theta_-^2 + \frac{c}{8}\theta_-^4\right)\right\} r dr, \tag{4.5}$$

we have $\mathcal{F}(t) \ge \mathcal{F}(0)$ and

$$\begin{aligned}
-2\zeta\int_{R_0}^{R_1}(\theta_-')^2 r dr &\ge 4\mathcal{F}(0) - \int_{R_0}^{R_1}\left(2L_4\theta_-\left[(\theta_-')^2 - \frac{4\theta_-^2}{r^2}\right] - \frac{8\zeta\theta_-^2}{r^2}\right) r dr \\
&\quad + \int_{R_0}^{R_1}\left(2a\theta_-^2 + \frac{c}{2}\theta_-^4\right) r dr.
\end{aligned} \tag{4.6}$$

We divide the argument into two cases: $L_4 < 0$ and $L_4 > 0$. Suppose first $L_4 < 0$. Then $\zeta > 0$ shows that $-\zeta\int_{R_0}^{R_1}(\theta_-')^2 r dr \ge -2\zeta\int_{R_0}^{R_1}(\theta_-')^2 r dr$. Using (4.6) in (4.4), we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{R_0}^{R_1} &\theta_-^2 r dr \\
&\ge \frac{3L_4}{2}\int_{R_0}^{R_1}(\theta_-')^2\theta_- r dr - 6L_4\int_{R_0}^{R_1}\frac{\theta_-^3}{r} dr - \int_{R_0}^{R_1} 4\zeta\frac{\theta_-^2}{r} + \left(a\theta_-^2 + \frac{c}{2}\theta_-^4\right) r dr \\
&\quad + 4\mathcal{F}(0) - \int_{R_0}^{R_1}\left(2L_4\theta_-\left[(\theta_-')^2 - \frac{4\theta_-^2}{r^2}\right] - \frac{8\zeta\theta_-^2}{r^2} - 2a\theta_-^2 - \frac{c}{2}\theta_-^4\right) r dr
\end{aligned}$$

which becomes:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{R_0}^{R_1} &\theta_-^2 r dr \\
&\ge -\frac{L_4}{2}\int_{R_0}^{R_1}\left((\theta_-')^2\theta_- r dr - 2L_4\frac{\theta_-^3}{r} - 4\zeta\frac{\theta_-^2}{r}\right) dr + 4\mathcal{F}(0) + a\int_{R_0}^{R_1}\theta_-^2 r dr \\
&\ge -\frac{L_4}{2}R_0\int_{R_0}^{R_1}(\theta_-')^2\theta_- dr + \frac{2L_4}{R_0}\int_{R_0}^{R_1}\theta_-^3 dr + 4\mathcal{F}(0) - |a|\int_{R_0}^{R_1}\theta_-^2 r dr.
\end{aligned} \tag{4.7}$$

Using Poincaré's inequality, we get

$$\int_{R_0}^{R_1}(\theta_-')^2\theta_- dr \ge \frac{4}{9}\int_{R_0}^{R_1}[(\theta_-^{3/2})']^2 dr \ge \frac{4\pi^2}{9(R_1 - R_0)^2}\int_{R_0}^{R_1}\theta_-^3 dr.$$

Therefore, if we choose R_0, R_1 so that

$$\frac{R_0^2\pi^2}{9(R_1 - R_0)^2} > 1, \tag{4.8}$$

the inequality (4.7) reduces to:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{R_0}^{R_1} &\theta_-^2 r dr \\
&\ge -\frac{L_4}{2}\left[\frac{4R_0\pi^2}{9(R_1 - R_0)^2} - \frac{4}{R_0}\right]\int_{R_0}^{R_1}\theta_-^3 dr - |a|\int_{R_0}^{R_1}\theta_-^2 r dr + 4\mathcal{F}(0) \\
&\ge M_0\left(\int_{R_0}^{R_1}\theta_-^2 r dr\right)^{\frac{3}{2}} - |a|\int_{R_0}^{R_1}\theta_-^2 r dr + 4\mathcal{F}(0).
\end{aligned} \tag{4.9}$$

Here

$$M_0 \stackrel{\text{def}}{=} -\frac{2L_4R_0}{\sqrt{R_1^4-R_0^4}}\left[\frac{\pi^2}{9(R_1-R_0)^2}-\frac{1}{R_0^2}\right].$$

Consequently, if one assumes $\int_{R_0}^{R_1}\theta_{0-}^2 r dr$ is suitably large, then (4.9) will force $\int_{R_0}^{R_1}\theta_-^2 r dr \to \infty$ in finite time, concluding the proof when $L_4 < 0$. The above argument with θ_- replaced by θ_+ will handle the case when $L_4 > 0$. □

Remark 4.2. Our technique does not seem to have a straightforward extension to domains which are not radially symmetric. In such domains, we do not know if a similar phenomenon occurs for large enough initial data.

5. The Physicality Preservation Argument

Our aim in this section is to prove Proposition 2.2, showing that certain eigenvalue constraints (the so-called physicality constraints) are preserved by the evolution equation (1.6). This issue is more subtle than the preservation of the L^∞-norm.

Proof of Proposition 2.2. Under the assumption $L_2 + L_3 = L_4 = 0$ and in $d = 2, 3$ system (1.6) becomes

$$\frac{\partial Q_{ij}}{\partial t} = 2L_1\Delta Q_{ij} - aQ_{ij} + b\left(Q_{ik}Q_{kj} - \frac{\operatorname{tr}(Q^2)}{d}\delta_{ij}\right) - c\operatorname{tr}(Q^2)Q_{ij}, \tag{5.1}$$

with $i, j = 1,\dots,d$. Note that when $d = 2$, the constant $2L_1$ is replaced by $\zeta = 2L_1 + L_2 + L_3 > 0$ in (5.1). Thus the argument below is also valid even if $L_2 + L_3 \neq 0$. For consistency, we only consider $L_2 + L_3 = 0$.

The proof will be done by using a nonlinear Trotter product formula (see for instance Chap. 15, Sec. 5 in Ref. 21). To briefly describe the idea, let us denote by $e^{2tL_1\Delta}R$ the solution of the heat equation in the whole space, starting from initial data R (where R is assumed to take values into the space of $d\times d$ matrices):

$$(e^{2tL_1\Delta}R)_{ij}(t,x) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{|x-y|^2}{8tL_1}}R_{ij}(y)dy, \quad i,j = 1,\dots,d, \tag{5.2}$$

and by $S(t,\bar S) \in \mathcal{S}^{(d)}$ the flow generated by the ODE part of (5.1), i.e. $S(t,\bar S)$ satisfies:

$$\begin{cases} \dfrac{\partial}{\partial t}S_{ij}(t,\bar S) = -aS_{ij} + b\left(S_{ik}S_{kj} - \dfrac{\operatorname{tr}(S^2)}{d}\delta_{ij}\right) - c\operatorname{tr}(S^2)S_{ij}, \\ S(0,\bar S)_{ij} = \bar S_{ij}, \end{cases} \tag{5.3}$$

with $i, j = 1,\dots,d$.

Then the Trotter formula provides a way of expressing the solution of (5.1) as a limit of successive superpositions of solutions of the heat equation part and the ODE part, namely by denoting $Q(t,x)$ the solution of (5.1) starting from initial data $Q_0(x)$ we have, loosely speaking:

$$Q(t,x) = \lim_{n\to\infty}(e^{2TL_1/n\Delta}S(T/n,\cdot))^nQ_0, \quad \forall t\in[0,T].$$

Let us note now that a set of the form

$$\{Q\in\mathbb{R}^{d\times d}, Q = Q^t; \beta\le\lambda_i(Q)\le\gamma, \text{ for all eigenvalues } \lambda_i(Q) \text{ of } Q\}$$

is convex (as the largest eigenvalue is a convex function of the matrix, while the smallest eigenvalue is a concave function, see for instance Ref. 5).

It is then clear that if we manage to show that both $e^{2tL_1\Delta}$ and $S(t,\cdot)$ preserve the closed convex hull of the range of the initial data then this will also hold for the limit $Q(t,x)$. The arguments consist of three steps.

Step 1: The convex hull preservation under the heat flow.

Denote

$$\Phi_n(y) \stackrel{\text{def}}{=} \begin{cases} (4\pi t)^{d/2}\left(\displaystyle\int_{B_n(0)} e^{-\frac{|y|^2}{8tL_1}}dy\right)^{-1} e^{-\frac{|y|^2}{8tL_1}} & \text{for } |y|\le n, \\ 0 & \text{for } |y|>n. \end{cases}$$

For any $f\in L^1(\mathbb{R}^d)$, we obtain that

$$\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} f(x-y)\Phi_n(y)dy \to e^{2tL_1\Delta}f(x), \tag{5.4}$$

pointwise as $n\to\infty$.

Now, let us observe that the measures $\mu_n(y) = \Phi_n(y)dy$ belong to the set $\mathcal{M}_{+1}(B_n(0))$ of regular Borel probability measures supported on $B_n(0)$. The extremal set of the convex set $\mathcal{M}_{+1}(B_n(0))$ consists of delta measures δ_x with $x\in B_n(0)$, where $\delta_x(E) = 1$ if and only if $x\in E$ for any Borel set $E\subset B_n(0)$; see for instance, (Example 8.16, p. 129 in Ref. 19). On the other hand, by Krein–Milman theorem (see also Chap. 8 in Ref. 19), we know that μ_n can be written as a limit of convex combinations of extremals in the weak-star topology of $\mathcal{M}_{+1}(B_n(0))$ interpreted as a subset of the dual space $[C(B_n(0))]^*$, i.e.

$$\sum_{j=1}^{J(k)}\theta_j^k\delta_{x_j^k} \stackrel{\star}{\rightharpoonup} \mu_n \quad \text{as } k\to\infty,$$

with the convexity condition

$$\sum_{j=1}^{J(k)}\theta_j^k = 1,$$

where $\theta_j^k\ge 0, \forall 1\le j\le J(k), k\in\mathbb{N}$. Therefore, for any $x\in\mathbb{R}^d$ and n large enough so that $|x|<n$, it holds

$$\lim_{k\to\infty}\sum_{j=1}^{J(k)}\theta_j^k f(x_j^k - x) = \int_{\mathbb{R}^d} f(x-y)d\mu_n(y)dy.$$

After passing to the limit $n\to\infty$, we henceforth get $(e^{2tL_1\Delta}f)(x)$ is in the convex hull of the image of the initial data f.

Step 2: The physicality preservation under the ODE.

We divide the argument into two cases.

The 2D case: We consider the ODE:

$$\frac{d}{dt}Q = -\frac{\partial f_B}{\partial Q} + \frac{1}{2}\operatorname{tr}\left(\frac{\partial f_B}{\partial Q}\right)\mathbb{I}, \tag{5.5}$$

for Q denoting 2×2 matrices, where we use the standard bulk term:

$$f_B(Q) = \frac{a}{2}\operatorname{tr}(Q^2) - \frac{b}{3}\operatorname{tr}(Q^3) + \frac{c}{4}(\operatorname{tr}(Q^2))^2. \tag{5.6}$$

Taking into account the specific form (5.12) of f_B, Eq. (5.11) becomes:

$$\frac{d}{dt}Q = -aQ + b\left(Q^2 - \frac{1}{2}\operatorname{tr}(Q^2)\mathbb{I}\right) - cQ\operatorname{tr}(Q^2). \tag{5.7}$$

Multiplying the equation scalarly by Q, and using that $\operatorname{tr}(Q)=0$ and also the fact, specific to 2×2 Q-tensors, that $\operatorname{tr}(Q^3)=0$ we obtain:

$$\frac{1}{2}\frac{d}{dt}|Q|^2 = -a|Q|^2 - c|Q|^4. \tag{5.8}$$

Let $g(|Q|) \stackrel{\text{def}}{=} -a|Q|^2 - c|Q|^4 = -c|Q|^2(|Q|^2 + \frac{a}{c})$. We consider the following two possibilities.

Case A: $a \geq 0$. Then $g(|Q|) < 0$, for $|Q| \neq 0$. Hence (5.8) implies $|Q(t)|^2 \leq |Q(0)|^2$.

Case B: $a < 0$. Then

$$g(|Q|) < 0, \quad \text{for} \quad |Q|^2 > -\frac{a}{c} > 0. \tag{5.9}$$

We claim that:

$$|Q(0)| \leq \sqrt{-\frac{a}{c}} \Rightarrow |Q(t)| \leq \sqrt{-\frac{a}{c}}, \quad \forall t > 0. \tag{5.10}$$

In order to prove the claim let us assume for contradiction that there exists a $\varepsilon > 0$ such that at some positive time $|Q(t)| = \sqrt{-\frac{a}{c}} + \varepsilon$ and let us denote by t_0 the smallest such positive time. Then Eq. (5.8) together with (5.9) imply that $\frac{d}{dt}|Q|^2 < 0$ hence there exists an earlier time $t_{-1} < t_0$ so that $|Q(t_{-1})| = \sqrt{-\frac{a}{c}} + \varepsilon$ contradicting our hypothesis on t_0 and proving the claim (5.10).

The 3D case: We consider the ODE:

$$\frac{d}{dt}Q = -\frac{\partial f_B}{\partial Q} + \frac{1}{3}\operatorname{tr}\left(\frac{\partial f_B}{\partial Q}\right)\mathbb{I}, \tag{5.11}$$

where we use the standard bulk term:

$$f_B(Q) = \frac{a}{2}\operatorname{tr}(Q^2) - \frac{b}{3}\operatorname{tr}(Q^3) + \frac{c}{4}(\operatorname{tr}(Q^2))^2. \tag{5.12}$$

Taking into account the specific form (5.12) of f_B, Eq. (5.11) becomes:

$$\frac{d}{dt}Q = -aQ + b\left(Q^2 - \frac{1}{3}\operatorname{tr}(Q^2)\mathbb{I}\right) - cQ\operatorname{tr}(Q^2). \tag{5.13}$$

Now take the scalar product of this equation with Q. (Recall, scalar product of matrices A, B is defined by $(A,B) \stackrel{\text{def}}{=} \operatorname{tr}(AB)$ and $|A| = \sqrt{\operatorname{tr}(A^2)}$.) Using additionally the fact that $\operatorname{tr}(Q)=0$ gives

$$\frac{1}{2}\frac{d}{dt}|Q|^2 = -a|Q|^2 + b\operatorname{tr}(Q^3) - c|Q|^4. \tag{5.14}$$

We recall that (see for instance Ref. 15) we have $|\operatorname{tr}(Q^3)| \leq \frac{|Q|^3}{\sqrt{6}}$ which used in (5.14) (under assumptions (1.3)) implies:

$$\frac{d}{dt}|Q|^2 \leq -a|Q|^2 + \frac{b}{\sqrt{6}}|Q|^3 - c|Q|^4. \tag{5.15}$$

Let us denote $h(Q) \stackrel{\text{def}}{=} -a|Q|^2 + \frac{b}{\sqrt{6}}|Q|^3 - c|Q|^4$. Then the roots of $\frac{h(Q)}{|Q|^2}$ are $\sqrt{\frac{2}{3}}s_\pm$, with

$$s_\pm = \frac{b \pm \sqrt{b^2 - 24ac}}{4c}. \tag{5.16}$$

Then

$$h(|Q|) < 0 \quad \text{for} \quad |Q| > \sqrt{\frac{2}{3}}s_+. \tag{5.17}$$

Taking into account (5.15) we claim that, if we denote by Q_0 the initial data of the ODE (5.13):

$$|Q_0|^2 \leq \frac{2}{3}s_+^2 \Rightarrow |Q(t)|^2 \leq \frac{2}{3}s_+^2, \quad \forall t > 0. \tag{5.18}$$

Indeed, if our claim were false, for any $\varepsilon > 0$, let us denote by $t_0(\varepsilon)$ the first time when $|Q|^2$ reaches the value $\frac{2}{3}s_+^2 + \varepsilon$, i.e.

$$|Q(t_0)|^2 = \frac{2}{3}s_+^2 + \varepsilon, \quad \text{and} \quad |Q(t)|^2 < \frac{2}{3}s_+^2 + \varepsilon, \quad \forall t < t_0.$$

Then (5.17) and (5.15) imply that $\frac{d}{dt}|Q(t_0)|^2 < 0$. Hence there exists a time $\tilde{t}_0 < t_0$, such that $|Q(\tilde{t}_0)| > \frac{2}{3}s_+^2 + \varepsilon$, which contradicts our choice of t_0. Thus for $|Q_0|^2 \leq \frac{2}{3}s_+^2$, Eq. (5.13) has a solution that is bounded, and the right-hand side of (5.13) is globally Lipschitz on the ball where the solution evolves. As a consequence, we obtain that for $|Q_0|^2 \leq \frac{2}{3}s_+^2$, Eq. (5.13) has a unique global solution evolving with the property that $|Q(t)|^2 \leq \frac{2}{3}s_+^2$.

Let us consider now the system:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\lambda_1[2c(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + a] + b\left(\frac{\lambda_1^2}{3} - \frac{2}{3}\lambda_2^2 - \frac{2}{3}\lambda_1\lambda_2\right), \\ \frac{d\lambda_2}{dt} &= -\lambda_2[2c(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + a] + b\left(\frac{\lambda_2^2}{3} - \frac{2}{3}\lambda_1^2 - \frac{2}{3}\lambda_1\lambda_2\right). \end{aligned} \tag{5.19}$$

The right-hand side of the system is a locally Lipschitz function so the system has a solution locally in time (in fact with some more work global in time and bounded, using arguments similar to the ones before for the matrix system).

On the other hand, let us note now that if we take

$$Q_0 = \begin{pmatrix} \lambda_1^0 & 0 & 0 \\ 0 & \lambda_2^0 & 0 \\ 0 & 0 & -\lambda_1^0 - \lambda_2^0 \end{pmatrix},$$

then

$$\bar{Q}(t) = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & -\lambda_1(t) - \lambda_2(t) \end{pmatrix}.$$

Hence if $\lambda_1(t), \lambda_2(t)$ are solutions of (5.19) with initial data $(\lambda_1^0, \lambda_2^0)$ then $\bar{Q}(t)$ is a solution of (5.13) with initial data Q_0. On the other hand, by uniqueness of solutions of (5.13), it must be the only solution corresponding to the diagonal initial data Q_0. Thus we have shown that a diagonal initial data will generate a diagonal solution.

For an arbitrary, non-diagonal initial data $\tilde{Q}_0$, since $\tilde{Q}_0$ is a symmetric matrix, there exists a matrix $R \in O(3)$, such that:

$$R\tilde{Q}_0 R^t = \begin{pmatrix} \tilde{\lambda}_1^0 & 0 & 0 \\ 0 & \tilde{\lambda}_2^0 & 0 \\ 0 & 0 & -\tilde{\lambda}_1^0 - \tilde{\lambda}_2^0 \end{pmatrix},$$

where $(\tilde{\lambda}_1^0, \tilde{\lambda}_2^0, -\tilde{\lambda}_1^0 - \tilde{\lambda}_2^0)$ are the eigenvalues of $\tilde{Q}_0$. If $Q(t)$ is a solution of (5.13) with initial data $\tilde{Q}_0$, then multiplying on the left by *the time-independent matrix* R, and on the right by *the time-independent matrix* R^t, using the fact that $RR^t = \mathbb{I}$ (as $R \in O(3)$), we obtain the following equation:

$$\begin{aligned}\frac{d}{dt}RQ(t)R^t &= -aRQ(t)R^t + b\left(RQ(t)R^tRQ(t)R^t - \frac{1}{3}\operatorname{tr}(RQ(t)R^tRQ(t)R^t)\mathbb{I}\right) \\ &\quad - cRQ(t)R^t\operatorname{tr}(RQ(t)R^tRQ(t)R^t).\end{aligned} \tag{5.20}$$

Hence if we denote by $M(t) \stackrel{\text{def}}{=} RQ(t)R^t$, we conclude that M satisfies Eq. (5.13) with initial data

$$M_0 \stackrel{\text{def}}{=} R\tilde{Q}_0R^t = \begin{pmatrix} \tilde{\lambda}_1^0 & 0 & 0 \\ 0 & \tilde{\lambda}_2^0 & 0 \\ 0 & 0 & -\tilde{\lambda}_1^0 - \tilde{\lambda}_2^0 \end{pmatrix}.$$

Since the initial data is diagonal, we infer by previous arguments that $M(t)$ is diagonal for all times and

$$M(t) = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & -\lambda_1(t) - \lambda_2(t) \end{pmatrix},$$

with $\lambda_1(t), \lambda_2(t)$ solutions of (5.19) with initial data $(\tilde{\lambda}_1^0, \tilde{\lambda}_2^0)$. Thus we obtain that

$$M(t) = RQ(t)R^t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & -\lambda_1(t) - \lambda_2(t) \end{pmatrix},$$

hence

$$Q(t) = R^t \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & -\lambda_1(t) - \lambda_2(t) \end{pmatrix} R.$$

This shows that we can reduce the study of the system (5.13) with an arbitrary initial data to the study of the system (5.19).

The bound (5.18) expressed in terms of eigenvalues λ_1, λ_2 becomes

$$2[(\lambda_1^0)^2 + (\lambda_2^0)^2 + (\lambda_1^0\lambda_2^0)] \le \frac{2}{3}s_+^2, \quad \forall t \ge 0. \tag{5.21}$$

Note that $\frac{3\lambda_i^2}{4} \le \lambda_i^2 + \mu^2 + \lambda_i\mu$, hence the last bound implies

$$2[(\lambda_1^0)^2 + (\lambda_2^0)^2 + \lambda_1^0\lambda_2^0)] \le \frac{2}{3}s_+^2 \Rightarrow |\lambda_1(t)|, \quad |\lambda_2(t)| \le \frac{2}{3}s_+, \quad \forall t \ge 0. \tag{5.22}$$

We consider now the difference $\lambda_1(t) - \lambda_2(t)$, and out of inspection from the system (5.19) we see that it satisfies an equation of the form:

$$\frac{d}{dt}(\lambda_1(t) - \lambda_2(t)) = (\lambda_1(t) - \lambda_2(t))G(\lambda_1(t), \lambda_2(t)),$$

for some function G. This shows that if $\lambda_1^0 \le \lambda_2^0$, then $\lambda_1(t) \le \lambda_2(t), \forall t > 0$. We assume without loss of generality that this is indeed the case.

We aim to show now that $\lambda_1(0) \ge -\frac{s_+}{3}$ implies $\lambda_1(t) \ge -\frac{s_+}{3}$ for all $t > 0$. We assume for contradiction that this is not the case and there exists a first time t_0 after which $\lambda_1(t) + \frac{s_+}{3}$ becomes negative, i.e. $\lambda_1(t_0) = -\frac{s_+}{3}$ and there exists a $\delta > 0$ so that $\lambda_1(t) < -\frac{s_+}{3}$ for $t \in (t_0, t_0 + \delta)$. The right-hand side of Eq. (5.19) evaluated at t_0 becomes:

$$\frac{2}{3}(cs_+ - b)\left(\lambda_2(t_0) + \frac{s_+}{3}\right)\left(\lambda_2(t_0) - \frac{2s_+}{3}\right). \tag{5.23}$$

Then Eq. (5.21) implies $\lambda_2(t_0) \in [-\frac{s_+}{3}, \frac{2s_+}{3}]$. If $\lambda_2(t_0) \in \{-\frac{s_+}{3}, \frac{2s_+}{3}\}$, then for all $t > 0$ we have $\lambda_1(t) = -\frac{s_+}{3}$, $\lambda_2(t) = \lambda_2(t_0)$, due to the fact that the pairs $(-\frac{s_+}{3}, \frac{2s_+}{3}), (-\frac{s_+}{3}, -\frac{s_+}{3})$ are stationary points of the system (5.19). Thus we assume without loss of generality that $\lambda_2(t_0) \in (-\frac{s_+}{3}, \frac{2s_+}{3})$ and henceforth, taking into account assumption (2.4), we infer that the expression in (5.23) is positive so $\frac{d\lambda_1}{dt}(t_0) > 0$, which contradicts our assumption that there exists a $\delta > 0$ so that $\lambda_1(t) < -\frac{s_+}{3}$ for $t \in (t_0, t_0 + \delta)$.

Thus we have shown that if $-\frac{s_+}{3} \le \lambda_1^0 \le \lambda_2^0 \le \frac{2s_+}{3}$, then $\lambda_1(t) \in [-\frac{s_+}{3}, \frac{2s_+}{3}]$ for all $t > 0$. The fact that $\lambda_1(t) \le \lambda_2(t)$ for all times ensures $-\frac{s_+}{3} \le \lambda_2(t)$ for all times.

Step 3: The Trotter product formula.

We use Proposition 5.3 on p. 313 in Ref. 21. To this end, we denote

$$V_n(t) \overset{\text{def}}{=} e^{s\Delta}S(s,\cdot)(e^{2TL_1/n\Delta}S(T/n,\cdot))^k Q_0,$$

for $t = \frac{kT}{n} + s$ with $0 \le s < \frac{T}{k}$. Then Proposition 5.3 ensures that we have

$$\|Q(t,\cdot) - V_n(t)\|_{H^k} \le C(\|Q_0\|_{H^k})n^{-\gamma}, \tag{5.24}$$

for $0 < \gamma < 1$, and all $t \in [0,T]$. □

Appendix A. Derivation of the Gradient Flow Equation

Our goal in this subsection is to derive (1.6), the equation for the gradient flow of $\mathcal{E}$.

Proposition A.1. *The gradient flow defined by* (1.5) *satisfies* (1.6).

Proof. Choosing a test function $\varphi \in C_c^\infty(\Omega, M^{d\times d}(\mathbb{R}))$ and integrating by parts gives:

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(Q+t\varphi)|_{t=0}\\
&\quad= \frac{d}{dt}\int_\Omega \mathcal{F}_{\text{el}}(Q+t\varphi)dx + \frac{d}{dt}\int_\Omega \mathcal{F}_{\text{bulk}}(Q+t\varphi)dx\\
&\quad= \int_\Omega 2L_1\partial_k\varphi_{ij}\partial_k Q_{ij} + L_2(\partial_j\varphi_{ik}\partial_k Q_{ij} + \partial_k\varphi_{ij}\partial_j Q_{ik})dx\\
&\qquad + L_3\int_\Omega \partial_j\varphi_{ij}\partial_k Q_{ik} + \partial_k\varphi_{ik}\partial_j Q_{ij}dx\\
&\qquad + L_4\int_\Omega \varphi_{lk}\partial_k Q_{ij}\partial_l Q_{ij} + Q_{lk}\partial_k\varphi_{ij}\partial_l Q_{ij} + Q_{lk}\partial_k Q_{ij}\partial_l\varphi_{ij}dx\\
&\qquad + \int_\Omega aQ_{ij}\varphi_{ij} - \frac{b}{3}(\varphi_{ik}Q_{kj}Q_{ji} + Q_{ik}\varphi_{kj}Q_{ji} + Q_{ik}Q_{kj}\varphi_{ji})\\
&\qquad + c\operatorname{tr}(Q^2)Q_{ij}\varphi_{ij}dx\\
&\quad= -2\int_\Omega (L_1\Delta Q_{ij} + L_2\partial_j\partial_k Q_{ik} + L_3\partial_j\partial_k Q_{ik})\varphi_{ij}dx\\
&\qquad - 2L_4\int_\Omega (\partial_l Q_{ij}\partial_k Q_{lk} + \partial_l\partial_k Q_{ij}Q_{lk} - \partial_i Q_{kl}\partial_j Q_{kl})\varphi_{ij}dx\\
&\qquad + \int_\Omega aQ_{ij}\varphi_{ij} - bQ_{jk}Q_{ki}\varphi_{ij} + c\operatorname{tr}(Q^2)Q_{ij}\varphi_{ij}dx.
\end{aligned}$$

Since φ is arbitrary this allows the identification:

$$\begin{aligned}
\left(\frac{\delta\mathcal{E}}{\delta Q}\right)_{ij} &= -2L_1\Delta Q_{ij} + aQ_{ij} - bQ_{jk}Q_{ki} + c\operatorname{tr}(Q^2)Q_{ij}\\
&\quad - 2(L_2+L_3)\partial_j\partial_k Q_{ik} - 2L_4\partial_l Q_{ij}\partial_k Q_{lk} - 2L_4\partial_l\partial_k Q_{ij}Q_{lk}\\
&\quad + L_4\partial_i Q_{kl}\partial_j Q_{kl}.
\end{aligned}$$

Substituting this in (1.5) and choosing μ to enforce the symmetry constraint $Q_{ij} = Q_{ji}$ forces

$$\mu_{ij} - \mu_{ji} = (L_2+L_3)(\partial_i\partial_k Q_{jk} - \partial_j\partial_k Q_{ik}).$$

Similarly, choosing λ to enforce the trace-free constraint $Q_{ii} = 0$ forces

$$\lambda = -\frac{b}{2}\operatorname{tr}(Q^2) - (L_2+L_3)\partial_l\partial_k Q_{lk} + \frac{L_4}{2}|\nabla Q|^2.$$

Substituting λ, μ and $\delta\mathcal{E}/\delta Q$ in (1.5) immediately gives (1.6). □

Appendix B. The Reduction of the Landau–de Gennes to Oseen–Frank in 2D

Our goal in this Appendix is to show that if Q takes a special form, then the Landau–de Gennes energy can be reduced to the Oseen–Frank energy functional. We recall that the 3D Oseen–Frank energy functional is

$$\begin{aligned}
W &= K_1(\operatorname{div} n)^2 + K_2|n\cdot\operatorname{curl} n|^2 + K_3|n\wedge\operatorname{curl} n|^2\\
&\quad + (K_2+K_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2],
\end{aligned} \tag{B.1}$$

where K_i are elastic constants measuring the relative strength of the various types of spatial variations of the unit vectors $n \in \mathbb{S}^2$ (see Ref. 11). In 2D we clarify that for a vector function n given by

$$n = (n_1, n_2, 0),$$

we have

$$\operatorname{curl} n = (0, 0, \partial_1 n_2 - \partial_2 n_1),$$

and hence

$$n\cdot\operatorname{curl} n = 0, \quad |n\wedge\operatorname{curl} n|^2 = |\operatorname{curl} n|^2.$$

On the other hand, $n_1^2 + n_2^2 = 1$ implies

$$(n_1,n_2,0)\cdot\partial_1(n_1,n_2,0) = (n_1,n_2,0)\cdot\partial_2(n_1,n_2,0) = 0,$$

and hence $(\partial_1 n_1, \partial_1 n_2, 0) = c(\partial_2 n_1, \partial_2 n_2, 0)$ for some $c \in \mathbb{R}$. Thus $\partial_1 n_1\partial_2 n_2 = \partial_2 n_1\partial_1 n_2$, which shows

$$\operatorname{tr}(\nabla n)^2 = (\operatorname{div} n)^2.$$

Consequently, the Oseen–Frank energy in 2D reduces to:

$$\begin{aligned} W_{2D} &= K_1(\operatorname{div} n)^2 + K_3|\operatorname{curl} n|^2 + (K_2+K_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2] \\ &= K_1(\operatorname{div} n)^2 + K_3|\operatorname{curl} n|^2. \end{aligned} \tag{B.2}$$

If Q takes the special form:

$$Q = s\left(n \otimes n - \frac{\mathbb{I}}{2}\right),$$

where s is a constant, then the 2D Landau–de Gennes energy functional reads:

$$\begin{aligned} \mathcal{E}(Q,\nabla Q) &= L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik} + L_4 Q_{lk}\partial_l Q_{ij}\partial_k Q_{ij} \\ &= 2L_1 s^2[|\operatorname{curl} n|^2 + \operatorname{tr}(\nabla n)^2] + L_2 s^2[|\operatorname{curl} n|^2 + \operatorname{tr}(\nabla n)^2] \\ &\quad + L_3 s^2[(\operatorname{div} n)^2 + |\operatorname{curl} n|^2] + L_4 s^3[|\operatorname{curl} n|^2 - \operatorname{tr}(\nabla n)^2] \\ &= (2L_1+L_2)s^2[|\operatorname{curl} n|^2 + \operatorname{tr}(\nabla n)^2] + L_3 s^2[(\operatorname{div} n)^2 + |\operatorname{curl} n|^2] \\ &\quad + L_4 s^3[|\operatorname{curl} n|^2 - \operatorname{tr}(\nabla n)^2] \\ &= (\tilde{L}_1 s^2 + L_3 s^2 - L_4 s^3)(\operatorname{div} n)^2 + (\tilde{L}_1 s^2 + L_3 s^2 + L_4 s^3)|\operatorname{curl} n|^2 \\ &\quad + (\tilde{L}_1 s^2 - L_4 s^3)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2] \\ &= (\tilde{L}_1 s^2 + L_3 s^2 - L_4 s^3)(\operatorname{div} n)^2 + (\tilde{L}_1 s^2 + L_3 s^2 + L_4 s^3)|\operatorname{curl} n|^2. \end{aligned} \tag{B.3}$$

Here we denote

$$\tilde{L}_1 = 2L_1 + L_2.$$

We let

$$K_1 = \tilde{L}_1 s^2 + L_3 s^2 - L_4 s^3, \quad K_3 = \tilde{L}_1 s^2 + L_3 s^2 + L_4 s^3, \tag{B.4}$$

then $\mathcal{E}(Q,\nabla Q)$ is reduced to $W_{2\mathrm{D}}$. And conversely, $\tilde{L}_1, L_3, L_4$ can be expressed in terms of K_i in the following way:

$$L_3 s^2 = K_1, \quad 2L_4 s^3 = K_3 - K_1, \quad \tilde{L}_1 s^2 = \frac{K_3 - K_1}{2}. \tag{B.5}$$

Remark B.1. Note that if $L_4 = 0$, then $K_1 \equiv K_3$ in (B.4), which indicates that the Oseen–Frank energy (B.2) cannot be completely recovered without L_4. Therefore, the cubic term is necessary.

Appendix C. Energy Coercivity in 2D

In this Appendix we prove that the condition (1.4) (reproduced as (C.1) below) is equivalent to coercivity in two dimensions, and quantitatively gives the estimate

(3.22) (reproduced as (C.2) below). As mentioned earlier, the three-dimensional analog can be found in Refs. 8 and 13.

Lemma C.1. *If $n = 2$ and the elastic constants L_1, L_2, L_3 satisfy*

$$L_1 + L_2 > 0, \quad L_1 + L_3 > 0, \tag{C.1}$$

then for all $x \in \Omega$ we have

$$(L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik})(x) \geq \nu|\nabla Q|^2(x), \tag{C.2}$$

where

$$\nu \stackrel{\text{def}}{=} \min\{L_1 + L_2, L_1 + L_3\} > 0. \tag{C.3}$$

Proof. Due to the special structure (3.6) of Q in 2D, the elastic energy can be rewritten as

$$\begin{aligned} &(L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik}) \\ &\quad = (2L_1 + L_2 + L_3)(|\nabla p|^2 + |\nabla q|^2) + 2(L_3 - L_2)\partial_1 p\partial_2 q + 2(L_2 - L_3)\partial_2 p\partial_1 q \\ &\quad = \chi^T \mathcal{B}\chi, \end{aligned} \tag{C.4}$$

where

$$\chi = (\partial_1 p, \partial_2 p, \partial_1 q, \partial_2 q)^T \in \mathbb{R}^4,$$

and

$$\mathcal{B} = \begin{pmatrix} 2L_1 + L_2 + L_3 & 0 & 0 & L_3 - L_2 \\ 0 & 2L_1 + L_2 + L_3 & L_2 - L_3 & 0 \\ 0 & L_2 - L_3 & 2L_1 + L_2 + L_3 & 0 \\ L_3 - L_2 & 0 & 0 & 2L_1 + L_2 + L_3 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

By a direct calculation, we see that the eigenvalues of $\mathcal{B}$ are

$$\lambda_1 = \lambda_2 = 2(L_1 + L_2) \quad \text{and} \quad \lambda_3 = \lambda_4 = 2(L_1 + L_3).$$

Consequently,

$$\begin{aligned} &(L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik}) \\ &\quad = \chi^T\mathcal{B}\chi \geq \min\{\lambda_1, \lambda_2\}|\chi|^2 = 2\nu[|\nabla p|^2 + |\nabla q|^2] = \nu|\nabla Q|^2, \end{aligned}$$

as desired. □

Appendix D. Calculations for the Hedgehog Ansatz

In this section we prove Lemma 4.1 deriving the evolution of θ that reduces the gradient flow dynamics in the case of the Hedgehog ansatz.

D.1. *Calculations for Hedgehog type solutions: L_1 and L_4 parts*

We begin by computing the first derivative of Q_{ij} in terms of θ:

$$Q_{ij,k} = \partial_k Q_{ij} = \theta' \frac{x_k}{|x|}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right) + \theta\left(\frac{\delta_{ik}x_j + \delta_{jk}x_i}{|x|^2} - \frac{2x_i x_j x_k}{|x|^4}\right).$$

Next we compute the second derivative of Q_{ij} in terms of θ:

$$\begin{aligned}
Q_{ij,kl} &= \theta'' \frac{x_k x_l}{|x|^2}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right) + \theta'\left(\frac{\delta_{kl}}{|x|} - \frac{x_k x_l}{|x|^3}\right)\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right)\\
&\quad + \theta' \frac{x_k}{|x|}\left(\frac{\delta_{il}x_j}{|x|^2} + \frac{\delta_{jl}x_i}{|x|^2} - \frac{2x_i x_j x_l}{|x|^4}\right) + \theta' \frac{x_l}{|x|}\left(\frac{\delta_{ik}x_j}{|x|^2} + \frac{\delta_{jk}x_i}{|x|^2} - \frac{2x_i x_j x_k}{|x|^4}\right)\\
&\quad + \theta\left[\frac{\delta_{ik}\delta_{jl}}{|x|^2} - \frac{2\delta_{ik}x_j x_l}{|x|^4} + \frac{\delta_{il}\delta_{jk}}{|x|^2} - \frac{2\delta_{jk}x_l x_i}{|x|^4}\right]\\
&\quad - \theta\left[\frac{2(\delta_{il}x_j x_k + \delta_{jl}x_i x_k + \delta_{kl}x_i x_j)}{|x|^4} - \frac{8x_i x_j x_k x_l}{|x|^6}\right].
\end{aligned}$$

Thus for the term $2L_4 Q_{ij,l}Q_{lk,k}$ in (1.6), we have:

$$\begin{aligned}
2L_4 Q_{ij,l}Q_{lk,k} &= 2L_4\left[\theta' \frac{x_l}{|x|}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right) + \theta\left(\frac{\delta_{il}x_j}{|x|^2} + \frac{\delta_{jl}x_i}{|x|^2} - \frac{2x_i x_j x_l}{|x|^4}\right)\right]\\
&\quad \times\left[\theta' \frac{x_k}{|x|}\left(\frac{x_l x_k}{|x|^2} - \frac{\delta_{lk}}{2}\right) + \theta\left(\frac{\delta_{lk}x_k}{|x|^2} + \frac{x_l\delta_{kk}}{|x|^2} - \frac{2x_l x_k x_k}{|x|^4}\right)\right]\\
&= 2L_4\left[\theta' \frac{x_l}{|x|}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right) + \theta\left(\frac{\delta_{il}x_j}{|x|^2} + \frac{x_i\delta_{jl}}{|x|^2} - \frac{2x_i x_j x_l}{|x|^4}\right)\right]\\
&\quad \times\left(\theta' \frac{x_l}{2|x|} + \theta\frac{x_l}{|x|^2}\right)\\
&= 2L_4\theta'\left(\frac{\theta'}{2} + \frac{\theta}{|x|}\right)\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right).
\end{aligned}$$

For the term $2L_4 Q_{kl}Q_{ij,kl}$, we get:

$$\begin{aligned}
&2L_4 Q_{kl}Q_{ij,kl}\\
&\quad = 2L_4\theta\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right)\left[\theta'' \frac{x_k x_l}{|x|^2} + \theta'\left(\frac{\delta_{kl}}{|x|} - \frac{x_k x_l}{|x|^3}\right)\right]\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right)\\
&\qquad + 2L_4\theta\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right)\left[\theta' \frac{x_k}{|x|}\left(\frac{\delta_{il}x_j}{|x|^2} + \frac{x_i\delta_{jl}}{|x|^2} - \frac{2x_i x_j x_l}{|x|^4}\right)\right]\\
&\qquad + 2L_4\theta\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right)\left[\theta' \frac{x_l}{|x|}\left(\frac{\delta_{ik}x_j}{|x|^2} + \frac{x_i\delta_{jk}}{|x|^2} - \frac{2x_i x_j x_k}{|x|^4}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\qquad + 2L_4\theta\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right)\theta\left[\frac{\delta_{ik}\delta_{jl}}{|x|^2} - \frac{2\delta_{ik}x_j x_l}{|x|^4} + \frac{\delta_{il}\delta_{jk}}{|x|^2} - \frac{2x_l x_i\delta_{jk}}{|x|^4}\right]\\
&\qquad - 2L_4\theta\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right)\theta\left[\frac{2(\delta_{il}x_j x_k + x_i\delta_{jl}x_k + x_i x_j\delta_{kl})}{|x|^4} - \frac{8x_i x_j x_k x_l}{|x|^6}\right]\\
&\quad = L_4\left(\theta\theta'' - \frac{\theta\theta'}{|x|} + \frac{4\theta^2}{|x|^2}\right)\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right).
\end{aligned}$$

For $-L_4 Q_{kl,i}Q_{kl,j}$, we have:

$$\begin{aligned}
-L_4 Q_{kl,i}Q_{kl,j} &= -L_4\left[\theta' \frac{x_i}{|x|}\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right) + \theta\left(\frac{\delta_{ki}x_l}{|x|^2} + \frac{x_k\delta_{il}}{|x|^2} - \frac{2x_k x_l x_i}{|x|^4}\right)\right]\\
&\quad \times\left[\theta' \frac{x_j}{|x|}\left(\frac{x_k x_l}{|x|^2} - \frac{\delta_{kl}}{2}\right) + \theta\left(\frac{\delta_{kj}x_l}{|x|^2} + \frac{x_k\delta_{jl}}{|x|^2} - \frac{2x_k x_l x_j}{|x|^4}\right)\right]\\
&= -L_4(\theta')^2\frac{x_i x_j}{2|x|^2} - \frac{2L_4\theta^2}{|x|^2}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right),
\end{aligned}$$

and

$$\frac{L_4}{2}|\nabla Q|^2\delta_{ij} = L_4\left[\frac{\theta^2}{|x|^2} + \frac{(\theta')^2}{4}\right]\delta_{ij}.$$

For terms related to L_1, we get:

$$\begin{aligned}
\Delta Q_{ij} &= Q_{ij,kk}\\
&= \theta''\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right) + \theta'\frac{1}{|x|}\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right)\\
&\quad + 2\theta' \frac{x_k}{|x|}\left(\frac{\delta_{ik}x_j}{|x|^2} + \frac{x_i\delta_{jk}}{|x|^2} - \frac{2x_i x_j x_k}{|x|^4}\right)\\
&\quad + 2\theta\left[\frac{\delta_{ik}\delta_{jk}|x|^2 - \delta_{ik}x_j x_k - x_k x_i\delta_{jk}}{|x|^4} - \frac{(\delta_{ik}x_j x_k + \delta_{jk}x_i x_k + \delta_{kk}x_i x_j)}{|x|^4}\right]\\
&\quad + \theta\frac{8x_i x_j}{|x|^4}\\
&= \left(\theta'' + \frac{\theta'}{|x|} - \frac{4\theta}{|x|^2}\right)\left(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\right).
\end{aligned}$$

D.2. *Terms related to $L_2 + L_3$*

There are two extra terms in this case, namely $2(L_2 + L_3)\partial_j\partial_k Q_{ik}$ and $-(L_2 + L_3)\partial_l\partial_k Q_{lk}\delta_{ij}$. For the former, we calculate

$$\begin{aligned}
Q_{ik,kj} &= \theta'' \frac{x_k x_j}{|x|^2}\left(\frac{x_i x_k}{|x|^2} - \frac{\delta_{ik}}{2}\right) + \theta'\left(\frac{\delta_{kj}}{|x|} - \frac{x_k x_j}{|x|^3}\right)\left(\frac{x_i x_k}{|x|^2} - \frac{\delta_{ik}}{2}\right)\\
&\quad + \theta' \frac{x_k}{|x|}\left(\frac{\delta_{ij}x_k}{|x|^2} + \frac{x_i\delta_{kj}}{|x|^2} - \frac{2x_i x_k x_j}{|x|^4}\right) + \theta' \frac{x_j}{|x|}\left(\frac{\delta_{ik}x_k}{|x|^2} + \frac{x_i\delta_{kk}}{|x|^2} - \frac{2x_i x_k x_k}{|x|^4}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\theta\left[\frac{\delta_{ik}\delta_{kj}}{|x|^2}-\frac{2\delta_{ik}x_kx_j}{|x|^4}+\frac{\delta_{ij}\delta_{kk}}{|x|^2}-\frac{2x_jx_i\delta_{kk}}{|x|^4}\right]\\
&-\theta\left[\frac{2(\delta_{ij}x_kx_k+\delta_{jk}x_ix_k+\delta_{kj}x_ix_k)}{|x|^4}-\frac{8x_ix_kx_kx_j}{|x|^6}\right]\\
&=\theta''\left(\frac{x_ix_j}{|x|^2}-\frac{x_ix_j}{2|x|^2}\right)+\theta'\left(\frac{x_ix_j}{|x|^3}-\frac{x_ix_j}{|x|^3}-\frac{\delta_{ij}}{2|x|}+\frac{x_ix_j}{2|x|^3}\right)\\
&+\theta'\left(\frac{\delta_{ij}}{|x|}+\frac{x_ix_j}{|x|^3}-\frac{2x_ix_j}{|x|^3}\right)+\theta'\left(\frac{x_ix_j}{|x|^3}+\frac{2x_ix_j}{|x|^3}-\frac{2x_ix_j}{|x|^3}\right)\\
&+\theta\left(\frac{\delta_{ij}}{|x|^2}-\frac{2x_ix_j}{|x|^4}+\frac{2\delta_{ij}}{|x|^2}-\frac{4x_ix_j}{|x|^4}-\frac{2\delta_{ij}}{|x|^2}-\frac{4x_ix_j}{|x|^4}+\frac{8x_ix_j}{|x|^4}\right)\\
&=\frac{\theta''}{2}\frac{x_ix_j}{|x|^2}+\frac{\theta'}{2|x|}\left(\frac{x_ix_j}{|x|^2}+\delta_{ij}\right)+\frac{\theta}{|x|^2}\left(\delta_{ij}-\frac{2x_ix_j}{|x|^2}\right).
\end{aligned}
$$

While for the latter, it holds:

$$
\begin{aligned}
Q_{lk,lk}&=\theta''\frac{x_kx_l}{|x|^2}\left(\frac{x_lx_k}{|x|^2}-\frac{\delta_{lk}}{2}\right)+\theta'\left(\frac{\delta_{kl}}{|x|}-\frac{x_kx_l}{|x|^3}\right)\left(\frac{x_lx_k}{|x|^2}-\frac{\delta_{lk}}{2}\right)\\
&+\theta'\frac{x_k}{|x|}\left(\frac{\delta_{ll}x_k}{|x|^2}+\frac{x_l\delta_{kl}}{|x|^2}-\frac{2x_lx_lx_k}{|x|^4}\right)+\theta'\frac{x_l}{|x|}\left(\frac{\delta_{lk}x_k}{|x|^2}+\frac{x_l\delta_{kk}}{|x|^2}-\frac{2x_lx_kx_k}{|x|^4}\right)\\
&+\theta\left[\frac{\delta_{lk}\delta_{lk}}{|x|^2}-\frac{2\delta_{lk}x_kx_l}{|x|^4}+\frac{\delta_{ll}\delta_{kk}}{|x|^2}-\frac{2x_lx_l\delta_{kk}}{|x|^4}\right]\\
&-\theta\left[\frac{2(\delta_{ll}x_kx_k+\delta_{lk}x_lx_k+\delta_{lk}x_lx_k)}{|x|^4}-\frac{8x_lx_lx_kx_k}{|x|^6}\right]\\
&=\frac{\theta''}{2}+\frac{3}{2|x|}\theta'.
\end{aligned}
$$

We conclude after putting them together that

$$
\begin{aligned}
&(L_2+L_3)(Q_{ik,kj}+Q_{jk,ki})-(L_2+L_3)Q_{lk,lk}\delta_{ij}\\
&\quad=(L_2+L_3)\left(\theta''+\frac{\theta'}{|x|}-\frac{4\theta}{|x|^2}\right)\left(\frac{x_ix_j}{|x|^2}-\frac{\delta_{ij}}{2}\right).
\end{aligned}
$$

It is noted that

$$
\left(\frac{x_ix_j}{|x|^2}-\frac{\delta_{ij}}{2}\right)\left(\frac{x_ix_j}{|x|^2}-\frac{\delta_{ij}}{2}\right)=S_{ij}S_{ij}=\frac{1}{2}.
$$

Hence summing up the above calculations, then taking the inner product with S, and denoting

$$
\zeta=2L_1+(L_2+L_3),
$$

we arrive at the following equation for the scalar unknown θ only:

$$
\partial_t\theta=L_4\left(\frac{(\theta')^2}{2}+\frac{\theta\theta'}{r}+\theta\theta''+\frac{6\theta^2}{r^2}\right)+\zeta\theta''+\frac{\zeta\theta'}{r}-\frac{4\zeta\theta}{r^2}-a\theta-\frac{c}{2}\theta^3.
$$

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