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A STOCHASTIC-LAGRANGIAN APPROACH TO THE NAVIER–STOKES EQUATIONS IN DOMAINS WITH BOUNDARY

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In this paper we derive a probabilistic representation of the deterministic 3-dimensional Navier–Stokes equations in the presence of spatial boundaries. The formulation in the absence of spatial boundaries was done by the authors in [*Comm. Pure Appl. Math.* **61** (2008) 330–345]. While the formulation in the presence of boundaries is similar in spirit, the proof is somewhat different. One aspect highlighted by the formulation in the presence of boundaries is the nonlocal, implicit influence of the boundary vorticity on the interior fluid velocity.

1. Introduction.

The (unforced) incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \nu \triangle u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

describe the evolution of the velocity field u of an incompressible fluid with kinematic viscosity $\nu > 0$ in the absence of external forcing. Here $u = u(x, t)$ with $t \geq 0$, $x \in \mathbb{R}^d$, $d \geq 2$. Equation (1.2) is the incompressibility constraint. Unlike compressible fluids, the pressure p in (1.1) does not have a physical meaning and is only a Lagrange multiplier that ensures incompressibility is preserved. While equations (1.1) and (1.2) can be formulated in any dimension $d \geq 2$, they are usually only studied in the physically relevant dimensions 2 or 3. The presentation of the Navier–Stokes equations above is in the absence of spatial boundaries; an issue that will be discussed in detail later.

When $\nu = 0$, (1.1) and (1.2) are known as the Euler equations. These describe the evolution of the velocity field of an (ideal) inviscid and incompressible fluid. Formally the difference between the Euler and Navier–Stokes equations is only the dissipative Laplacian term. Since the Laplacian is exactly the generator a Brownian motion, one would expect to have an exact stochastic representation of (1.1) and (1.2) which is physically meaningful, that is, can be thought of as an appropriate average of the inviscid dynamics and Brownian motion.

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The difficulty, however, in obtaining such a representation is because of both the nonlinearity and the nonlocality of equations (1.1) and (1.2). In 2D, an exact stochastic representation of (1.1) and (1.2) dates back to Chorin [14] in 1973 and was obtained using vorticity transport and the Kolmogorov equations. In three dimensions, however, this method fails to provide an exact representation because of the vortex stretching term.

In 3D, a variety of techniques has been used to provide exact stochastic representations of (1.1) and (1.2). One such technique (Le Jan and Sznitman [26]) uses a backward branching process in Fourier space. This approach has been extensively studied and generalized [3, 4, 32, 35, 36] by many authors (see also [37]). A different and more recent technique due to Busnello, Flandoli and Romito [6] (see also [5]) uses noisy flow paths and a Girsanov transformation. A related approach in [11] is the stochastic-Lagrangian formulation, exact stochastic representation of solutions to (1.1) and (1.2) which is essentially the averaging of noisy particle trajectories and the inviscid dynamics. Stochastic variational approaches (generalizing Arnold's [1] deterministic variational formulation for the Euler equations) have been used by [13, 16] and a related approach using stochastic differential geometry can be found in [19].

One common setback in all the above methods is the inability to deal with boundary conditions. The main contribution of this paper adapts the stochastic-Lagrangian formulation in [11] (where the authors only considered periodic boundary conditions or decay at infinity) to the situation with boundaries. The usual probabilistic techniques used to transition to domains with boundary involve stopping the processes at the boundary. This introduces two major problems with the techniques in [11]. First, stopping introduces spatial discontinuities making the proof used in [11] fail and a different approach is required. Second and more interesting is the fact that merely stopping does *not* give the no-slip (0-Dirichlet) boundary condition as one would expect. One needs to also create trajectories at the boundary which essentially propagate the influence of the vorticity at the boundary to the interior fluid velocity.

1.1. *Plan of the paper.*

This paper is organized as follows. In Section 2 a brief introduction to the stochastic-Lagrangian formulation without boundaries is given. In Section 3 we motivate and state the stochastic-Lagrangian formulation in the presence of boundaries (Theorem 3.1). In Section 4 we recall certain standard facts about backward Itô integrals which will be used in the proof of Theorem 3.1. In Section 5 we prove Theorem 3.1. Finally, in Section 6 we discuss stochastic analogues of vorticity transport and inviscid conservation laws.

2. The stochastic-Lagrangian formulation without boundaries.

In this section, we provide a brief description of the stochastic-Lagrangian formulation in the absence of boundaries. For motivation, let us first study a Lagrangian description of the Euler equations [equations (1.1) and (1.2) with $\nu = 0$; we will usually use a superscript of 0 to denote quantities relating to the Euler equations]. Let $d = 2, 3$ denote the spatial dimension and X_t^0 be the flow defined by

$$\dot{X}_t^0 = u_t^0(X_t^0), \tag{2.1}$$

with initial data $X_0^0(a) = a$, for all $a \in \mathbb{R}^d$. To clarify our notation, X^0 is a function of the initial data $a \in \mathbb{R}^d$ and time $t \in [0, \infty)$. We usually omit the spatial variable and use X_t^0 to denote $X^0(\cdot, t)$, the slice of X^0 at time t. Time derivatives will always be denoted by a dot or ∂_t instead of a t subscript.

One can immediately check (see, e.g., [7]) that u satisfies the incompressible Euler equations if and only if $\ddot{X}^0$ is a gradient composed with X. By Newton's second law, this admits the physical interpretation that the Euler equations are equivalent to assuming that the force on individual particles is a gradient.

One would naturally expect that solutions to the Navier–Stokes equations can be obtained similarly by adding noise to particle trajectories and averaging. However, for noisy trajectories, an assumption on $\ddot{X}^0$ will be problematic. In the incompressible case, we can circumvent this difficultly using the Weber formula [38] [equation (2.2) below]. Indeed, a direct computation (see, e.g., [7]) shows that for divergence free u, the assumption that $\ddot{X}^0$ is a gradient is equivalent to

$$u_t^0 = \mathbf{P}[(\nabla^* A_t^0)(u_0^0 \circ A_t^0)], \tag{2.2}$$

where $\mathbf{P}$ denotes the Leray–Hodge projection [10, 15, 28] onto divergence free vector fields, the notation ∇^* denotes the transpose of the Jacobian and for any $t \geq 0$, $A_t^0 = (X_t^0)^{-1}$ is the spatial inverse of the map X_t^0 [i.e., $A_t^0(X_t^0(a)) = a$ for all $a \in \mathbb{R}^d$ and $X_t^0(A_t^0(x)) = x$ for all $x \in \mathbb{R}^d$].

From this we see that the Euler equations are formally equivalent to equations (2.1) and (2.2). Since these equations no longer involve second (time) derivatives of the flow X^0, one can consider noisy particle trajectories without any analytical difficulties. In fact, adding noise to (2.1) and averaging out the noise in (2.2) gives the equivalent formulation of the Navier–Stokes equations stated below.

THEOREM 2.1 (Constantin, Iyer [11]). *Let $d \in \{2, 3\}$ be the spatial dimension, $\nu > 0$ represent the kinematic viscosity and u_0 be a divergence free, periodic, Hölder $2 + \alpha$ function and W be a d-dimensional Wiener process. Consider the system*

$$dX_t = u_t(X_t)\, dt + \sqrt{2\nu}\, dW_t, \tag{2.3}$$

$$X_0(a) = a \qquad \forall a \in \mathbb{R}^d, \tag{2.4}$$

$$u_t = E\mathbf{P}[(\nabla^* A_t)(u_0 \circ A_t)], \tag{2.5}$$

where, as before, for any $t \geq 0$, $A_t = X_t^{-1}$ denotes the spatial inverse[3] *of X_t. Then u is a classical solution of the Navier–Stokes equations* (1.1) *and* (1.2) *with initial data u_0 and periodic boundary conditions if and only if u is a fixed point of the system* (2.3)–(2.5).

[3]It is well known (see, e.g., Kunita [25]) that the solution to (2.3) and (2.4) gives a stochastic flow of diffeomorphisms and, in particular, guarantees the existence of the spatial inverse of X.

REMARK. The flows X, A above are now a function of the initial data $a \in \mathbb{R}^d$, time $t \in [0, \infty)$ and the probability variable $\varpi \in \Omega$. We always suppress the probability variable, use X_t to denote $X(\cdot, t)$ and omit the spatial variable when unnecessary. The function u is a deterministic function of space and time and, as above, we use u_t to denote the function $u(\cdot, t)$.

We now briefly explain the idea behind the proof of Theorem 2.1 given in [11] and explain why this method can not be used in the presence of spatial boundaries. Consider first the solution of the SDE (2.3) with initial data (2.4). Using the Itô–Wentzel formula [25], Theorem 4.4.5, one can show that any (spatially regular) process θ which is constant along trajectories of X satisfies the SPDE

$$d\theta_t + (u_t \cdot \nabla)\theta_t\, dt - \nu \triangle \theta_t\, dt + \sqrt{2\nu} \nabla \theta_t\, dW_t = 0. \tag{2.6}$$

Since the process A (which, as before, is defined to be the spatial inverse of X) is constant along trajectories of X, the process θ defined by

$$\theta_t = \theta_0 \circ A_t \tag{2.7}$$

is constant along trajectories of X. Thus, if θ_0 is regular enough (C^2), then θ satisfies SPDE (2.6). Now, if u is deterministic, taking expected values of (2.6) we see that $\bar{\theta}_t = E\theta_0 \circ A_t$ satisfies

$$\partial_t \bar{\theta}_t + (u_t \cdot \nabla)\bar{\theta}_t - \nu \triangle \bar{\theta}_t = 0 \tag{2.8}$$

with initial condition $\bar{\theta}|_{t=0} = \theta_0$.

REMARK. Note that when $\nu = 0$, A is deterministic so $\bar{\theta} = E\theta = \theta$. Further, equation (2.6) reduces to the transport equation for which writing the solution as $\theta_t = \theta_0 \circ A_t$ is exactly the method of characteristics. When $\nu > 0$, the above procedure is an elegant generalization, termed as the "method of random characteristics" (see [11, 20, 33] for further information).

Once explicit equations for A and $u_0 \circ A$ have been established, a direct computation using Itô's formula shows that u given by (2.5) satisfies the Navier–Stokes equations (1.1) and (1.2). This was the proof used in [11].

REMARK. This point of view also yields a natural understanding of generalized relative entropies [8, 12, 29, 30]. Eyink's recent work [17] adapted this framework to magnetohydrodynamics and related equations by using the analogous Weber formula [24, 34]. We also mention that Zhang [39] considered a backward analogue and provided short elegant proofs to classical existence results to (1.1) and (1.2).

3. The formulation for domains with boundary. In this section we describe how (2.3)–(2.5) can be reformulated in the presence of boundaries. We begin by describing the difficulty in using the techniques from [11] described in Section 2.

Let $D \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Even if we insist $u = 0$ on the boundary of D, we note that the noise in (2.3) is independent of space and thus, insensitive to the presence of the boundary. Consequently, some trajectories of the stochastic flow X will leave the domain D and for any $t > 0$, the map X_t will (surely) not be spatially invertible. This renders (2.7) meaningless.

In the absence of spatial boundaries, equation (2.7) dictates that $\bar{\theta}(x,t)$ is determined by averaging the initial data over all trajectories of X which reach x at time t. In the presence of boundaries, one must additionally average the boundary value of all trajectories reaching (x,t), starting on ∂D at any intermediate time (Figure 1). As we will see later, this means the analogue of (2.7) in the presence of spatial boundaries is a spatially discontinuous process. This renders (2.6) meaningless, giving a second obstruction to using the methods of [11] in the presence of boundaries.

While the method of random characteristics has the above inherent difficulties in the presence of spatial boundaries, equation (2.8) is exactly the Kolmogorov Backward equation ([31], Section 8.1). In this case, an expected value representation in the presence of boundaries is well known. More generally, the Feynman–Kac ([31], Section 8.2) formula, at least for linear equations with a potential term, has been successfully used in this situation. A certain version of this method (Section 3.1), without making the usual time reversal substitution, is essentially the same as the method of random characteristics. It is this version that will yield the natural generalization of (2.3)–(2.5) in domains with boundary (Theorem 3.1). Before turning to the Navier–Stokes equations, we provide a brief discussion on the relation between the Feynman–Kac formula and the method of random characteristics.

3.1. *The Feynman–Kac formula and the method of random characteristics*. Both the Feynman–Kac formula and the method of random characteristics have

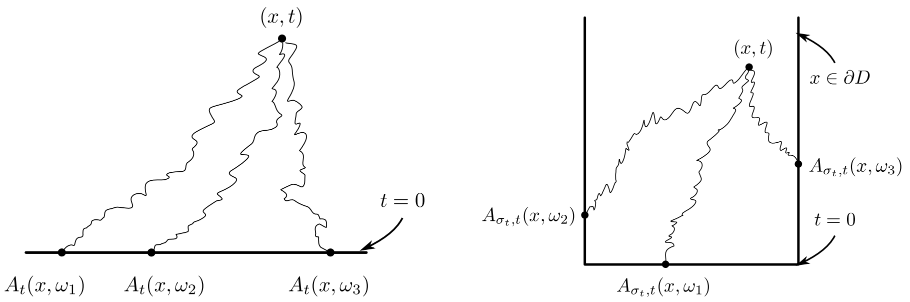

FIG. 1. *Three sample realizations of A without boundaries* (left) *and with boundaries* (right).

their own advantages and disadvantages: The method of random characteristics only involves forward SDE's and obtains the solution of (2.8) at time t with only the knowledge of the initial data and "X at time t" (or more precisely, the solution at time t of the equation (2.3) with initial data specified at time 0). However, this method involves computing the spatial inverse of X, which analytically and numerically involves an additional step.

On the other hand, to compute the solution of (2.8) at time t via the probabilistic representation using the Kolmogorov backward equation (or equivalently, the Feynman–Kac formula with a 0 potential term) when u is time dependent involves *backward* SDE's and further requires the knowledge of the solution to (2.3) with initial conditions specified at all times $s \le t$. However, this does not require computation of spatial inverses and, more importantly, yields the correct formulation in the presence of spatial boundaries.

Now, to see the relation between the method of random characteristics and the Feynman–Kac formula, we rewrite (2.3) in integral form and keep track of solutions starting at all times $s \ge 0$. For any $s \ge 0$, we define the process $\{X_{s,t}\}_{t\ge s}$ to be the flow defined by

$$X_{s,t}(x) = x + \int_s^t u_r \circ X_{s,r}(x)\,dr + \sqrt{2\nu}(W_t - W_s). \tag{3.1}$$

Now, as always, we let $A_{s,t} = X_{s,t}^{-1}$. Then formally composing (3.1) with $A_{s,t}$ and using the semigroup property $X_{s,t} \circ X_{r,s} = X_{r,t}$ gives the self-contained backward equation for $A_{s,t}$

$$A_{s,t}(x) = x - \int_s^t u_r \circ A_{r,t}(x)\,dr - \sqrt{2\nu}(W_t - W_s). \tag{3.2}$$

Now (2.7) can be written as

$$\theta_t = \theta_0 \circ A_{0,t} \tag{3.3}$$

and using the semigroup property $A_{r,s} \circ A_{s,t} = A_{r,t}$ we see that

$$\theta_t = \theta_s \circ A_{s,t}. \tag{3.4}$$

This formal calculation leads to a natural generalization of (2.7) in the presence of boundaries. As before, let $D \subset \mathbb{R}^d$ be a domain with Lipschitz boundary and assume, for now, that u is a Lipschitz function defined on *all* of $\mathbb{R}^d$. Let $A_{s,t}$ be the flow defined by (3.2) and for $x \in D$, we define the *backward exit time* $\sigma_t(x)$ by

$$\sigma_t(x) = \inf\{s \mid s \in [0,t] \text{ and } \forall r \in (s,t],\ A_{r,t}(x) \in D\}. \tag{3.5}$$

Let $g : \partial D \times [0,\infty) \to \mathbb{R}$ and $\theta_0 : D \to \mathbb{R}$ be two given (regular enough) functions and define the process θ_t by

$$\theta_t(x) = \begin{cases} g_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x), & \text{if } \sigma_t(x) > 0,\\ \theta_0 \circ A_{0,t}(x), & \text{if } \sigma_t(x) = 0.\end{cases} \tag{3.6}$$

Note that when $\sigma_t(x) > 0$, equation (3.6) is consistent with (3.4). Thus, (3.6) is the natural generalization of (2.7) in the presence of spatial boundaries and we expect $\bar{\theta}_t = E\theta_t$ satisfies the PDE (2.8) with initial data $\bar{\theta}_0 = \theta_0$ and boundary conditions $\theta = g$ on $\partial D \times [0, \infty)$. Indeed, this is essentially the expected value representation obtained via the Kolmogorov backward equations.

If an extra term $c_t(x)\bar{\theta}_t(x)$ is desired on the left-hand side of (2.8), then we only need to replace (3.6) by

$$\theta_t(x) = \begin{cases} \exp\Bigl(-\int_{\sigma_t(x)}^t c_s(A_{s,t})\,ds\Bigr) g_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x), & \text{if } \sigma_t(x) > 0, \\ \exp\Bigl(-\int_0^t c_s(A_{s,t})\,ds\Bigr) \theta_0 \circ A_{0,t}(x), & \text{if } \sigma_t(x) = 0 \end{cases}$$

provided c is bounded below. This is essentially the Feynman–Kac formula and its application to the Navier–Stokes equations is developed in the next section.

Note that the backward exit time σ is usually discontinuous in the spatial variable. Thus, even with smooth g, θ_0, the process θ need not be spatially continuous. As mentioned earlier, equation (2.6) will now become meaningless and we will not be able to obtain a SPDE for θ. However, equation (2.8), which describes the evolution of the expected value $\bar{\theta} = E\theta$, can be directly derived using the backward Markov property and Itô's formula (see, e.g., [18]). We will not provide this proof here but will instead provide a proof for the more complicated analogue for the Navier–Stokes equations described subsequently.

3.2. *Application to the Navier–Stokes equations in domains with boundary.* First note that if $g = 0$ in (3.6), then the solution to (2.8) with initial data θ_0 and 0-Dirichlet boundary conditions will be given by

$$\bar{\theta}_t = E\chi_{\{\sigma_t=0\}}\theta_0 \circ A_{0,t} \qquad [\text{i.e., } \bar{\theta}_t(x) = E\chi_{\{\sigma_t(x)=0\}}\theta_0 \circ A_{0,t}(x)]. \tag{3.7}$$

Recall the no-slip boundary condition for the Navier–Stokes equations is exactly a 0-Dirichlet boundary condition on the velocity field. Let u be a solution to the Navier–Stokes equations in D with initial data u_0 and no-slip boundary conditions. Now, following (3.7), we would expect that analogous to (2.5), the velocity field u can be recovered from the flow $A_{s,t}$ [equation (3.2)], the backward exit time σ_t [equation (3.5)] and the initial data u_0 by

$$u_t = \mathbf{P}E\chi_{\{\sigma_t=0\}}(\nabla^* A_{0,t})u_0 \circ A_{0,t}. \tag{3.8}$$

This, however, is *false*. In fact, there are two elementary reasons one should expect (3.8) to be false. First, absorbing Brownian motion at the boundaries will certainly violate incompressibility. The second and more fundamental reason is that experiments and physical considerations lead us to expect production of vorticity at the boundary. This is exactly what is missing from (3.8). The correct representation is provided in the following result.

THEOREM 3.1. *Let $u \in C^1([0,T); C^2(D)) \cap C([0,T]; C^1(\bar{D}))$ be a solution of the Navier–Stokes equations (1.1) and (1.2) with initial data u_0 and no-slip boundary conditions. Let A be the solution to the backward SDE (3.2) and σ be the backward exit time defined by (3.5). There exists a function $\tilde{w} : \partial D \times [0,T] \to \mathbb{R}^3$ such that for*

$$w_t(x) = \begin{cases} (\nabla^* A_{0,t}(x))u_0 \circ A_{0,t}(x), & \text{when } \sigma_t = 0, \\ (\nabla^* A_{\sigma_t(x),t}(x))\tilde{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x), & \text{when } \sigma_t > 0, \end{cases} \tag{3.9}$$

we have

$$u_t = \mathbf{P}E w_t. \tag{3.10}$$

Conversely, given a function $\tilde{w} : \partial D \times [0,T] \to \mathbb{R}^d$, suppose there exists a solution to the stochastic system (3.2), (3.9), (3.10). If further $u \in C^1([0,T); C^2(D)) \cap C([0,T]; C^1(\bar{D}))$, then u satisfies the Navier–Stokes equations (1.1)–(1.2) with initial data u_0 and vorticity boundary conditions

$$\nabla \times u = \nabla \times E w \qquad \text{on } \partial D \times [0,T]. \tag{3.11}$$

The proof of Theorem 3.1 is presented in Section 5. We conclude this section with a few remarks.

REMARK 3.2. By $\nabla^* A_{\sigma_t(x),t}(x)$ in equation (3.9) we mean $[\nabla^* A_{s,t}(x)]_{s=\sigma_t(x)}$. That is, $\nabla^* A_{\sigma_t(x),t}(x)$ refers to the transpose of the Jacobian of A, evaluated at initial time $\sigma_t(x)$, final time t and position x (see [22, 23, 25] for existence). This is different from the transpose of the Jacobian of the function $A_{\sigma_t(\cdot),t}(\cdot)$ which does not exist as the function is certainly not differentiable in space.

REMARK 3.3 (Regularity assumptions). In order to simplify the presentation, our regularity assumptions on u are somewhat generous. Our assumptions on u will immediately guarantee that u has a Lipschitz extension to $\mathbb{R}^d$. Now the process A, defined to be a solution to (3.2) with this Lipschitz extension of u, can be chosen to be a (backward) stochastic flow of diffeomorphisms [25]. Thus, ∇A is well defined and further defining σ by (3.5) is valid. Finally, since the statement of Theorem 3.1 only uses values of $A_{s,t}$ for $s \geq \sigma_t$, the choice of the Lipschitz extension of u will not matter. See also Remark 5.3.

REMARK 3.4. Note that our statement of the converse above *does not* explicitly give any information on the Dirichlet boundary values of u. Of course, the normal component of u must vanish at the boundary of D since u is the Leray–Hodge projection of a function. But an explicit local relation between $\tilde{w}$ and the boundary values of the tangential component of u cannot be established. We remark, however, that while the vorticity boundary condition (3.11) is somewhat artificial, it is enough to guarantee uniqueness of solutions to the initial value problem for the Navier–Stokes equations.

REMARK 3.5 (Choice of $\tilde{w}$). We explain how $\tilde{w}$ can be chosen to obtain the no-slip boundary conditions. We will show (Lemma 5.1) that for w defined by (3.9), the expected value $\bar{w} \stackrel{\text{def}}{=} Ew$ solves the PDE

$$\partial_t \bar{w}_t + (u_t \cdot \nabla)\bar{w}_t - \nu\Delta\bar{w}_t + (\nabla^* u_t)\bar{w}_t = 0 \tag{3.12}$$

with initial data

$$\bar{w}|_{t=0} = u_0. \tag{3.13}$$

As shown before, $\nabla^* u_t$ in (3.12) denotes the transpose of the Jacobian of u_t. Now, if $u = \mathbf{P}\bar{w}$, then we will have $\nabla \times u = \nabla \times \bar{w}$ in D and by continuity, on the boundary of D. Thus, to prove existence of the function $\tilde{w}$, we solve the PDE (3.12) with initial conditions (3.13) and *vorticity* boundary conditions

$$\nabla \times \bar{w}_t = \nabla \times u_t \qquad \text{on } \partial D. \tag{3.14}$$

We chose $\tilde{w}$ to be the Dirichlet boundary values of this solution.

To elaborate on Remark 3.5, we trace through the influence of the vorticity on the boundary on the velocity in the interior. First, the vorticity at the boundary influences $\bar{w}$ by entering as a boundary condition on the first derivatives for the PDE (3.12). Now, to obtain u we need to find $\tilde{w}$, the (Dirichlet) boundary values of (3.12) and use this to weight trajectories that start on the boundary of D. The process of finding $\tilde{w}$ is essentially passing from Neumann boundary values of a PDE to the Dirichlet boundary values which is usually a nonlocal pseudo-differential operator. Thus, while the procedure above is explicit enough, the boundary vorticity influences the interior velocity in a highly implicit, nonlocal manner.

REMARK 3.6 (Uniqueness of $\tilde{w}$). Our choice of $\tilde{w}$ is not unique. Indeed, if $\bar{w}^1$ and $\bar{w}^2$ are two solutions of (3.12)–(3.14), then we must have $\bar{w}^1 - \bar{w}^2 = \nabla q$, where q satisfies the equation

$$\nabla(\partial_t q + (u \cdot \nabla)q - \nu\Delta q) = 0 \tag{3.15}$$

with initial data $\nabla q_0 = 0$. Since we do not have boundary conditions on q, we can certainly have nontrivial solutions to this equation. Thus, our choice of $\tilde{w}$ is only unique up to addition by the gradient of a solution to (3.15).

4. Backward Itô integrals. While the formulation of Theorem 3.1 involves only regular (forward) Itô integrals, the proof requires backward Itô integrals and processes adapted to a two parameter filtration. The need for backward Itô integrals stems from equation (3.2) which, as mentioned earlier, is the evolution of A, *backward* in time. This is, however, obscured because our diffusion coefficient is constant making the martingale term exactly the increment of the Wiener process

and can be explicitly computed without any backward (or even forward) Itô integrals.

To elucidate matters, consider the flow X' given by

$$X'_{s,t}(a) = a + \int_s^t u_r \circ X'_{s,r}(a)\,dr + \int_s^t \sigma_r \circ X'_{s,r}(a)\,dW_r. \tag{4.1}$$

If, as usual, $A'_{s,t} = (X'_{s,t})^{-1}$, then substituting formally[4] $a = A'_{s,t}(x)$ and assuming the semigroup property gives the equation

$$A'_{s,t}(x) = x - \int_s^t u_r \circ A'_{r,t}(x)\,dr - \int_s^t \sigma_r \circ A'_{r,t}(x)\,dW_r \tag{4.2}$$

for the process $A'_{s,t}$. The need for backward Itô integrals is now evident; the last term above does not make sense as a forward Itô integral since $A'_{r,t}$ is not $\mathcal{F}_r$ measurable. This term, however, is well defined as a backward Itô integral; an integral with respect to a decreasing filtration where processes are sampled at the right endpoint. Since forward Itô integrals are more predominant in the literature, we recollect a few standard facts about backward Itô integrals in this section. A more detailed account, with proofs, can be found in [18, 25], for instance.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{W_t\}_{t\geq 0}$ be a d-dimensional Wiener process on Ω and let $\mathcal{F}_{s,t}$ be the σ-algebra generated by the increments $W_{t'} - W_{s'}$ for all $s \leq s' \leq t' \leq t$, augmented so that the filtration $\{\mathcal{F}_{s,t}\}_{0\leq s\leq t}$ satisfies the usual conditions.[5] Note that for $s \leq s' \leq t' \leq t$, we have $\mathcal{F}_{s',t'} \subset \mathcal{F}_{s,t}$. Also $W_t - W_s$ is $\mathcal{F}_{s,t}$-measurable and is independent of both the past $\mathcal{F}_{0,s}$, and the future $\mathcal{F}_{t,\infty}$.

We define a (two parameter) family of random variables $\{\xi_{s,t}\}_{0\leq s\leq t}$ to be a (two parameter) process adapted to the (two parameter) filtration $\{\mathcal{F}_{s,t}\}_{0\leq s\leq t}$, if for all $0 \leq s \leq t$, the random variable $\xi_{s,t}$ is $\mathcal{F}_{s,t}$-measurable. For example, $\xi_{s,t} = W_t - W_s$ is an adapted process. More generally, if u and σ are regular enough *deterministic* functions, then the solution $\{X'_{s,t}\}_{0\leq s\leq t}$ of the (forward) SDE (4.1) is an adapted process.

Given an adapted (two parameter) process ξ and any $t \geq 0$, we define the backward Itô integral $\int_\cdot^t \xi_{r,t}\,dW_r$ by

$$\int_s^t \xi_{r,t}\,dW_r = \lim_{\|P\|\to 0} \sum_i \xi_{t_{i+1},t}(W_{t_{i+1}} - W_{t_i}),$$

where $P = (r = t_0 < t_1 \cdots < t_N = t)$ is a partition of $[r,t]$ and $\|P\|$ is the length of the largest subinterval of P. The limit is taken in the L^2 sense, exactly as with forward Itô integrals (see, e.g., [21], page 148, [27], page 35, [25], page 111).

[4]The formal substitution does *not* give the correct answer when σ is not spatially constant. This is explained subsequently and the correct equation is (4.3) below.

[5]By "usual conditions" in this context, we mean that for all $s \geq 0$, $\mathcal{F}_{s,s}$ contains all $\mathcal{F}_{0,\infty}$-null sets. Further, $\mathcal{F}_{s,t}$ is right-continuous in t and left-continuous in s. See [21], Definition 2.25, for instance.

The standard properties (existence, Itô isometry, martingale properties) of the backward Itô integral are, of course, identical to those of the forward integral. The only difference is in the sign of the Itô correction. Explicitly, consider the process $\{A'_{s,t}\}_{0\le s\le t}$ satisfying the backward Itô differential equation (4.2). If $\{f_{s,t}\}_{0\le s\le t}$ is adapted, C^2 in space and continuously differentiable with respect to s, then the process $B_{s,t} = f_{s,t} \circ A_{s,t}$ satisfies the backward Itô differential equation

$$B_{t,t} - B_{s,t} = \int_s^t \left[\partial_r f_{r,t} + (u_r \cdot \nabla) f_{r,t} - \frac{1}{2} a_r^{ij} \partial_{ij} f_{r,t}\right] \circ A_{r,t}\, dr + \int_s^t [\nabla f_{r,t} \sigma_r] \circ A_{r,t}\, dW_r,$$

where $a_r^{ij} = \sigma_r^{ik} \sigma_r^{jk}$ with the Einstein sum convention.

Though we only consider solutions to (4.1) for constant diffusion coefficient, we briefly address one issue when σ is not constant. Our motivation for the equation (4.2) was to make the substitution $x = A'_{s,t}(x)$ and formally use the semigroup property. This, however, does not yield the correct equation when σ is not constant and the equation for $A'_{s,t} = (X'_{s,t})^{-1}$ involves an additional correction term. To see this, we discretize the forward integral in (4.1) (in time) and substitute $a = A'_{s,t}(x)$. This yields a sum sampled at the *left* endpoint of each time step. While this causes no difficulty for the bounded variation terms, the martingale term is a discrete approximation to a *backward* integral and hence, must be sampled at the *right* endpoint of each time step. Converting this to sum sampled at the right endpoint via a Taylor expansion of σ is what gives this extra correction. Carrying through this computation (see, e.g., [25], Section 4.2) yields the equation

$$A'_{s,t}(x) = x - \int_s^t u_r \circ A'_{r,t}(x)\, dr - \int_s^t \sigma_r \circ A'_{r,t}(x)\, dW_r + \int_s^t (\partial_j \sigma_r^{i,k} \circ A'_{r,t}(x))(\sigma_r^{j,k} \circ A'_{r,t}(x)) e_i\, dr, \tag{4.3}$$

where $\{e_i\}_{1\le i\le d}$ are the elementary basis vectors and $\sigma^{i,j}$ denotes the i, jth entry in the $d \times d$ matrix σ.

We recall that the proof of the (forward) Itô formula involves approximating f by its Taylor polynomial about the left endpoint of the partition intervals. Analogously, the backward Itô formula involves approximating f by Taylor polynomial about the *right* endpoint of partition intervals, which accounts for the reversed sign in the Itô correction.

Finally, we remark that for any fixed $t \ge 0$, the solution $\{A_{s,t}\}_{0\le s\le t}$ of the backward SDE (3.2) is a backward strong Markov process [the same is true for solutions to (4.3)]. The backward Markov property states that $r < s < t$ then

$$E_{\mathcal{F}_{s,t}} f \circ A_{r,t}(x) = E_{A_{s,t}(x)} f \circ A_{r,t}(x) = [E f \circ A_{r,s}(y)]_{y=A_{s,t}(x)},$$

where $E_{\mathcal{F}_{s,t}}$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_{s,t}$ and $E_{A_{s,t}(x)}$ the conditional expectation with respect to the σ-algebra generated by the process $A_{s,t}(x)$.

For the strong Markov property (we define σ to be a *backward t-stopping time*[6] if almost surely $\sigma \le t$) and for all $s \le t$, the event $\{\sigma \ge s\}$ is $\mathcal{F}_{s,t}$-measurable. Now if σ is any backward t-stopping time with $r \le \sigma \le t$ almost surely, the backward strong Markov property states

$$E_{\mathcal{F}_{\sigma,t}} f \circ A_{r,t}(x) = E_{A_{\sigma,t}} f \circ A_{r,t}(x) = [E f \circ A_{r,s}(y)]_{\substack{s=\sigma,\\ y=A_{\sigma,t}(x).}}$$

The proofs of the backward Markov properties is analogous to the proof of the forward Markov properties and we refer the reader to [18], for instance.

5. The no-slip boundary condition. In this section we prove Theorem 3.1. First, we know from [22, 23] that spatial derivatives of A can be interpreted as the limit (in probability) of the usual difference quotient. In fact, for regular enough velocity fields u (extended to all of $\mathbb{R}^d$), the process A can, in fact, be chosen to be a flow of diffeomorphisms of $\mathbb{R}^d$ (see, e.g., [25]) in which case A is surely differentiable in space. Interpreting the Jacobian of A as either the limit (in probability) of the usual difference quotient or as the Jacobian of the stochastic flow of diffeomorphism, we know [22, 23, 25] that ∇A satisfies the equation

$$\nabla A_{s,t}(x) = I - \int_s^t \nabla u_r|_{A_{r,t}(x)} \nabla A_{r,t}(x)\, dr, \tag{5.1}$$

obtained by formally differentiating (3.2) in space. Here I denotes the $d \times d$ identity matrix. We reiterate that equation (5.1) is an ODE as the Wiener process is independent of the spatial parameter.

LEMMA 5.1. *Let D, u, T be as in Theorem 3.1, σ be the backward exit time from D [equation (3.5)] and A be the solution to (3.2) with respect to the backward stopping time σ.*

(1) *Let $\bar{w} \in C^1([0,T); C^2(D)) \cap C([0,T]; C^1(\bar{D}))$ be the solution of (3.12) with initial data (3.13) and boundary conditions*

$$\bar{w} = \tilde{w} \qquad \text{on } \partial D. \tag{5.2}$$

Then, for w defined by (3.9), we have $\bar{w} = Ew$.

(2) *Let w be defined by (3.9) and $\bar{w} = Ew$ as above. If for all $t \in (0,T]$, $\bar{w}_t \in \mathcal{D}(A_{\cdot,t})$ and $\bar{w}$ is C^1 in time, then $\bar{w}$ satisfies*

$$\partial_t \bar{w} + L_t \bar{w} + (\nabla^* u)\bar{w} = 0, \tag{5.3}$$

[6] Our use of the term backward t-stopping time is analogous to s-stopping time in [18], page 24.

where L_t is defined by

$$L_t\phi(x) = \lim_{s\to t^-} \frac{\phi(x) - E\phi(A_{s\vee\sigma_t(x),t}(x))}{t-s} \tag{5.4}$$

and $\mathcal{D}(A_{\cdot,t})$ is the set of all ϕ for which the limit on the right-hand side exists. Further, $\bar{w}$ has initial data u_0 and boundary conditions (5.2).

Before proceeding any further, we first address the relationship between the two assertions of the lemma. We claim that if $\bar{w} \in C^1((0,T); C^2(D))$, then equation (5.3) reduces to equation (3.12). This follows immediately from the next proposition.

PROPOSITION 5.2. *If $\phi \in C^2(D)$, then for any $t \in (0,T]$, $\phi \in \mathcal{D}(A_{\cdot,t})$ and further, $L_t\phi = (u_t\cdot\nabla)\phi - \nu\triangle\phi$.*

PROOF. Omitting the spatial variable for notational convenience, the backward Itô formula gives

$$\begin{aligned}\phi - \phi\circ A_{s\vee\sigma_t,t} &= \phi\circ A_{t,t} - \phi\circ A_{s\vee\sigma_t,t}\\ &= \int_{s\vee\sigma_t}^t [(u_r\cdot\nabla)\phi|_{A_{r,t}} - \nu\triangle\phi|_{A_{r,t}}]\,dr + \sqrt{2\nu}\int_{s\vee\sigma_t}^t \nabla\phi|_{A_{r,t}}\,dW_r.\end{aligned}$$

Since $s\vee\sigma_t$ is a backward t-stopping time, the second term above is a martingale. Thus

$$\begin{aligned}L_t\phi &= \lim_{s\to t^-} E\frac{1}{t-s}\int_s^t \chi_{\{r\geq\sigma_t\}}[(u_r\cdot\nabla)\phi|_{A_{r,t}} - \nu\triangle\phi|_{A_{r,t}}]\,dr\\ &= (u_t\cdot\nabla)\phi - \nu\triangle\phi\end{aligned}$$

since the process A has continuous paths and $\sigma_t < t$ on the interior of D. □

REMARK 5.3. One can weaken the regularity assumptions on u in the statement of Theorem 3.1 by instead assuming for all $t\in(0,T]$, $u_t \in \mathcal{D}(A_{\cdot,t})$ and is C^1 in time, as with the second assertion of Lemma 5.1. However, while the formal calculus remains essentially unchanged, there are a couple of technical points that require attention. First, when assumptions on smoothness of u up to the boundary is relaxed (or when ∂D is irregular), a Lipschitz extension of u need not exist. In this case, we can no longer use (3.5) to define σ. Further, we can not regard the process A as a stochastic flow of diffeomorphisms and some care has to be taken when differentiating it. These issues can be addressed using relatively standard techniques and once they are sorted out, the proof of Theorem 3.1 remains unchanged.

Now we prove the first assertion of Lemma 5.1.

PROOF. Recall that $\nabla^* A_{s,t}$ is differentiable in s. Differentiating (5.1) in s and transposing the matrices gives

$$\partial_s \nabla A_{s,t}(x) = \nabla^* A_{s,t}(x)\nabla^* u_s|_{A_{s,t}(x)}. \tag{5.5}$$

Let $t\in(0,T]$, $x\in D$ and σ' be any backward t-stopping time with $\sigma' \geq \sigma_t(x)$ almost surely. Omitting the spatial variable for convenience, the backward Itô formula and equations (3.12) and (5.5) give

$$\begin{aligned}&\bar{w}_t - \nabla^*A_{\sigma',t}\bar{w}_{\sigma'}\circ A_{\sigma',t}\\ &\quad= \nabla^*A_{t,t}\bar{w}_t\circ A_{t,t} - \nabla^*A_{\sigma',t}\bar{w}_{\sigma'}\circ A_{\sigma',t}\\ &\quad= \int_{\sigma'}^t \partial_r\nabla^*A_{r,t}\bar{w}_r\circ A_{r,t}\\ &\qquad + \int_{\sigma'}^t \nabla^*A_{r,t}\big(\partial_r\bar{w}_r + (u_r\cdot\nabla)\bar{w}_r - \nu\triangle\bar{w}_r\big)\circ A_{r,t}\,dr\\ &\qquad + \sqrt{2\nu}\int_{\sigma'}^t (\nabla^*A_{r,t})(\nabla^*\bar{w}_r)\circ A_{r,t}\,dW_r\\ &\quad= \int_{\sigma'}^t \nabla^*A_{r,t}\big((\nabla^*u_r)\bar{w}_r + \partial_r\bar{w}_r + (u_r\cdot\nabla)\bar{w}_r - \nu\triangle\bar{w}_r\big)\circ A_{r,t}\,dr\\ &\qquad + \sqrt{2\nu}\int_{\sigma'}^t (\nabla^*A_{r,t})(\nabla^*\bar{w}_r)\circ A_{r,t}\,dW_r\\ &\quad= \sqrt{2\nu}\int_{\sigma'}^t (\nabla^*A_{r,t})w_r\circ A_{r,t}\,dW_r.\end{aligned}$$

Thus, taking expected values gives

$$\bar{w}_t(x) = E\nabla^*A_{\sigma',t}(x)\bar{w}_{\sigma'}\circ A_{\sigma',t}(x). \tag{5.6}$$

Recall that when $\sigma_t(x) > 0$, $A_{\sigma_t(x),t}(x)\in\partial D$. Thus, choosing $\sigma' = \sigma_t(x)$ and using the boundary conditions (5.2) and initial data (3.13), we have

$$\bar{w}_{\sigma_t(x)}\circ A_{\sigma_t(x),t} = \begin{cases}\tilde{w}_{\sigma_t(x)}\circ A_{\sigma_t(x),t}, & \text{if } \sigma_t(x) > 0,\\ u_0\circ A_{\sigma_t(x),t}, & \text{if } \sigma_t(x) = 0.\end{cases} \tag{5.7}$$

Substituting this in (5.6) completes the proof. □

In order to prove the second assertion in Lemma 5.1, we will directly prove (5.6) using the backward strong Markov property. Before beginning the proof, we establish a few preliminaries.

Let D, u, T, σ, A, w, $\bar{w}$ be as in the second assertion of Lemma 5.1. Given $x\in D$ and a $d\times d$ matrix M, define the process $\{B_{s,t}(x,M)\}_{\sigma_t(x)\leq s\leq t\leq T}$ to be the solution of the ODE

$$B_{s,t}(x,M) = M - \int_s^t \nabla u_r|_{A_{r,t}(x)} B_{r,t}(x,M)\,dr.$$

If I denotes the $d \times d$ identity matrix, then by (5.1) we have $B_{s,t}(x, I) = \nabla A_{s,t}(x)$ for any $\sigma_t(x) \le s \le t \le T$. Further, since the evolution equation for B is linear, we see

$$B_{s,t}(x, M) = B_{s,t}(x, I)M = \nabla A_{s,t}(x)M. \tag{5.8}$$

Note that for any fixed $t \in (0, T]$, the process $\{\nabla A_{s,t}\}_{0 \le s \le t}$ is *not* a backward Markov process. Indeed, the evolution of $\nabla A_{s,t}$ at any time $s \le t$ depends on the time s through the process $A_{s,t}$ appearing on the right-hand side in (5.1). However, process $(A_{s,t}, \nabla A_{s,t})$ [or equivalently the process $(A_{s,t}, B_{s,t})$] is a backward Markov process since the evolution of this system now only depends on the state. This leads us to the following identity which is the essence of proof of the second assertion in Lemma 5.1.

LEMMA 5.4. *Choose any backward t-stopping time σ' with $\sigma' \ge \sigma_t(x)$ almost surely. Then*

$$\begin{aligned} &E_{\mathcal{F}_{\sigma',t}} B^*_{\sigma_t(x),t}(x, I)\bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) \\ &\qquad = \big[E B^*_{\sigma_r(y),r}(y, M)\bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y)\big]_{\substack{r=\sigma', y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}} \end{aligned} \tag{5.9}$$

holds almost surely.

This follows from an appropriate application of the backward strong Markov property. While this is easily believed, checking that the strong Markov property applies in this situation requires a little work and will distract from the heart of the matter. Thus, we momentarily postpone the proof of Lemma 5.4 and proceed with the proof of the second assertion of Lemma 5.1.

PROOF OF LEMMA 5.1. We recall $\bar{w} = Ew$ where w is defined by (3.9). By our assumption on u and ∂D, the boundary conditions (5.2) and initial data (3.13) are satisfied. For convenience, when $y \in \partial D$, $t > 0$, we define $w_t(y) = \tilde{w}(y)$ and when $t = 0$, $y \in \bar{D}$, we define $w_0(y) = u_0(y)$.

Let $x \in D$, $t \in (0, T]$ as used before. Let σ' be any backward t-stopping time with $\sigma' \ge \sigma_t(x)$. First, if $\sigma' = \sigma_t(x)$ almost surely, then, since the point $(A_{\sigma_t(x),t}, t)$ belongs to the parabolic boundary $\partial_p(D \times [0, T]) \stackrel{\text{def}}{=} (\partial D \times [0, T]) \cup (D \times \{0\})$, our boundary conditions and initial data will guarantee (5.6).

Now, for arbitrary $\sigma' \ge \sigma_t(x)$, we will use Lemma 5.4 to deduce (5.6) directly. Indeed,

$$\begin{aligned} \bar{w}_t(x) &= E\nabla^* A_{\sigma_t(x),t}(x)\bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t} \\ &= E E_{\mathcal{F}_{\sigma',t}} B^*_{\sigma_t(x),t}(x, I)\bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) \end{aligned}$$

$$\begin{aligned} &= E\Big(\big[E B^*_{\sigma_r(y),r}(y, M)\bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y)\big]_{\substack{r=\sigma', y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\Big) \\ &= E\Big(\big[M^* E B^*_{\sigma_r(y),r}(y, I)\bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y)\big]_{\substack{r=\sigma', y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\Big) \\ &= E\nabla^* A_{\sigma',t}(x)\bar{w}_{\sigma'} \circ A_{\sigma',t}(x), \end{aligned}$$

showing that (5.6) holds for any backward t stopping time $\sigma' \ge \sigma_t(x)$.

Now, choose $\sigma' = s \vee \sigma_t(x)$ for $s < t$. Note that for any $x \in D$, we must have $\sigma_t(x) < t$ almost surely. Thus, omitting the spatial coordinate for convenience, we have

$$\begin{aligned} 0 &= \lim_{s \to t^-} \frac{\bar{w}_t - \bar{w}_t}{t - s} = \lim_{s \to t^-} \frac{1}{t - s}(\bar{w}_t - E\nabla^* A_{s\vee\sigma_t,t}\bar{w}_{s\vee\sigma_t} \circ A_{s\vee\sigma_t,t}) \\ &= \lim_{s \to t^-} \Big(\frac{1}{t - s}[\bar{w}_t - E\bar{w}_t \circ A_{s\vee\sigma_t,t}] \\ &\qquad + \frac{1}{t - s}E(\bar{w}_t - \bar{w}_{s\vee\sigma_t}) \circ A_{s\vee\sigma_t,t} \\ &\qquad + \frac{1}{t - s}E(I - \nabla^* A_{s\vee\sigma_t,t})\bar{w}_{s\vee\sigma_t} \circ A_{s\vee\sigma_t,t}\Big) \\ &= L_t\bar{w}_t + \partial_t\bar{w}_t + (\nabla^* u_t)\bar{w}_t, \end{aligned}$$

on the interior of D. The proof is complete. □

It remains to prove Lemma 5.4.

PROOF OF LEMMA 5.4. Define the stopped processes $A'_{s,t}(x) = A_{\sigma_t(x)\vee s,t}(x)$ and $B'_{s,t}(x, M) = B_{\sigma_t(x)\vee s,t}(x, M)$. Define the process C by

$$C_{s,t}(x, M, \tau) = \big(A'_{s,t}(x), B'_{s,t}(x, M), \tau + t - \sigma_t(x) \vee s\big).$$

Note that for any given $s \le t$, we know that $\sigma_t(x)$ need not be $\mathcal{F}_{s,t}$ measurable. However, $\sigma_t(x) \vee s$ *is* an $\mathcal{F}_{s,t}$ measurable backward t-stopping time. Thus, $A'_{s,t}$, $B'_{s,t}$ and, consequently, $C_{s,t}$ are all $\mathcal{F}_{s,t}$ measurable.

Now we claim that almost surely, for $0 \le r \le s \le t \le T$, we have the backward semigroup identity

$$C_{r,s} \circ C_{s,t} = C_{r,t}. \tag{5.10}$$

To prove this, consider first the third component of the left-hand side of (5.10):

$$C^{(3)}_{r,s} \circ C_{s,t}(x, M, \tau) = \big(\tau + t - \sigma_t(x) \vee s\big) + s - \sigma_s(A'_{s,t}(x)) \vee s. \tag{5.11}$$

Consider the event $\{s > \sigma_t(x)\}$. By the semigroup property for A and strong existence and uniqueness of solutions to (3.2), we have $\sigma_s(A_{s,t}(x)) = \sigma_t(x)$ almost surely. Thus, almost surely on $\{s > \sigma_t(x)\}$, we have

$$\begin{aligned} C^{(3)}_{r,s} \circ C_{s,t}(x, M, \tau) &= (\tau + t - s) + s - \sigma_t(x) \vee s \\ &= \tau + t - \sigma_t(x) \vee r = C^{(3)}_{r,t}(x, M, \tau). \end{aligned}$$

Now consider the event $\{s \leq \sigma_t\}$. We know $A'_{s,t}(x) \in \partial D$ and so $\sigma_s(A'_{s,t}(x)) = s$. This gives

$$C^{(3)}_{r,s} \circ C_{s,t}(x, M, \tau) = \big(\tau + t - \sigma_t(x)\big) + s - s = \tau + t - \sigma_t(x) \vee r = C^{(3)}_{r,t}(x)$$

almost surely on $\{s \leq \sigma_t(x)\}$. Therefore, we have proved almost sure equality of the third components in equation (5.10).

For the first component $C^{(1)}_{s,t} = A'_{s,t}$, consider as before the case $s > \sigma_t(x)$. In this case $A'_{s,t} = A_{s,t}$ and the semigroup property of A gives equality of the first components in (5.10) almost surely on $\{s > \sigma_t(x)\}$. When $s \leq \sigma_t(x)$, as before, $A'_{s,t} \in \partial D$ and $\sigma_s(A'_{s,t}(x)) = s$. Thus,

$$A'_{r,s} \circ A'_{s,t}(x) = A_{s,s} \circ A_{\sigma_t(x),t}(x) = A_{\sigma_t(x),t}(x) = A'_{r,t}(x)$$

almost surely on $s \leq \sigma_t(x)$. This shows almost sure equality of the first components in equation (5.10). Almost sure equality of the second components follows similarly, completing the proof of (5.10).

Now, for $0 \leq r \leq s \leq t \leq T$, the random variable $C_{s,t}$ is $\mathcal{F}_{s,t}$ measurable and so must be independent of $\mathcal{F}_{r,s}$. This, along with (5.10), will immediately guarantee the Markov property for C. Since the filtration $\mathcal{F}_{\cdot,\cdot}$ satisfies the usual conditions and for any fixed t the function $s \mapsto C_{s,t}$ is continuous, C satisfies the strong Markov property (see, e.g., [18], Theorem 2.4).

Thus, for any fixed $t \in [0, T]$ and any Borel function φ, the strong Markov property gives

$$\begin{aligned} E_{\mathcal{F}_{\sigma',t}} \varphi(C_{0,t}(x, I, 0)) &= [E\varphi(C_{r,t}(y, M, \tau))]_{\substack{r=\sigma', \\ (y,M,\tau)=C_{0,\sigma'}(x,I,0)}} \\ &= [E\varphi(C_{r,t}(y, M, \tau))]_{\substack{r=\sigma', y=A_{\sigma',t}(x), \\ M=B_{\sigma',t}(x,I), \tau=\sigma_r(x),}} \end{aligned}$$

almost surely for any $x \in \mathbb{R}^d$, $M \in \mathbb{R}^{d^2}$, $\tau \geq 0$. Choosing $\varphi(x, M, \tau) = M^* \bar{w}_{t-\tau}(x)$ proves (5.9). $\square$

Now a direct computation shows that if $\bar{w}$ satisfies (3.12), then $u = \mathbf{P}\bar{w}$ satisfies (1.1) regardless of our choice of $\tilde{w}$. Of course, we will only get the no-slip boundary conditions with the correct choice of $\tilde{w}$. We first obtain the PDE for u.

LEMMA 5.5. *If $\bar{w}$ satisfies (3.12) and $u = \mathbf{P}\bar{w}$, then u satisfies (1.1) and (1.2).*

PROOF. By definition of the Leray–Hodge projection, $u = w + \nabla q$ for some function q and equation (1.2) is automatically satisfied. Thus, using equation (3.12) we have

$$\begin{aligned} &\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \triangle u_t + (\nabla^* u_t) u_t \\ &\quad + \partial_t \nabla q_t + (u_t \cdot \nabla) \nabla q_t + (\nabla^* u_t) \nabla q_t - \nu \triangle \nabla q_t = 0. \end{aligned} \tag{5.12}$$

Defining p by

$$\nabla p = \nabla\big(\tfrac{1}{2}|u|^2 + \partial_t q_t + (u_t \cdot \nabla) q_t - \nu \triangle q_t\big),$$

equation (5.12) becomes (1.1). $\square$

Now to address the no-slip boundary condition. The curl of $\bar{w}$ satisfies the vorticity equation which is how the vorticity enters our boundary condition.

LEMMA 5.6. *Let $\bar{w}$ be a solution of (3.12). Then $\xi = \nabla \times \bar{w}$ satisfies the vorticity equation*

$$\partial_t \xi + (u \cdot \nabla)\xi - \nu \triangle \xi = \begin{cases} 0, & \text{if } d = 2, \\ (\xi \cdot \nabla) u, & \text{if } d = 3. \end{cases} \tag{5.13}$$

PROOF. We only provide the proof for $d = 3$. For this proof we will use subscripts to indicate the component instead of time as we usually do. If $i, j, k \in \{1, 2, 3\}$ are all distinct, let ε_{ijk} denote the signature of the permutation $(1, 2, 3) \mapsto (i, j, k)$. For convenience, we let $\varepsilon_{ijk} = 0$ if i, j, k are not all distinct. Using the Einstein summation convention, $\xi = \nabla \times \bar{w}$ translates to $\xi_i = \varepsilon_{ijk} \partial_j \bar{w}_k$ on components. Thus, taking the curl of (3.12) gives

$$\partial_t \xi_i + (u \cdot \nabla)\xi_i - \nu \triangle \xi_i + \varepsilon_{ijk}\, \partial_j u_m\, \partial_m \bar{w}_k + \varepsilon_{ijk}\, \partial_k u_m\, \partial_j \bar{w}_m = 0 \tag{5.14}$$

because $\varepsilon_{ijk}\, \partial_j \partial_k u_m \bar{w}_m = 0$. Making the substitutions $j \mapsto k$ and $k \mapsto j$ in the last sum above we have

$$\begin{aligned} \varepsilon_{ijk}\, \partial_j u_m\, \partial_m \bar{w}_k + \varepsilon_{ijk}\, \partial_k u_m\, \partial_j \bar{w}_m &= \varepsilon_{ijk}\, \partial_j u_m (\partial_m \bar{w}_k - \partial_k \bar{w}_m) \\ &= \varepsilon_{ijk}\, \partial_j u_m \varepsilon_{nmk} \xi_n \\ &= (\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn})\, \partial_j u_m \xi_n \\ &= -\partial_j u_i \xi_j, \end{aligned}$$

where δ_{ij} denotes the Kronecker delta function and the last equality follows because $\partial_j u_j = 0$. Thus, (5.14) reduces to (5.13). $\square$

Theorem 3.1 now follows from the above lemmas.

PROOF OF THEOREM 3.1. First, suppose u is a solution of the Navier–Stokes equations, as in the statement of the theorem. We choose $\tilde{w}$ as explained in Remark 3.5. Notice that our assumptions on u and D will guarantee a classical solution to (3.12)–(3.14) exists on the interval $[0, T]$ and thus, such a choice is possible.

By Lemma 5.1 we see that for w defined by (3.9), the expected value $\bar{w} = Ew$ satisfies (3.12) with initial data (3.13) and boundary conditions (5.2). By our choice of $\tilde{w}$ and uniqueness to the Dirichlet problem (3.12), (3.13) and (5.2), we must have the vorticity boundary condition (3.14).

Now, let $\xi = \nabla \times \bar{w}$ and $\omega = \nabla \times u$. By Lemma 5.6, we see that ξ satisfies the vorticity equation (5.13). Since u satisfies (1.1) and (1.2), it is well known (see, e.g., [15, 28] or the proof of Lemma 5.6) that ω also satisfies

$$\partial_t \omega_t + (u_t \cdot \nabla)\omega_t - \nu \triangle \omega_t = \begin{cases} 0, & \text{if } d = 2, \\ (\omega_t \cdot \nabla) u_t, & \text{if } d = 3. \end{cases} \tag{5.15}$$

From (3.14) we know $\xi = \omega$ on $\partial D \times [0, T]$. By (3.13), we see that $\xi_0 = \nabla \times u_0 = \omega_0$ and hence, $\xi = \omega$ on the parabolic boundary $\partial_p(D \times [0, T])$.

The above shows that ω and ξ both satisfy the same PDE [equations (5.13) or (5.15)] with the same initial data and boundary conditions and so we must have $\xi = \omega$ on $D \times [0, T]$. Thus, $\nabla \times \bar{w} = \nabla \times u$ in $D \times [0, T]$ showing u and $\bar{w}$ differ by a gradient. Since $\nabla \cdot u = 0$ and $u = 0$ on $\partial D \times [0, T]$, we must have $u = \mathbf{P}\bar{w}$ proving (3.10).

Conversely, assume we have a solution to the system (3.2), (3.9) and (3.10). As stated above, Lemma 5.1 shows $\bar{w} = Ew$ satisfies (3.12) with initial data (3.13). By Lemma 5.5 we know u satisfies the equation (1.1) and (1.2) with initial data u_0. Finally, since equation (3.10) shows $\nabla \times u = \nabla \times \bar{w}$ in $D \times [0, T]$ and by continuity, we have the boundary condition (3.11). $\square$

6. Vorticity transport and ideally conserved quantities. The vorticity is a quantity which is of fundamental importance, both for the physical and theoretical aspects of fluid dynamics. To single out one among the numerous applications of vorticity, we refer the reader to two classical criterion which guarantee global and existence and regularity of the Navier–Stokes equations provided the vorticity is appropriately controlled: the first due to Beale, Kato and Majda [2] and the second due to Constantin and Fefferman [9].

For the Euler equations, exact identities and conservation laws governing the evolution of vorticity are well known. For instance, vorticity transport [equation (6.1)] shows that the vorticity at time t followed along streamlines is exactly the initial vorticity stretched by the Jacobian of the flow map. Similarly, the conservation of circulation [equation (6.9)] shows that the line integral of the velocity (which, by Stokes theorem, is a surface integral of the vorticity) computed along a closed curve that is transported by the fluid flow is constant in time.

Prior to [11], these identities were unavailable for the Navier–Stokes equations. In [11], the authors provide analogues of these identities for the Navier–Stokes equations in the absence of boundaries. These identities, however, do not always prevail in the presence of boundaries.

In this section we illustrate the issues involved by considering three inviscid identities. All three identities generalize perfectly to the viscous situations without boundaries. In the presence of boundaries, the first identity (vorticity transport) generalizes perfectly, the second identity (Ertel's Theorem) generalizes somewhat unsatisfactorily and the third identity (conservation of circulation) has no nontrivial generalization in the presence of boundaries.

6.1. *Vorticity transport*. Let u^0 be a solution to the Euler equations with initial data u_0. Let X^0 the inviscid flow map defined by (2.1) and for any $t \geq 0$, let $A_t^0 = (X_t^0)^{-1}$ be the spatial inverse of the diffeomorphism X_t^0. The vorticity transport (or Cauchy formula) states

$$\omega_t^0 = \begin{cases} \omega_0^0 \circ A_t^0, & \text{if } d = 2, \\ [(\nabla X_t^0)\omega_0^0] \circ A_t^0, & \text{if } d = 3, \end{cases} \tag{6.1}$$

where we recall that the vorticity ω^0 is defined by $\omega^0 = \nabla \times u^0$ and where $\omega_0^0 = \nabla \times u_0$ is the initial vorticity.

In [11], the authors obtained a natural generalization of (6.1) for the Navier–Stokes equations in the absence of spatial boundaries. If u solves (1.1) and (1.2) with initial data u_0 and X is the noisy flow map defied by (2.3)–(2.4), then $\omega = \nabla \times u$ is given by

$$\omega_t = \begin{cases} E\omega_0 \circ A_t, & \text{if } d = 2, \\ E((\nabla X_t)\omega_0) \circ A_t, & \text{if } d = 3. \end{cases} \tag{6.2}$$

We now provide the generalization of this in the presence of boundaries. Note that for any $t \geq 0$, $(\nabla X_t) \circ A_t = (\nabla A_t)^{-1}$, so we can rewrite (6.2) completely in terms of the process A. Now, as usual, we replace $A = X^{-1}$ with the solution of (3.2) with respect to the minimal existence time σ. We recall that in Theorem 3.1, in addition to "starting trajectories at the boundary," we had to correct the expression for the velocity by the boundary values of a related quantity (the vorticity). For the vorticity, however, we need no additional correction and the interior vorticity is completely determined given A, σ and the vorticity on the parabolic boundary[7] $\partial_p(D \times [0, T])$.

PROPOSITION 6.1. *Let u be a solution to (1.1) and (1.2) in D with initial data u_0 and suppose $\omega = \nabla \times u \in C^1([0, T); C^2(D)) \cap C([0, T] \times \bar{D})$. Let $\tilde{\omega}$ denote the values of ω on the parabolic boundary $\partial_p(D \times [0, T])$. Explicitly, $\tilde{\omega}$ is defined by*

$$\tilde{\omega}(x, t) = \begin{cases} \omega_0(x), & \text{if } x \in D \text{ and } t = 0, \\ \omega_t(x), & \text{if } x \in \partial D. \end{cases}$$

[7] Recall the parabolic boundary $\partial_p(D \times [0, T])$ is defined to be $(D \times \{0\}) \cup (\partial D \times [0, T))$.

Then,

$$\omega_t(x) = \begin{cases} E\big[\tilde\omega_{\sigma_t(x)}\big(A_{\sigma_t(x),t}(x)\big)\big], & \text{if } d=2,\\ E\big[\big(\nabla A_{\sigma_t(x),t}(x)\big)^{-1}\tilde\omega_{\sigma_t(x)}\big(A_{\sigma_t(x),t}(x)\big)\big], & \text{if } d=3.\end{cases} \tag{6.3}$$

PROPOSITION 6.2. *More generally, suppose $\tilde\omega$ is any function defined on the parabolic boundary of $D\times[0,T]$ and let ω be defined by (6.3). If for all $t\in(0,T]$, $\omega_t\in\mathcal{D}(A_{\cdot,t})$ and ω is C^1 in time, then ω satisfies*

$$\partial_t\omega_t + L_t\omega_t = \begin{cases} 0, & \text{if } d=2,\\ (\omega_t\cdot\nabla)u_t, & \text{if } d=3,\end{cases}$$

with $\omega=\tilde\omega$ on the parabolic boundary. Here, L_t is the generator of $A_{\cdot,t}$; $\mathcal{D}(A_{\cdot,t})$ is the domain of L_t. These are defined in the statement of Lemma 5.1.

Of course, Proposition 6.2, along with Proposition 5.2 and uniqueness of (strong) solutions to (5.15), will prove Proposition 6.1. However, direct proofs of both Proposition 6.2 and Proposition 6.1 are short and instructive and we provide independent proofs of each.

PROOF OF PROPOSITION 6.1. We only provide the proof when $d=3$. As shown before, differentiating (5.1) in space and taking the matrix inverse of both sides gives

$$\partial_r(\nabla A_{r,t}(x))^{-1} = -(\nabla A_{r,t}(x))^{-1}\nabla u_r|_{A_{r,t}(x)}, \tag{6.4}$$

almost surely. Now choose any $x\in D$, $t>0$ and any backward t-stopping time $\sigma'\geq\sigma_t(x)$. Omitting the spatial parameter for notational convenience, the backward Itô formula gives

$$\begin{aligned}
&\omega_t - (\nabla A_{\sigma',t})^{-1}\omega_{\sigma'}\circ A_{\sigma',t}\\
&\quad= (\nabla A_{t,t})^{-1}\omega_t\circ A_{t,t} - (\nabla A_{\sigma',t})^{-1}\omega_{\sigma'}\circ A_{\sigma',t}\\
&\quad= \int_{\sigma'}^t \partial_r(\nabla A_{r,t})^{-1}\omega_r\circ A_{r,t}\,dr\\
&\qquad+ \int_{\sigma'}^t (\nabla A_{r,t})^{-1}\big(\partial_r\omega_r + (u_r\cdot\nabla)\omega_r - \nu\triangle\omega_r\big)\circ A_{r,t}\,dr\\
&\qquad+ \sqrt{2\nu}\int_{\sigma'}^t (\nabla A_{r,t})^{-1}(\nabla\omega_r)\circ A_{r,t}\,dW_r\\
&= \int_{\sigma'}^t -(\nabla A_{r,t})^{-1}\nabla u_r|_{A_{r,t}}\omega_r\circ A_{r,t}\,dr\\
&\qquad+ \int_{\sigma'}^t (\nabla A_{r,t})^{-1}((\omega_r\cdot\nabla)u_r)\circ A_{r,t}\,dr\\
&\qquad+ \sqrt{2\nu}\int_{\sigma'}^t (\nabla A_{r,t})^{-1}(\nabla\omega_r)\circ A_{r,t}\,dW_r\\
&\quad= \sqrt{2\nu}\int_{\sigma'}^t (\nabla A_{r,t})^{-1}(\nabla\omega_r)\circ A_{r,t}\,dW_r.
\end{aligned}$$

Thus, taking expected values gives

$$\omega_t = E[(\nabla A_{\sigma',t})^{-1}\omega_{\sigma'}\circ A_{\sigma',t}]. \tag{6.5}$$

Choosing $\sigma'=\sigma_t(x)$ and using the fact that $A_{\sigma_t(x),t}(x)$ always belongs to the parabolic boundary finishes the proof. □

PROOF OF PROPOSITION 6.2. Again, we only consider the case $d=3$. We will prove equation (6.5) directly and then deduce (5.15). Let the process B be as in the proof of the second assertion of Lemma 5.1 and use B^{-1} to denote the process consisting of *matrix* inverses of the process B. Pick $x\in D$, $t\in(0,T]$ and a backward t-stopping time $\sigma'\geq\sigma_t(x)$. Using (5.9) we have

$$\begin{aligned}
\omega_t(x) &= E\big[\big(\nabla A_{\sigma_t(x),t}(x)\big)^{-1}\tilde\omega_{\sigma_t(x)}\big(A_{\sigma_t(x),t}(x)\big)\big]\\
&= E E_{\mathcal{F}_{\sigma',t}}\big[B^{-1}_{\sigma_t(x),t}(x,I)\tilde\omega_{\sigma_t(x)}\circ A_{\sigma_t(x),t}(x)\big]\\
&= E\Big(\big[E B^{-1}_{\sigma_r(y),r}(y,M)\tilde\omega_{\sigma_r(y)}\circ A_{\sigma_r(y),r}(y)\big]_{\substack{r=\sigma',y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\Big)\\
&= E\Big(\big[M^{-1}E B^{-1}_{\sigma_r(y),r}(y,I)\tilde\omega_{\sigma_r(y)}\circ A_{\sigma_r(y),r}(y)\big]_{\substack{r=\sigma',y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\Big)\\
&= E[(\nabla A_{\sigma',t}(x))^{-1}\omega_{\sigma'}\circ A_{\sigma',t}(x)],
\end{aligned}$$

proving (6.5).

As stated before, choose $s\leq t$ and $\sigma'=\sigma_t(x)\vee s$. Omitting the spatial parameter for notational convenience gives

$$\begin{aligned}
0 &= \lim_{s\to t^-}\frac{\omega_t-\omega_t}{t-s} = \lim_{s\to t^-}\frac{1}{t-s}[\omega_t - E(\nabla A_{\sigma_t\vee s,t})^{-1}\omega_{\sigma_t\vee s}\circ A_{\sigma_t\vee s,t}]\\
&= \lim_{s\to t^-}\Big(\frac{1}{t-s}[\omega_t - E\omega_t\circ A_{\sigma_t\vee s,t}]\\
&\qquad+\frac{1}{t-s}E[\omega_t-\omega_{\sigma_t\vee s}]\circ A_{\sigma_t\vee s,t}\\
&\qquad+\frac{1}{t-s}E[I-(\nabla A_{\sigma_t\vee s,t})^{-1}]\omega_{\sigma_t\vee s}\circ A_{\sigma_t\vee s,t}\Big)\\
&= L_t\omega_t + \partial_t\omega_t - (\nabla u_t)\omega_t.
\end{aligned}$$

□

We remark that the vorticity transport in Propositions 6.1 or 6.2 can be used to provide a stochastic representation of the Navier–Stokes equations. To see this, first note that the proofs of Propositions 6.1 and 6.2 are independent of Theorem 3.1. Next, since u is divergence free, taking the curl twice gives the negative laplacian. Thus, provided boundary conditions on u are specified, we can obtain u from ω by

$$u_t = (-\triangle)^{-1}\nabla \times \omega_t. \tag{6.6}$$

Therefore, in Theorem 3.1 we can replace (3.10) by (6.3) and (6.6), where $\tilde{\omega}$ is the vorticity on the parabolic boundary and we impose 0-Dirichlet boundary conditions on (6.6).

6.2. *Ertel's theorem.* As shown above, we use a superscript of 0 to denote the appropriate quantities related to the Euler equations. For this section we also assume $d = 3$. Ertel's theorem says that if θ^0 is constant along trajectories of X^0, then so is $(\omega^0 \cdot \nabla)\theta^0$. Hence, $\phi^0 = (\omega^0 \cdot \nabla)\theta^0$ satisfies the PDE

$$\partial_t \phi^0 + (u \cdot \nabla)\phi^0 = 0.$$

For the Navier–Stokes equations, we first consider the situation without boundaries. Let u solve (1.1) and (1.2), X be defined by (2.3), A be the spatial inverse of X and define ξ by

$$\xi_t(x) = (\nabla A_t(x))^{-1}\omega_0 \circ A_t(x),$$

where $\omega_0 = \nabla \times u_0$ is the initial vorticity. From (6.2) we know that $\omega = \nabla \times u = E\xi$. Now we can generalize Ertel's theorem as follows:

PROPOSITION 6.3. *Let θ be a $C^1(\mathbb{R}^d)$ valued process. If θ is constant along trajectories of the (stochastic) flow X, then so is $(\xi \cdot \nabla)\theta$. Hence, $\phi = E(\xi \cdot \nabla)\theta$ satisfies the PDE*

$$\partial_t \phi_t + (u_t \cdot \nabla)\phi_t - \nu\triangle\phi_t = 0, \tag{6.7}$$

with initial data $(\omega_0 \cdot \nabla)\theta_0$.

PROOF. If θ is constant along trajectories of X, we must have $\theta_t = \theta_0 \circ A_t$ almost surely. Thus,

$$(\xi_t \cdot \nabla)\theta_t = (\nabla\theta_t)\xi_t = \nabla\theta_0|_{A_t}(\nabla A_t)(\nabla A_t)^{-1}\omega_0 \circ A_t = (\xi_0 \cdot \nabla\theta_0) \circ A_t,$$

which is certainly constant along trajectories of X. The PDE for ϕ now follows immediately. $\square$

Now, in the presence of boundaries, this needs further modification. Let A be a solution to (3.2) and σ be the backward exit time of A from D. The notion of "constant along trajectories" now corresponds to processes θ defined by

$$\theta_t(x) = \tilde{\theta}_{\sigma_t(x)}(A_{\sigma_t(x),t}), \tag{6.8}$$

for some function $\tilde{\theta}$ defined on the parabolic boundary of D.

Irrespective of the regularity of D and $\tilde{\theta}$, the process θ will not be continuous in space, let alone differentiable. The problem arises because while A is regular enough in the spatial variable, the existence time σ_t is not. Thus, we are forced to avoid derivatives on σ in the statement of the theorem, leading to a somewhat unsatisfactory generalization.

PROPOSITION 6.4. *Let $\tilde{\theta}$ be a C^1 function defined on the parabolic boundary of $D \times [0, T]$ and let $\tilde{\theta}'$ be any C^1 extension of $\tilde{\theta}$, defined in a neighborhood of the parabolic boundary of $D \times [0, T]$. If θ is defined by (6.8), then*

$$\phi_t(x) = E[(\xi_t \cdot \nabla)(\tilde{\theta}'_s \circ A_{s,t})(x)]_{s=\sigma_t(x)}$$

satisfies the PDE (6.7) with initial data $(\omega_0 \cdot \nabla)\tilde{\theta}_0$ and boundary conditions $\phi_t(x) = (\omega_t \cdot \nabla)\tilde{\theta}'(x)$ for $x \in \partial D$.

REMARK. A satisfactory generalization in the scenario with boundaries would be to make sense of $E(\xi_t \cdot \nabla)\theta_t$ (despite the spatial discontinuity of θ) and reformulate Proposition 6.4 accordingly.

Note that when $D = \mathbb{R}^d$, then $\sigma_t \equiv 0$ and hence, $\phi_t = E(\xi_t \cdot \nabla)\theta_t$. In this case Proposition 6.4 reduces to Proposition 6.3. The proof of Proposition 6.4 is identical to that of Proposition 6.3 and the same argument obtains

$$[(\xi_t \cdot \nabla)(\tilde{\theta}'_s \circ A_{s,t})(x)]_{s=\sigma_t(x)} = [(\xi_s \cdot \nabla)\tilde{\theta}'_s(y)]_{\substack{s=\sigma_t(x),\\ y=A_{\sigma_t(x),t}(x)}},$$

which immediately implies (6.7).

6.3. *Circulation.* The circulation is the line integral of the velocity field along a closed curve. For the Euler equations, the circulation along a closed curve that is transported by the flow is constant in time. Explicitly, let u^0, X^0, A^0, u_0 be as in the previous subsection. Let Γ be a rectifiable closed curve, then for any $t \geq 0$,

$$\oint_{X_t^0(\Gamma)} u_t^0 \cdot dl = \oint_\Gamma u_0^0 \cdot dl. \tag{6.9}$$

For the Navier–Stokes equations, without boundaries, a generalization of (6.9) was considered in [11]. Let u solve (1.1) and (1.2), X be defined by (2.3) and (2.4) and A be the spatial inverse of X. Then

$$\oint_\Gamma u_t \cdot dl = E\oint_{A_t(\Gamma)} u_0 \cdot dl. \tag{6.10}$$

A proof of this (in the absence of boundaries) follows immediately from Theorem 2.1. Indeed,

$$\begin{aligned} E\oint_{A_t(\Gamma)} u_0 \cdot dl &= E\oint_\Gamma (\nabla^* A_t)u_0 \circ A_t \cdot dl \\ &= E\oint_\Gamma \mathbf{P}[(\nabla^* A_t)u_0 \circ A_t] \cdot dl = \oint_\Gamma u_t \cdot dl, \end{aligned} \tag{6.11}$$

where the first equality follows by definition of line integrals, the second because the line integral of gradients along closed curves is 0 and the last by Fubini and (2.5).

Equation (6.10) does not make sense in the presence of boundaries, as the curves one integrates over will no longer be rectifiable!

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