A STOCHASTIC-LAGRANGIAN PARTICLE SYSTEM FOR THE NAVIER-STOKES EQUATIONS.

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ABSTRACT. This paper is based on a formulation of the Navier-Stokes equations developed in `arxiv:math.PR/0511067` (to appear in Commun. Pure Appl. Math), where the velocity field of a viscous incompressible fluid is written as the expected value of a stochastic process. In this paper, we take N copies of the above process (each based on independent Wiener processes), and replace the expected value with $\frac{1}{N}$ times the sum over these N copies. (We remark that our formulation requires one to keep track of N stochastic flows of diffeomorphisms, and not just the motion of N particles.)

We prove that in two dimensions, this system has (time) global solutions with initial data in the space $C^{1,\alpha}$ which consists of differentiable functions whose first derivative is α Hölder continuous (see Section 3 for the precise definition). Further, we show that as $N \to \infty$ the system converges to the solution of Navier-Stokes equations on any finite interval $[0, T]$. However for fixed N, we prove that this system retains roughly $O(\frac{1}{N})$ times its original energy $t \to \infty$. For general flows, we only provide a lower bound to this effect. In the special case of shear flows, we compute the behaviour as $t \to \infty$ explicitly.

1. INTRODUCTION

The Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u - \nu \triangle u + \nabla p = 0 \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

describe the evolution of a velocity field of an incompressible fluid with kinematic viscosity $\nu > 0$. These equations have been used to model numerous physical problems, for example air flow around an airplane wing, ocean currents and meteorological phenomena to name a few [3, 14, 16]. The mathematical theory (existence and regularity [4, 13]) of these equations have been extensively studied and is still one of the outstanding open problems in modern PDE's [6, 7].

The questions addressed in this paper are motivated by a formalism of (1.1)–(1.2) developed in [8] (equations (2.1)–(2.2)). This formalism essentially superimposes Brownian motion onto particle trajectories, and then averages with respect to the Wiener measure. In this paper, we take N independent copies of the Wiener process and replace the expected value in the above formalism with $\frac{1}{N}$ times the sum over these N independent copies (see equations (2.5)–(2.7) for the exact details). In the original, the formulation the random trajectory of a induced by a single Brownian

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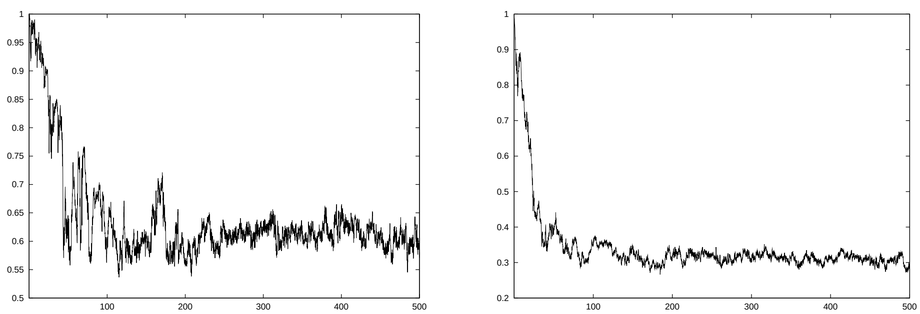
motion interacts with its own law. This is essentially a self-consistent, mean-field interaction. In this paper, we replace this with N copies or replica whose average is used to approximate the interaction with the processes own law. This technique has been extensively used in numerical computation (e.g. [2, 15, 17]). We draw attention to the fact that in our formulation, we are required to keep track of N stochastic flows of diffeomorphisms, and not just the motion of N different particles, as is the conventional approach.

We study both the behaviour as $N \to \infty$ and $t \to \infty$ of this system obtained. The behaviour as $N \to \infty$ is as expected: In two dimensions on any finite time interval $[0, T]$, the system converges as $N \to \infty$ to the solution of the true Navier-Stokes equations at rate roughly $O(\frac{1}{\sqrt{N}})$. In three dimensions, we can only guarantee this if we have certain apriori bounds on the solution (Theorem 4.1). These apriori bounds are of course guaranteed for short time, but are unknown (in the 3-dimensional setting) for long time [6, 7].



(a) $N = 2$ (b) $N = 8$

FIGURE 1. Graph of $\|\omega_t^N\|_{L^2}^2$ vs time

At first glance, the behaviour as $t \to \infty$ for fixed N is less intuitive. For the 2-dimensional problem, Figures 1(a) and 1(b) show a graph of $\|\omega_t^N\|_{L^2}^2$ vs time, with $N = 2$ and $N = 8$ respectively. A little reflection shows that this behavior is not completely surprising. The disposition occurs through the averaging of different copies of the flow. With only N copies, one can only produce dissipation of order $1/\sqrt{N}$.

In the Section 5 we and obtain a sharp lower bound to this effect. We show (Theorem 5.2) that

$$\limsup_{t \to \infty} \mathbb{E} \|\nabla u_t\|_{L^2}^2 \geqslant \frac{1}{NL^2} \|u_0\|_{L^2}^2 .$$

Further, we explicitly compute the $t \to \infty$ behaviour in the special case of shear flows and verify that our lower bound is sharp.

Finally we remark that we considered the analogue of the system above for the one dimensional Burgers equations. As is well known the viscous burgers equations have global strong solutions. However preliminary numerical simulations show that the system above forms shocks almost surely, even for very large N. We are currently working on understanding how to continue this system past these shocks, in a manner analogous to the entropy solutions for the inviscid Burgers equations, and studying its behaviour as $t \to \infty$ and $N \to \infty$.

2. The particle system

In this section we construct a particle system for the Navier-Stokes equations based on stochastic Lagrangian trajectories. We begin by describing a stochastic Lagrangian formulation of the Navier-Stokes equations developed in [8, 10].

Let W be a standard 2 or 3-dimensional Brownian motion, and u_0 some given divergence free $C^{2,\alpha}$ initial data. Let $\mathbb{E}$ denote the expected value with respect to the Wiener measure and $\mathbf{P}$ be the Leray-Hodge projection onto divergence free vector fields. Consider the system of equations

$$dX_t(x) = u_t(X_t(x))\,dt + \sqrt{2\nu}\,dW_t, \qquad X_0(x) = x, \tag{2.1}$$

$$u_t(x) = \mathbb{E}\,\mathbf{P}\left[(\nabla^* Y_t)(u_0 \circ Y_t)\right](x), \qquad Y_t = X_t^{-1}. \tag{2.2}$$

With a slight abuse of notation, we denote by X_t the map from initial conditions to the value at time t. Hence X_t is a stochastic flow of diffeomorphisms with X_0 equal to the identity and Y_t the "spatial" inverse. In other words, $Y_t\colon X_t(x) \mapsto x$. Also by $\nabla^* Y_t$ we mean the transpose of the Jacobian of map Y_t. We impose periodic boundary conditions on the displacement $\lambda_t(y) = X_t(y) - y$, and on the Leray-Hodge projection $\mathbf{P}$.

In [8, 10] it was shown that the system (2.1)–(2.2) is equivalent to the Navier-Stokes equations in the following sense: If the initial data is regular ($C^{2,\alpha}$), then the pair X, u is a solution to the system (2.1)–(2.2) if and only if u is a (classical) solution to the incompressible Navier-Stokes equations with periodic boundary conditions and initial data u_0.

We digress briefly and comment on the physical significance of (2.1)–(2.2). Note first that that equation (2.2) is *algebraically* equivalent to the equations

$$u_t = -\triangle^{-1} \nabla \times \omega_t \tag{2.3}$$

$$\omega_t = [(\nabla X_t)\omega_0] \circ Y_t \tag{2.4}$$

This follows by direct computation, and was shown [5] and [8] for instance. We recall that (2.4) is the usual vorticity transport equation for the Euler equations, and (2.3) is just the Biot-Savart formula.

Thus in particular when $\nu = 0$, the system (2.5)–(2.7) is exactly the incompressible Euler equations. Hence the system (2.1)–(2.2) essentially does the following: We add Brownian motion to Lagrangian trajectories. Then recover the velocity u in the same manner as for the Euler equations, but additionally average out the noise.

We remark that the system (2.1)–(2.2) is non-linear in the sense of McKean [18]. The drift of the flow X depends on its distribution. However in this case, the law of X alone is not enough to compute the drift u. This is because of the presence of the $\nabla^* Y$ term in (2.2), which requires knowledge of spatial covariances, in addition to the law of X. In other words, one needs that law of the entire flow of diffeomorphism and not just the law of the one-point motions.

We now motivate our particle system. For the formulation (2.1)–(2.2) above, the natural numerical scheme would be to use the law of large numbers to compute the expected value. Let (W^i) be a sequence of independent Wiener processes, and consider the system

$$dX_t^{i,N} = u_t^N\left(X_t^{i,N}\right)\,dt + \sqrt{2\nu}\,dW_t^i \tag{2.5}$$

$$Y_t^{i,N} = \left(X_t^{i,N}\right)^{-1} \tag{2.6}$$

$$u_t^N = \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* Y_t^{i,N})(u_0 \circ Y_t^{i,N})\right] \tag{2.7}$$

with initial data $X_0(x) = x$. We impose again periodic boundary conditions on the initial data u_0, the displacement $\lambda_t(x) = X_t(x) - x$, and the Leray-Hodge projection $\mathbf{P}$.

We remark that the algebraic equivalence of (2.2) and (2.3)–(2.4) is still valid in this setting. Thus the system (2.5)–(2.7) could equivalently be formulated by replacing equation (2.7) with the more familiar equations

$$\omega_t^N = \frac{1}{N}\sum_{i=1}^N \left[(\nabla X_t^{i,N})\omega_0\right] \circ Y_t^{i,N} \tag{2.8}$$

and

$$u_t^N = -\triangle^{-1}\nabla \times \omega_t^N. \tag{2.9}$$

Finally we mention again that the presence of $\nabla^* Y^{i,N}$ in (2.7) (or the presence of $\nabla X^{i,N}$ in (2.8)) requires the N *entire* flows $X^{i,N}$. One can of course rewrite the above system involving only the laws of $X^{i,N}$ and $\nabla X^{i,N}$, however knowing only the one point motions of the flow $X^{i,N}$ or the law of $X^{i,N}$ alone will not give a closed system of equations.

In the next section, we show the existence of global solutions to (2.5)–(2.7) in two dimensions. In section 4, we show that the solution to (2.5)–(2.7) converges to the solution of the Navier-Stokes equations as $N \to \infty$. Finally in Section 5 we study the behaviour of the system (2.5)–(2.7) as $t \to \infty$ (for fixed N), and partially explain the behaviour shown in Figures 1(a) and 1(b).

3. Global existence of the particle system in two dimensions.

In this section we prove that the particle system (2.5)–(2.7) has global solutions in two dimensions. Once we are guaranteed global in time solutions, we are able to study the behaviour as $t \to \infty$, which we do in Section 5. We remark also that as a consequence of Theorem 3.5 (proved here), our convergence result as $N \to \infty$ (Theorem 4.1) applies on any finite time interval $[0, T]$ in the two dimensional situation.

We first establish some notational convention: We let $L > 0$ be a length scale, and assume work with the spatial domain is $[0, L]^d$, where $d \geqslant 2$ is the spatial dimension. We define the non-dimensional L^p and Hölder norms by

$$\|u\|_{L^p} = L^{\frac{-d}{p}}\left(\int_{[0,L]^d} |u|^p\right)^{\frac{1}{p}}$$

$$\|u\|_{L^\infty} = \sup_{x\in[0,L]^d} |u(x)|$$

$$|u|_\alpha = L^\alpha \sup_{x,y\in[0,L]^d} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

$$\|u\|_{k,\alpha} = \sum_{|m|=k} L^k |D^m u|_\alpha + \sum_{|m|<k} L^{|m|} \|D^m u\|_{L^\infty}$$

We denote the Hölder space $C^{k,\alpha}$ and Lebesgue space L^p to be the space of functions u which are periodic on $[0,L]^d$ with $\|u\|_{k,\alpha} < \infty$ and $\|u\|_{L^p} < \infty$ respectively.

Let $W^1, \dots, W^N$ be N independent (2 dimensional) Wiener processes, with filtration $\mathcal{F}_t$.

Proposition 3.1. *Let $t_0 \geqslant 0$, and $u^1_{t_0}, \dots, u^N_{t_0}$ be $\mathcal{F}_{t_0}$-measurable, periodic mean 0 functions such that the norms $\|u^i_{t_0}\|_{1,\alpha}$ are almost surely bounded. Then there exist $T = T(\alpha, \|u^1_{t_0}\|_{1,\alpha}, \dots, \|u^N_{t_0}\|_{1,\alpha})$ such that the system*

$$X^i_{t_0,t}(x) = x + \int_{t_0}^t u_s(X^i_{t_0,s}(x))\, ds + \sqrt{2\nu}(W^i_t - W^i_{t_0}) \tag{3.1}$$

$$Y^i_{t_0,t} = (X^i_{t_0,t})^{-1} \tag{3.2}$$

$$u_t(x) = \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* Y^i_{t_0,t})\,(u^i_{t_0} \circ Y^i_{t_0,t})\right](x) \tag{3.3}$$

has a solution on the interval $[t_0, T]$. Further there exists $U = U(\alpha, \|u^i_{t_0}\|_{1,\alpha})$ such that

$$\sup_{t\in[t_0,T]} \left\|\mathbf{P}\left[(\nabla^* Y^i_{t_0,t})\, u^i_{t_0} \circ Y^i_{t_0,t}\right]\right\|_{1,\alpha} \leqslant U \tag{3.4}$$

almost surely.

Proposition 3.1 is proved in Appendix A

Definition 3.2. We call X^i, u in Proposition 3.1 the solution of the system (3.1)–(3.3) with initial data $u^1_{t_0}, \dots, u^N_{t_0}$.

The functions X^i, Y^i are not periodic themselves, but have periodic displacements: Namely, if we define

$$\lambda^i_t(y) = X^i_t(y) - y \tag{3.5}$$

$$\mu^i_t(x) = Y^i_t(x) - x\ , \tag{3.6}$$

then μ^i, λ^i are periodic.

We remark that if $t_0 = 0$ and the $\omega^i_{t_0}$'s are all equal, then the system (3.1)–(3.3) reduces to (2.5)–(2.7). However when formulated as above, solutions can be continued past time T by restarting the flows X^i, as in the Lemma below.

Lemma 3.3. *Say $t_0 < t_1 < t_2$, and $X^i_{t_0,s}$, u_s solve* (3.1)–(3.3) *on $[t_0, t_2]$, with initial data $u^i_{t_0}$. For any $t > t_0$ define*

$$u^i_t = \mathbf{P}\left[(\nabla^* Y^i_{t_0,t})\, u^i_{t_0} \circ Y^i_{t_0,t}\right].$$

Let $\tilde{X}^i_{t_1,s}$, $\tilde{u}_s$ solve (3.1)–(3.3) *on $[t_1, t_2]$ with initial data $u^i_{t_1}$. Then for all $s \in [t_1, t_2]$ we have $\tilde{u}^i_s = u^i_s$ and*

$$X^i_{t_0,s} = \tilde{X}^i_{t_1,s} \circ X^i_{t_0,t_1}$$

almost surely.

Proof. The proof is identical to the proof of Proposition 3.3.1 in [10], and we do not provide it here. □

For the remainder of this section, we assume without loss of generality that $t_0 = 0$ (we allow of course $\mathcal{F}_0$ to be non-trivial). For ease of notation, we use X_s to denote $X_{0,s}$. We now prove that the system (3.1)–(3.3) has global solutions in two dimensions. This essentially follows from a Beale-Kato-Majda type condition [1], and the two dimensional vorticity transport.

Lemma 3.4. *If u is divergence free and periodic in $\mathbb{R}^d$, then for any $\alpha \in (0,1)$, there exists a constant $c = c(\alpha, d)$ such that*

$$\|\nabla u\|_{L^\infty} \leqslant c \, \|\omega\|_{L^\infty} \left(1 + \ln^+ \left(\frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}} \right) \right)$$

where $\omega = \nabla \times u$.

The lemma is a standard fact about singular integral operators, and we provide a proof in Appendix B for completeness.

Theorem 3.5. *In 2 dimensions, the system (3.1)–(3.3) has time global solutions provided the initial data $u_0^1, \dots u_0^N$ is periodic, $\mathcal{F}_0$-measurable with $\|u_0^i\|_{1,\alpha}$ bounded almost surely. In particular we have global solutions to (3.1)–(3.3) in two dimensions, if $u_0^1 = \cdots = u_0^N = u_0$ is deterministic, Hölder $1+\alpha$ and periodic.*

Proof. Taking the curl of (3.3) gives the familiar Cauchy formula [5, 8, 9]

$$\omega_t = \frac{1}{N} \sum_{i=1}^N \left[(\nabla X_t^i) \, \omega_0^i \right] \circ Y_t^i \tag{3.7}$$

which in two dimensions reduces to

$$\omega_t = \frac{1}{N} \sum_{i=1}^N \omega_0^i \circ Y_t^i. \tag{3.8}$$

Taking Hölder norms gives

$$\|\omega_t\|_\alpha \leqslant \frac{c}{N} \sum_{i=1}^N \left\|\omega_0^i\right\|_\alpha (1 + \left\|\nabla Y^i\right\|_{L^\infty}^\alpha) \quad \text{a.s.} \tag{3.9}$$

Now differentiating (3.1) gives

$$\nabla X_t^i = I + \int_0^t (\nabla u_s \circ X_s^i) \, \nabla X_s^i \, ds \quad \text{a.s.}$$

Taking the L^∞ norm, and applying Gronwall's Lemma shows

$$\left\|\nabla X_t^i\right\|_{L^\infty} \leqslant \exp\left(c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right) \quad \text{a.s.}$$

Recall $\nabla \cdot u = 0$, and hence $\det(\nabla X^i) = 1$ almost surely. Thus the entries of ∇Y are a polynomial (of degree 1) in the entries of ∇X. This immediately gives

$$\left\|\nabla Y_t^i\right\|_{L^\infty} \leqslant \exp\left(c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right) \tag{3.10}$$

almost surely. Combining this with (3.9) gives us the apriori bound

$$\|\omega_t\|_\alpha \leqslant \frac{c}{N} \sum_{i=1}^N \left\|\omega_0^i\right\|_\alpha \exp\left(c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right) \tag{3.11}$$

Applying Lemma 3.4 gives us

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqslant c\,\|\omega\|_{L^\infty}\left(1+\ln^+\left(\frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}}\right)\right)\\ &\leqslant c\,\|\omega\|_{L^\infty}\left(1+\ln^+\left(\frac{|\omega|_\alpha}{\frac{1}{N}\sum_{i=1}^N\left\|\omega_0^i\right\|_{L^\infty}}\right)+\ln^+\left(\frac{\frac{1}{N}\sum_{i=1}^N\left\|\omega_0^i\right\|_{L^\infty}}{\|\omega\|_{L^\infty}}\right)\right)\end{aligned}$$

Note that the function $x\ln^+\frac{1}{x}$ is bounded, so the last term on the right can be bounded above by some constant c_0. For the remainder of the proof, we let $c_0 = c_0(\|\omega_0^i\|_\alpha, \alpha)$ denote a constant (with dimensions that of ω) which changes from line to line. Thus

$$\begin{aligned}\|\nabla u_t\|_{L^\infty} &\leqslant c_0 + c\,\|\omega_t\|_{L^\infty}\left(1+\ln^+\left(\frac{|\omega_t|_\alpha}{\frac{1}{N}\sum_{i=1}^N\left\|\omega_0^i\right\|_{L^\infty}}\right)\right)\\ &\leqslant c_0 + c\,\|\omega_t\|_{L^\infty}\left(1+\int_0^t\|\nabla u_s\|_{L^\infty}\,ds\right)\end{aligned}$$

and hence

$$\|\nabla u_t\|_{L^\infty}\left(1+\int_0^t\|\nabla u_s\|_{L^\infty}\,ds\right)^{-1}\leqslant c_0 + c\,\|\omega_t\|_{L^\infty}.$$

Integrating gives us the apriori bound

$$\begin{aligned}\int_0^t\|\nabla u_t\|_{L^\infty} &\leqslant \exp\left(c_0 t + c\int_0^t\|\omega_t\|_{L^\infty}\right)-1\\ &\leqslant c_0 t e^{c_0 t}\end{aligned}\tag{3.12}$$

since (3.8) implies $\|\omega_t\|_{L^\infty}\leqslant\frac{1}{N}\sum\|\omega_0^i\|_{L^\infty}$.

Now if we set $\omega_t^i = \omega_0^i\circ Y_t^i$, then (3.10) and the apriori bound (3.12) gives

$$\left\|\omega_t^i\right\|_\alpha\leqslant\left\|\omega_0^i\right\|_\alpha\exp\left(c_0 t e^{c_0 t}\right)\tag{3.13}$$

If u_t^i is as in Lemma 3.3, then (3.13) shows

$$\left\|\nabla u_t^i\right\|_\alpha\leqslant c\left\|\omega_t^i\right\|_\alpha\leqslant c\,\|\omega_0\|_\alpha\exp\left(c_0 t e^{c_0 t}\right).\tag{3.14}$$

Since the mean velocity is a conserved quantity, a bound on $\|\nabla u_t^i\|_\alpha$ immediately gives a bound on $\|u_t^i\|_{1,\alpha}$, which in conjunction with local existence (Proposition 3.1), and Lemma 3.3 concludes the proof. □

4. Convergence as $N\to\infty$

In this section, we fix a time interval $[0,T]$, and show that the particle system (2.5)–(2.7) converges to the solution to the Navier-Stokes equations as $N\to\infty$. The rate of convergence is $O(\frac{1}{\sqrt{N}})$, which is too slow to be of practical interest. For this reason we do not concern ourselves with the strongest possible convergence result under minimal assumptions. Instead we select a convergence result with a short elementary proof, under assumptions that are immediately guaranteed by local existence.

Theorem 4.1. *For each i, N, let $X^{i,N}$, u^N be a solution to the particle system (2.5)–(2.7) with initial data u_0 on some time interval $[0,T]$. Let u be a solution* *to the Navier-Stokes equations (with the same initial data) on the interval* $[0,T]$*. Suppose* U *is such that*

$$\sup_{t\in[0,T]} \|\nabla u_t\|_{L^2} \leqslant \frac{U}{L} \quad \text{and} \quad \sup_{t\in[0,T]} \left\|\nabla u_t^{i,N}\right\|_{L^2} \leqslant \frac{U}{L} \quad a.s.$$

Then $(u^N) \to u$ *in the following sense:*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left\|u_t^N - u_t\right\|_{L^2} = 0$$

We remark that given $C^{1,\alpha}$ initial data, local existence (Proposition 3.1) guarantees that the conditions of this theorem are satisfied on some small interval $[0,T]$. In two dimensions, Theorem 3.5 shows that the conditions of this theorem are satisfied on any interval finite $[0,T]$.

The proof will follow almost immediately from the following Lemma.

Lemma 4.2. *Let* $u_t^{i,N} = \mathbf{P}[(\nabla^* Y_t^{i,N})u_0 \circ Y_t^{i,N}]$ *be the* i^{th} *summand in* (2.7)*. Then* $u^{i,N}$ *satisfies the SPDE*

$$du_t^{i,N} + \left[(u_t^N\cdot\nabla)u_t^{i,N} - \nu\triangle u_t^{i,N} + (\nabla^* u_t^N)u_t^{i,N} + \nabla p_t^{i,N}\right] dt + {} + \sqrt{2\nu}\nabla u_t^{i,N} dW_t^i = 0 \tag{4.1}$$

and u^N *satisfies the SPDE*

$$du_t^N + \left[(u_t^N\cdot\nabla)u_t^N - \nu\triangle u_t^N + \nabla p_t^N\right] dt + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \nabla u_t^{i,N} dW_t^i = 0 \tag{4.2}$$

Remark. We draw attention to the fact that the pressure term in (4.2) has bounded variation in time.

Proof. We first recall a fact from [8, 10] (see also [11]). If X is the stochastic flow

$$dX_t = u_t\,dt + \sqrt{2\nu}\,dW_t$$

and $Y_t = X_t^{-1}$ is the spatial inverse. Then the process $\theta = f(Y)$ satisfies the SPDE

$$d\theta_t + (u_t\cdot\nabla)\theta_t\,dt - \nu\triangle\theta_t\,dt + \sqrt{2\nu}\nabla\theta_t\,dW_t = 0 \tag{4.3}$$

This immediately shows that $Y^{i,N}$ and $v^{i,N} = u_0 \circ Y^{i,N}$ both satisfy the SPDE (4.3). For notational convenience, we momentarily drop the N as a superscript and use the notation $v^{i,j}$ to denote the j^{th} component of v^i.

Now we set $w^i = (\nabla^* Y^i)v^i$ and apply Itô's formula:

$$\begin{aligned}
dw^{i,j} &= d(\partial_j Y^i)\cdot v^i + (\partial_j Y^i)\cdot dv^i + d\left\langle \partial_j Y^{i,k}, v^{i,k}\right\rangle \\
&= (\partial_j Y^i)\cdot\left[-(u\cdot\nabla)v^i + \nu\triangle v^i\right] dt - \sqrt{2\nu}\partial_j Y^i\cdot\left(\nabla v^i\,dW^i\right) + \\
&\qquad + v^i\cdot\left[-((\partial_j u)\cdot\nabla)Y^i - (u\cdot\nabla)\partial_j Y^i + \nu\triangle\partial_j Y^i\right] dt - \\
&\qquad - \sqrt{2\nu}v^i\cdot\left(\nabla\partial_j Y^i\,dW^i\right) + 2\nu\partial_{jl}^2 Y^{i,k}\partial_l v^{i,k}\,dt \\
&= \left[-(u\cdot\nabla)w^i + \nu\triangle w^i - (\nabla^* u)\cdot w^i\right] dt - \sqrt{2\nu}\nabla w^i\,dW^i
\end{aligned}$$

Restoring the dependence on N to our notation since $u^{i,N} = \mathbf{P}w^{i,N}$, we know that

$$u^{i,N} = w^{i,N} + \nabla q^{i,N}$$

for some function $q^{i,N}$. Thus

$$du^{i,N} = dw^{i,N} + d(\nabla q^{i,N})$$

$$
\begin{aligned}
&= \left[-(u\cdot\nabla)w^{i,N} + \nu\triangle w^{i,N} - (\nabla^* u)w^{i,N}\right] dt - \sqrt{2\nu}\nabla w^{i,N} dW^i \\
&= \left[-(u\cdot\nabla)u^{i,N} + \nu\triangle u^{i,N} - (\nabla^* u)u^{i,N}\right] dt - \sqrt{2\nu}\nabla u^{i,N} dW^i + \\
&\quad + \left[-(u\cdot\nabla)(\nabla q^{i,N}) + \nu\triangle(\nabla q^{i,N}) - (\nabla^* u)(\nabla q^{i,N})\right] dt - \\
&\quad - \sqrt{2\nu}\nabla(\nabla q^{i,N})dW^i + d(\nabla q^{i,N}).
\end{aligned}
$$

If we define $P^{i,N}$ by

$$P_t^{i,N} = \int_0^t \left[(u_s\cdot\nabla)q_s^{i,N} - \nu\triangle q_s^{i,N}\right] ds + \int_0^t \sqrt{2\nu}\nabla q_s^{i,N}\cdot dW_s^i + q_t^{i,N}$$

then we have

$$
\begin{aligned}
(4.4)\quad du^{i,N} + \left[(u\cdot\nabla)u^{i,N} - \nu\triangle u^{i,N} + (\nabla^* u)u^{i,N}\right] dt \\
+ d(\nabla P^{i,N}) + \sqrt{2\nu}\nabla u^{i,N} dW^i = 0.
\end{aligned}
$$

Notice that $u^{i,N}$ is divergence free by definition, and thus $\nabla u^{i,N} dW_i$ is also divergence free. Thus $d(\nabla P^{i,N})$, the only other term with non-zero quadratic variation, must have a divergence free martingale part. Since the martingale part of $d(\nabla P^{i,N})$ is also a gradient, it must be 0. Thus

$$d(\nabla P^{i,N}) = \nabla p^{i,N}\, dt$$

for some function $p^{i,N}$. This proves (4.1).

The identity (4.2) now follows by summing (4.1) in i, dividing by N, and defining p^N by

$$p^N = \frac{1}{2}\nabla|u|^2 + \frac{1}{N}\sum_{i=1}^N p^{i,N} \qquad \square$$

Proof of Theorem 4.1. Let u be a solution of the Navier-Stokes equations, with initial data u_0, and set $v^N = u^N - u$. Then v^N satisfies the SPDE

$$dv_t^N + (v_t^N\cdot\nabla)u_t\,dt + (u_t\cdot\nabla)v_t^N\,dt - \nu\triangle v_t^N\,dt + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \nabla u_t^{i,N} dW_t^i.$$

Thus by Itô's formula,

$$
\begin{aligned}
\frac{1}{2}d\left\|v_t^N\right\|_{L^2}^2 + \left(v_t^N, (v_t^N\cdot\nabla)u_t\right) dt + \nu\left\|\nabla v_t^N\right\|_{L^2}^2 dt + \\
+ \frac{2\nu}{N}\sum_{i=1}^N v_t^N\cdot(\nabla u_t^{i,N}\, dW_t^{i,k}) = \frac{\nu}{N^2}\sum_{i=1}^N \left\|\nabla u_t^{i,N}\right\|_{L^2}^2 dt.
\end{aligned}
$$

Here the notation (f,g) denotes the L^2 innerproduct of f and g. Taking expected values gives us

$$\frac{1}{2}\partial_t\mathbb{E}\left\|v_t^N\right\|_{L^2}^2 \leqslant \frac{U}{L}\mathbb{E}\left\|v^N\right\|_{L^2}^2 + \frac{\nu U^2}{NL^2}$$

and by Gronwall's lemma we have

$$\mathbb{E}\left\|v_t^N\right\|_{L^2}^2 \leqslant \frac{2\nu U^2}{L^2 N} t e^{\frac{Ut}{L}}$$

concluding the proof. $\square$

5. Convergence as $t \to \infty$

In this section, we fix N, and consider the behaviour of the system (2.5)–(2.7) as $t \to \infty$. We show that the system (2.5)–(2.7) does not dissipate all its energy as $t \to \infty$. Roughly speaking we show

$$\limsup_{t\to\infty} \mathbb{E}\,\|\nabla u_t\|_{L^2}^2 \geqslant O\left(\tfrac{1}{N}\right).$$

This is in contrast to the true (unforced) Navier-Stokes equations, which dissipate all of its energy as $t \to \infty$ (provided of course the solutions are defined globally in time).

In general we are unable to compute exact asymptotic behaviour of the system (2.5)–(2.7) as $t \to \infty$. But in the special case of shear flows, we compute this exactly, and show that the system eventually converges to a constant, retaining exactly $\frac{1}{N}$ times its initial energy.

For the remainder of this section, we pick a fixed $N \in \mathbb{N}$ and for notational convenience we omit the superscript N. We begin by computing exactly the asymptotic behaviour of the system (2.5)–(2.7) in the special case of shear flows.

Proposition 5.1. *Suppose the initial data $u_0(x) = (\phi_0(x_2), 0)$ for some $C^{1,\alpha}$ periodic function ϕ. If u is the velocity field that solves the system* (2.5)–(2.7) *with initial data u_0, then*

$$\lim_{t\to\infty} \mathbb{E}\,\omega_t(x)^2 = \frac{1}{N}\,\|\omega_0\|_{L^2}^2 \tag{5.1}$$

where $\omega = \nabla \times u$ is the vorticity.

Proof. Let X^i, Y^i be the flows in the system (2.5)–(2.7), and as before define u^i to be the i^{th} summand in (2.7), and $\omega^i = \omega_0 \circ Y^i$.

First note that the SPDE's for u^i and u (equations (4.1) and (4.2)) are all translation invariant. Thus since the initial data is independent of x_1, the same must be true for all time. Since u^i, u are divergence free, the second coordinate must necessarily be 0, and the form of the initial data is preserved. Namely,

$$u_t(x) = (\phi_t(x_2), 0) \quad \text{and} \quad u^i_t(x) = (\phi^i_t(x_2), 0)$$

for some $C([0,\infty), C^{1,\alpha})$ periodic functions ϕ^i, ϕ.

Now the definition of X^i shows that

$$X^i_t(y) = \begin{pmatrix} y_1 + \lambda^{i,1}_t(y_2) \\ y_2 + \sqrt{2\nu} W^{i,2}_t \end{pmatrix}$$

and hence

$$Y^i_t(x) = \begin{pmatrix} x_1 + \mu^{i,1}_t(x_2) \\ x_2 - \sqrt{2\nu} W^{i,2}_t \end{pmatrix}.$$

Recall λ^i, μ^i are as in (3.5), (3.6), and here the notation $\lambda^{i,1}$ to denotes the first coordinate of λ^i. This immediately shows

$$\begin{aligned} \omega^i_t(x) &= \omega_0 \circ Y^i_t(x) \\ &= -\partial_2 \phi_0(x_2 - \sqrt{2\nu} W^{i,2}_t) \end{aligned}$$

where $W^{i,2}$ again denotes the second coordinate of the Brownian motion W^i.

Now using standard mixing properties of Brownian motion [12, Section 1.3], (or explicitly computing in this case) we know that for every $x \in [0, L]^2$

$$\lim_{t\to\infty} \mathbb{E}\,\omega_t^i(x)^2 = \lim_{t\to\infty} \mathbb{E}\left[-\partial_2\phi_0(x_2 - \sqrt{2\nu}W_t^{i,2})\right]^2 = \|\omega_0\|_{L^2}^2 \tag{5.2}$$

and

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\,\omega_t^i(x)\omega_t^j(x) &= \lim_{t\to\infty} \mathbb{E}\left[\partial_2\phi_0(x_2 - \sqrt{2\nu}W_t^{i,2})\partial_2\phi_0(x_2 - \sqrt{2\nu}W_t^{j,2})\right]\\ &= \lim_{t\to\infty}\left[\mathbb{E}\partial_2\phi_0(x_2 - \sqrt{2\nu}W_t^{i,2})\right]\left[\mathbb{E}\partial_2\phi_0(x_2 - \sqrt{2\nu}W_t^{j,2})\right]\\ &= \left(\frac{1}{L^2}\int_{[0,L]^2}\partial_2\phi_0\right)^2\\ &= 0 \qquad (5.3)\end{aligned}$$

when $i \neq j$.

Now by two dimensional Cauchy formula (3.8)

$$\omega_t = \frac{1}{N}\sum_{i=1}^N \omega_0 \circ Y_t^i.$$

(since in our case, $\omega_0^1 = \cdots = \omega_0^N = \omega_0$). Thus

$$\mathbb{E}\,\omega_t(x)^2 = \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\,\omega_0\circ Y_t^i(x)^2 + \frac{2}{N^2}\sum_{i=2}^N\sum_{j=1}^{i-1}\mathbb{E}\,\omega_0\circ Y_t^i(x)\omega_0\circ Y_t^j(x) \tag{5.4}$$

and using (5.2) and (5.3) the proof is complete. □

We remark that all we need for (5.1) to hold is the identities (5.2) and (5.3). Equality (5.2) is guaranteed provided reasonable ergodic properties of the flow X_t^i are known. Equality (5.3) is guaranteed provided the flows X_t^i and X_t^j eventually decorrelate.

While we are unable to guarantee these properties for a more general class of flows, we conclude this section by proving a weaker version of (5.1) for two dimensional flows with general initial data.

Theorem 5.2. *Let X^i, u be a solution to the system* (2.5)–(2.7) *with (spatial) mean zero initial data $u_0 \in C^{1,\alpha}$ and periodic boundary conditions. Suppose further $u \in C([0,\infty), C^{1,\alpha})$. Then*

$$\limsup_{t\to\infty} \mathbb{E}\,\|\nabla u_t\|_{L^2}^2 \geqslant \frac{1}{NL^2}\,\|u_0\|_{L^2}^2 \tag{5.5}$$

Note that the assumption $u \in C([0,\infty), C^{1,\alpha})$ is satisfied in the two dimensional situation with $C^{1,\alpha}$ initial data (Theorem 3.5). The proof we provide below will also work in the three dimensional situation, as long as global existence and well-posedness of (2.5)–(2.7) is known.

As is standard with the Navier-Stokes equations, the condition that u_0 is (spatially) mean zero is not a restriction. By changing coordinates to a frame moving with the mean of the initial velocity, we can arrange that the initial data (in the new frame) has spatial mean 0.

Finally we remark that the lower bound in inequality (5.5) is sharp, since in the special case of shear flows we have the *equality* (5.1). However we are unable to obtain a bound on $\liminf \mathbb{E}\|\nabla u_t\|_{L^2}^2$.

Proof of Theorem 5.2. As before let $u^i_t = \mathbf{P}\left[(\nabla^* Y^i_t) u_0 \circ Y^i_t\right]$. Using Lemma 4.2 and Itô's formula we have

$$\frac{1}{2} d\left|u^i\right|^2 + u^i \cdot \left[(u\cdot\nabla)u^i - \nu\triangle u^i + (\nabla^* u)u^i + \nabla p^i\right]\,dt + \\ + \sqrt{2\nu}\,u^i\cdot(\nabla u^i\,dW^i) = \nu\left|\nabla u^i\right|^2\,dt \tag{5.6}$$

Note that

$$\int u^i\cdot((\nabla^* u)u^i) = \int((\nabla u)u^i)\cdot u^i = \int u^i\cdot(u\cdot\nabla)u^i = 0$$

Thus integrating (5.6) in space, and using the fact that u^i is divergence free gives

$$d\left\|u^i\right\|_{L^2}^2 = 0$$

and hence $\|u^i_t\|_{L^2} = \|u^i_0\|_{L^2}$ almost surely.

Now suppose that for some $\varepsilon > 0$, there exists t_0 such that for all $t > t_0$

$$\mathbb{E}\left\|\nabla u_t\right\|_{L^2}^2 \leqslant \frac{1}{NL^2} - \varepsilon.$$

Using Itô's formula and (4.2) gives

$$\frac{1}{2}d\left|u\right|^2 + u\cdot\left[(u\cdot\nabla)u - \nu\triangle u + \nabla p\right]\,dt + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N u\cdot(\nabla u^i\,dW^i) \\ = \frac{\nu}{N^2}\sum_{i=1}^N\left|\nabla u^i\right|^2\,dt$$

Integrating in space, and taking expected values gives

$$\begin{aligned}\frac{1}{2\nu}\partial_t\mathbb{E}\left\|u_t\right\|_{L^2}^2 &= \mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^N\left\|\nabla u^i_t\right\|_{L^2}^2 - \left\|\nabla u_t\right\|_{L^2}^2\right]\\ &\geqslant \mathbb{E}\left[\frac{1}{N^2L^2}\sum_{i=1}^N\left\|u^i_t\right\|_{L^2}^2 - \left\|\nabla u_t\right\|_{L^2}^2\right]\\ &= \frac{1}{N^2L^2}\sum_{i=1}^N\left\|u_0\right\|_{L^2}^2 - \mathbb{E}\left\|\nabla u_t\right\|_{L^2}^2\\ &= \frac{1}{NL^2}\left\|u_0\right\|_{L^2}^2 - \mathbb{E}\left\|\nabla u_t\right\|_{L^2}^2\\ &\geqslant \varepsilon\end{aligned}$$

for $t \geqslant t_0$. Here we used the Poincaré inequality to obtain the second inequality above. Note that we have assumed that the initial data has (spatial) mean 0. Since the (spatial) mean is conserved by the system (2.5)–(2.7), u_t also has (spatial) mean zero, and our application of the Poincaré inequality is valid.

Now, the above inequality immediately implies $\mathbb{E}\|u_t\|_{L^2}^2$ becomes arbitrarily large as $t \to \infty$. This is a contradiction because

$$\left\|u_t\right\|_{L^2} = \left\|\frac{1}{N}\sum_{i=1}^N u^i_t\right\|_{L^2} \leqslant \frac{1}{N}\sum_{i=1}^N\left\|u^i_t\right\|_{L^2} = \left\|u_0\right\|_{L^2}$$

holds almost surely. ∎

Appendix A. Local existence.

In this appendix we provide the proof of Proposition 3.1. A similar proof appeared in [9] (see also [5]), and the proof provided here is based on similar ideas. We present the proof here because we require local existence for $C^{1,\alpha}$ initial data (the proof in [9] used $C^{2,\alpha}$), and to ensure that bounds and existence time therein are independent of N.

Without loss, we assume $t_0 = 0$, and $u_0^1, \dots, u_0^N$ to be $\mathcal{F}_0$ measurable. Let U be a large constant and T a small time, both of which will be specified later.

Define $\mathcal{U} = \mathcal{U}(T, U)$ be the set of all time continuous $\mathcal{F}_t$-adapted $C^{1,\alpha}$ valued divergence free and spatially periodic processes u such that

$$u_0 = \frac{1}{N}\sum_{i=1}^N u_0^i \quad \text{and} \quad \sup_{t\in[0,T]} \|u_t\|_{1,\alpha} \leqslant U$$

hold almost surely. Also, we define $\mathcal{M} = \mathcal{M}(T)$ to be the set of all time continuous $\mathcal{F}_t$-adapted $C^{1,\alpha}$ valued spatially periodic processes μ such that

$$\mu_0 = 0 \quad \text{and} \quad \sup_{t\in[0,T]} \|\nabla\mu_t\|_\alpha \leqslant \frac{1}{2}$$

hold almost surely.

Now given $u \in \mathcal{U}$ let $X^{i,u}$ be the flow solving the SDE

$$dX_t^{i,u} = u_t(X_t^{i,u})\,dt + \sqrt{2\nu}\,dW_t^i$$

with initial data $X_0^{i,u}(y) = y$. As before, define $Y_t^{i,u} = (X_t^{i,u})^{-1}$, and define

$$\lambda_t^{i,u}(y) = X_t^{i,u}(y) - y$$
$$\mu_t^{i,u}(x) = Y_t^{i,u}(x) - x$$

to be the Eulerian and Lagrangian displacements respectively.

Finally define the (non-linear) operator W by

$$W(u)_t = \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[\left(\nabla^* Y_t^{i,u}\right)\left(u_0^i \circ Y_t^{i,u}\right)\right]$$

Clearly a fixed point of W will produce a solution to the system (3.1)–(3.3). Thus the proof will be complete if we show that for an appropriate choice of T and U, W maps $\mathcal{U}$ into itself, and is a contraction with respect to the weaker norm

$$\|u\|_{\mathcal{U}} = \sup_{t\in[0,T]} \|u_t\|_\alpha$$

We first show W maps $\mathcal{U}$ into itself, using the two lemmas below.

Lemma A.1. *There exists $c = c(\alpha)$ such that*

$$\|W(u)\|_{1,\alpha} \leqslant c\left[\max_{1\leqslant i\leqslant N}\left(1 + \left\|\nabla\mu^{i,u}\right\|_\alpha\right)^{2+\alpha}\right]\frac{1}{N}\sum_{i=1}^N \left\|u_0^i\right\|_{1,\alpha} \quad a.s.$$

Proof. First recall $\mathbf{P}$ vanishes on gradients. Thus

$$\mathbf{P}\left[(\nabla^* Y)\,v\right] = -\mathbf{P}\left[(\nabla^* v)\,Y\right]. \tag{A.1}$$

Now

$$\partial_i \mathbf{P}\left[(\nabla^* Y)v\right] = \mathbf{P}\left[(\nabla^* Y)\partial_i v + (\nabla^*\partial_i Y)v\right]$$

$$= \mathbf{P}\left[(\nabla^* Y)\partial_i v - (\nabla^* v)\partial_i Y\right]$$

where we used (A.1) for the second term. Note that the right hand side involves only first order derivatives. Since $\mathbf{P}$ is a standard Calderón-Zygmund singular integral operator, which is bounded on Hölder spaces, we obtain the estimate

$$\|\mathbf{P}\left[(\nabla^* Y)v\right]\|_{1,\alpha} \leqslant c\,\|\nabla^* Y\|_\alpha \|v\|_{1,\alpha}$$

for some constant $c = c(\alpha)$.

Applying this estimate to W, we have

$$\|W(u)_t\|_{1,\alpha} \leqslant \frac{c}{N}\sum_{i=1}^N \left\|\nabla^* Y_t^{i,u}\right\|_\alpha \left\|u_0^i \circ Y_t^{i,u}\right\|_{1,\alpha} \quad \text{a.s.} \tag{A.2}$$

from which the Lemma follows. □

Lemma A.2. *There exists $T = T(U,\alpha)$ such that $\lambda^{i,u}, \mu^{i,u} \in \mathcal{M}(T)$.*

We note that the diffusion coefficient is spatially constant, and thus we get the desired (almost sure) control on $\nabla\lambda$. Since $\nabla\cdot u = 0$, $\det(\nabla X^{i,u}) = 1$, giving the desired control on $\nabla\mu$. The details are standard, and we do not provide the them here (see for instance [9, 12]).

Now choosing $U = c(\frac{3}{2})^{2+\alpha}\frac{1}{N}\sum\|u_0^i\|_{1,\alpha}$, and then choosing T as in Lemma A.2, Lemma A.1 shows that W maps $\mathcal{U}(U,T)$ into itself. Note that each summand on the right of (A.2) is bounded by U, which will prove the bound (3.4). Further, given a uniform (in i) bound on $\|u_0^i\|_{1,\alpha}$, our choice of U can be made independent of N.

It remains to show that W is a contraction. By definition of W we have

$$\begin{aligned}
W(u)_t - W(v)_t &= \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* Y_t^{i,u})u_0^i\circ Y_t^{i,u} - (\nabla^* Y_t^{i,v})u_0^i\circ Y_t^{i,v}\right]\\
&= \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* Y_t^{i,u})\left(u_0^i\circ Y_t^{i,u} - u_0^i\circ Y_t^{i,v}\right)\right] +\\
&\quad + \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[\left(\nabla^* Y_t^{i,u} - \nabla^* Y_t^{i,v}\right)u_0^i\circ Y_t^{i,v}\right]\\
&= \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* Y_t^{i,u})\left(u_0^i\circ Y_t^{i,u} - u_0^i\circ Y_t^{i,v}\right)\right] -\\
&\quad - \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[\nabla^*(u_0^i\circ Y_t^{i,v})\left(Y_t^{i,u} - Y_t^{i,v}\right)\right]
\end{aligned}$$

where we used the identity (A.1) to obtain the last equality. Now we recall that $\mu^{i,u}, \mu^{i,v} \in \mathcal{M}$, and take C^α norms. This gives

$$\|W(u)_t - W(v)_t\|_\alpha \leqslant \frac{c}{LN}\sum_{i=1}^N \left\|u_0^i\right\|_{1,\alpha}\left\|Y_t^{i,u} - Y_t^{i,v}\right\|_\alpha \quad \text{a.s.} \tag{A.3}$$

Now from the definition of $Y^{i,u}$ and $Y^{i,v}$ we have

$$Y_t^{i,u} - Y_t^{i,v} = \int_0^t \left[u_s(Y_s^{i,u}) - v_s(Y_s^{i,v})\right]\,ds.$$

Taking C^α norms, and applying Gronwall's inequality, and absorbing the exponential in time factor into the constant c gives

$$\left\| Y_t^{i,u} - Y_t^{i,v} \right\|_\alpha \leqslant c \int_0^t \|u_s - v_s\|_\alpha \, ds \quad \text{a.s.}$$

Returning to (A.3) we have

$$\|W(u)_t - W(v)_t\|_\alpha \leqslant \frac{ct}{L} \sup_{s \leqslant t} \|u_s - v_s\|_\alpha \frac{1}{N} \sum_{i=1}^N \left\| u_0^i \right\|_{1,\alpha} \quad \text{a.s.}$$

Choosing t small enough one can ensure W is a contraction mapping. A standard iteration now shows the existence of a fixed point of W, concluding the proof of Proposition 3.1.

Appendix B. Logarithmic L^∞ bound on singular integral operators.

In this appendix we provide a proof of Lemma 3.4. We restate it here for the readers convenience.

Lemma (3.4)**.** *If u is divergence free and periodic in $\mathbb{R}^d$, then for any $\alpha \in (0,1)$, there exists a constant $c = c(\alpha, d)$ such that*

$$\|\nabla u\|_{L^\infty} \leqslant c \|\omega\|_{L^\infty} \left(1 + \ln^+ \left(\frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}} \right) \right)$$

where $\omega = \nabla \times u$.

Proof. Let K be a standard Calderón-Zygmund periodic kernel, and we define the operator T by

$$Tf = K * f$$

then we will prove that

$$\|Tf\|_{L^\infty} \leqslant c \|f\|_{L^\infty} \left(1 + \ln^+ \left(\frac{|f|_\alpha}{\|f\|_{L^\infty}} \right) \right) \tag{B.1}$$

This immediately implies Lemma 3.4 because we know $\nabla u = -\nabla(\triangle^{-1})\nabla \times \omega$, and $-\nabla(\triangle^{-1})\nabla\times$ is a Calderón-Zygmund type singular integral operator.

Now we prove (B.1). We assume for convenience that all functions are periodic on the cube $[0,1]^d$. We also recall that the kernel K satisfies the properties

1. $K(y) \leqslant c|y|^d$ when $|y| \leqslant \frac{1}{2}$.
2. $\int_{|y|=r} K(y) \, d\sigma(y) = 0$ for any $r \in (0, \frac{1}{2})$.

Pick any $\varepsilon \in (0, \frac{1}{2})$. Then

$$\begin{aligned} Tf(x) &= \int_{[0,1]^d} K(y) f(x-y) \, dy \\ &\leqslant \int_{|y|<\varepsilon} K(y) f(x-y) \, dy + \int_{|y| \geqslant \varepsilon} K(y) f(x-y) \, dy \end{aligned} \tag{B.2}$$

Using property 1 about K, we bound the second integral by

$$\begin{aligned} \left| \int_{|y| \geqslant \varepsilon} K(y) f(x-y) \, dy \right| &\leqslant c \|f\|_{L^\infty} \int_{r=\varepsilon}^1 \frac{1}{r^d} r^{d-1} \, dr \\ &\leqslant c \|f\|_{L^\infty} \ln\left(\tfrac{1}{\varepsilon}\right) \end{aligned}$$

Using property 2 about K, we bound the first integral in (B.2) by

$$\begin{aligned}\left|\int_{|y|<\varepsilon} K(y)f(x-y)\,dy\right| &= \left|\int_{|y|<\varepsilon} K(y)(f(x-y)-f(x))\,dy\right| \\ &\leqslant c\,|f|_\alpha \int_{|y|<\varepsilon} |y|^{\alpha-d}\,dy \\ &= c\,|f|_\alpha\,\varepsilon^\alpha\end{aligned}$$

Combining estimates we have

$$\|Tf\|_{L^\infty} \leqslant c\left[\varepsilon^\alpha\,|f|_\alpha + \|f\|_{L^\infty}\ln\left(\tfrac{1}{\varepsilon}\right)\right]$$

Choosing

$$\varepsilon = \min\left\{\frac{1}{2}, \left(\frac{\|f\|_{L^\infty}}{|f|_\alpha}\right)^{1/\alpha}\right\}$$

finishes the proof. □

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