THE UNIVERSITY OF CHICAGO

A STOCHASTIC LAGRANGIAN FORMULATION OF THE INCOMPRESSIBLE NAVIER-STOKES AND RELATED TRANSPORT EQUATIONS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
GAUTAM IYER

CHICAGO, ILLINOIS
JUNE 2006

Table of Contents

ACKNOWLEDGEMENTS

I would like to thank my advisor Peter Constantin and my second advisor Lenya Ryzhik for all their help and guidance. I would also like to thank my wife Jenée Iyer for putting up with my prolonged absence while writing up my papers.

CHAPTER 1
INTRODUCTION

Formalisms representing solutions of partial differential equations as the expected value of functionals of stochastic processes dates back to Einstein and Feynman in physics, and Kolmogorov and Kac in mathematics. The theory for linear parabolic equations is well developed [12, 16]. The theory for nonlinear partial differential equations is not as well developed and generally involves either branching processes [19], or implicit, fixed point representations.

In this dissertation we provide representations of the Navier-Stokes and related nonlinear transport equations as the expected value of functionals of stochastic processes. Our formulation is an implicit fixed point representation and is often referred to as 'nonlinear in the sense of McKean': the drift term in the stochastic differential equation is computed as the expected value of an expression involving the flow it drives.

Connections between stochastic evolution and deterministic Navier-Stokes equations have been established in the seminal work of Chorin [6]. In two dimensions, the nonlinear equation obeyed by the vorticity has the form a Fokker-Planck (forward Kolmogorov) equation. The Biot-Savart law relates the vorticity to the velocity in a linear fashion. These facts are used by Chorin to formulate the random vortex method to represent the vorticity of the Navier-Stokes equation using random walks and a particle limit. The random vortex method has been proved to converge by Goodman [13] and Long [21], (see also [22]). A stochastic representation of the Navier-Stokes equations for two dimensional flows using similar ideas but without discretization is given in [4].

A heuristic representation of the Navier-Stokes equations in three dimensions using ideas of random walks and particles limits was done by Peskin [23]. There are many examples of modeling approaches using stochastic representations in physical situations, for instance [7, 24].

The three dimensional situation is complicated by the fact that there are no obvious Fokker-Planck like equations describing the solutions. LeJan and Sznitman [20] used a backward-in-time branching process in Fourier space to express the velocity field of a three dimensional viscous fluid as the average of a stochastic process. Their approach did not involve a limiting process, and this led to a new existence theorem. This was later generalized [2] to a physical space analogue.

More recently, Busnello, Flandoli and Romito developed a representation for the 3-dimensional Navier-Stokes equations using noisy flow paths, similar to the ones considered here. They used Girsanov transformations to recover the velocity field of the fluid from magnetization variables, and generalized their method to work for any Fokker-Planck-type equation. Their method however does not admit a self contained local existence proof.

The first result we prove in this dissertation (Chapter 2) provides a stochastic representation of deterministic Navier-Stokes equations using noisy flow paths. The idea behind this representation consists basically of two steps: In the first step a Weber formula is used to express the velocity field of the inviscid equation in terms of the particle trajectories of the inviscid equation *without* involving time derivatives. The second step is to replace classical Lagrangian trajectories by a stochastic flow with a constant diffusion coefficient, and the (deterministic) velocity field. Averaging the inviscid Weber formula produces the solution. It is essential that time derivatives do not appear in the expressions to be averaged.

Explicitly, the we consider the system of equations

$$\begin{aligned} dX &= u\,dt + \sqrt{2\nu}\,dW \\ A &= X^{-1} \\ u &= \mathbf{EP}\left[(\nabla^* A)\,(u_0 \circ A)\right] \end{aligned}$$

with initial data

$$X(a,0) = a.$$

Here $\mathbf{P}$ is the Leray-Hodge projection onto divergence free vector fields, W a 3-dimensional Wiener process, and $\mathbf{E}$ denotes the expected value with respect to the Wiener measure. We will show that u satisfies the incompressible Navier-Stokes equations with initial data u_0.

Constantin [10] showed that when $\nu = 0$, the above system is equivalent to the incompressible Euler equations (we provide a proof of this in Chapter 2), and X is nothing but the flow map of the inviscid fluid. Thus when $\nu > 0$, the map X can be interpreted as the noisy flow map of the fluid.

With this as motivation, we generalize the method of characteristics to solve elliptic second order PDE's by adding noise to the characteristics of the first order PDE, and averaging it out. This is similar in nature to the Kolmogorov forward equations, however representing the system in the framework of random characteristics has the advantage of providing a physical meaning to the stochastic representation. This is developed in the beginning of Chapter 2, and we conclude Chapter 2 by using this idea to provide stochastic representations of solutions to related non-linear hydrodynamic type equations.

In Chapter 3, we provide a self-contained proof of the existence and regularity (in periodic domains) of the Navier-Stokes equations using our stochastic formulation. We conclude Chapter 3 by considering a stochastic model where the drift of the flow X above is computed from the flow map X *without* taking expected values. The significance of this model is the self consistent description: the computation of the (random) drift does in terms of the flow X can be done for each realisation of the Wiener process, without knowledge of other realizations.

This system does not give us the Navier-Stokes equations, however we show that this system is a super-linear approximation of the Navier-Stokes equations for short time. Surprisingly this system behaves more like the Euler equations, and we conclude by proving that this system is nothing but a random translate of the Euler equations.

CHAPTER 2
THE STOCHASTIC LAGRANGIAN FORMULATION

Our primary goal in this Chapter will be to develop a physically meaningful stochastic representation of the Navier-Stokes and related non-linear transport equations. In Section 2.1, we extend the method of characteristics to second order elliptic PDE's, which is the foundation of our representation of non-linear transport equations.

As we will see in this Chapter, our representations of non-linear transport equations involve an implicit fixed point formulation. The typical situation we will consider will be one where the drift of a stochastic flow is the expected value of a functional of the flow itself. We briefly address the physical significance of this in Section 2.6, in the context of the reaction diffusion equations. We explore this further in the next Chapter (in Section 3.3) in the context of the Navier-Stokes equations.

2.1 The method of random characteristics

Consider the first order PDE

$$\partial_t \theta + (u \cdot \nabla\theta) = f(\theta). \tag{2.1.1}$$

To obtain a solution of this, we consider the flow given by

$$\dot{X} = u(X) \tag{2.1.2}$$

$$X_0(a) = a, \tag{2.1.3}$$

and the spatially parametrized ODE

$$\dot{\vartheta} = f(\vartheta) \tag{2.1.4}$$

$$\vartheta_0(a) = \theta_0(a). \tag{2.1.5}$$

A direct computation (the method of characteristics) shows that $\theta = \vartheta(X^{-1})$ is a solution of (2.1.1) with initial data θ_0.

We will now use stochastic techniques obtain an analogue of the above method for second order elliptic PDE's. The constant coefficient *linear* case can be obtained by adding a martingale to the flow X, and is essentially the same as the Kolmogorov forward [16] equation. The variable coefficient (linear) requires a correction term in the drift b, and finally the non-linear case can only be treated using an implicit fixed point representation. We consider the linear case in this section, and will consider the non-linear case later in this chapter.

Throughout this section we will assume that X is a stochastic flow of homeomorphisms that satisfies the SDE

$$dX = b\,dt + \sigma\,dW \tag{2.1.6}$$

with initial data

$$X_0(a) = a \tag{2.1.7}$$

where b, σ are C^2 in space, C^0 in time, and W is an n-dimensional Wiener process. We will also use (a_{ij}) to denote the matrix $\sigma\sigma^*$.

Lemma 2.1.1. *Let A_t be any (spatially C^2) process such that $A_t \circ X_t$ that is constant in time. Then exists a process B of bounded variation such that*

$$A_t = B_t - \int_0^t (\nabla A_s)\sigma\,dW_s \tag{2.1.8}$$

Proof. Applying the generalized Itô formula to $A \circ X$ we have

$$\begin{aligned} 0 &= \int_{t'}^t A(X_s, ds) + \int_{t'}^t \nabla A\big|_{X_s,s}\,dX_s + \tfrac{1}{2}\int_{t'}^t \partial^2_{ij}A\big|_{X_s,s} d\left\langle X^{(i)}\,,\,X^{(j)}\right\rangle_s + \\ &\qquad + \left\langle \int_{t'}^t \partial_i A(X_s, ds)\,,\, X^{(i)}_t - X^{(i)}_{t'} \right\rangle \\ &= \int_{t'}^t A(X_s, ds) + \int_{t'}^t \left[\nabla A\big|_{X_s,s} b + \tfrac{1}{2} a_{ij}\partial^2_{ij}A\big|_{X_s,s}\right] ds + \\ &\qquad + \int_{t'}^t \nabla A\big|_{X_s,s}\sigma\,dW_s + \left\langle \int_{t'}^t \partial_i A(X_s, ds)\,,\, X^{(i)}_t - X^{(i)}_{t'} \right\rangle . \end{aligned} \tag{2.1.9}$$

Notice that the second and fourth terms on the right are of bounded variation. Since the above equality holds for any t', t, and X_s is flow of homeomorphisms, the lemma follows. ☐

Proposition 2.1.2. *Let A_t be a spatially C^2 process. Then $A_t \circ X_t$ is constant in time if and only if*

$$dA_t + (b\cdot\nabla)A_t\,dt - \tfrac{1}{2}a_{ij}\partial^2_{ij}A_t\,dt - \partial_j A_t(\partial_i\sigma_{jk})\sigma_{ik}\,dt + (\nabla A_t)\sigma\,dW_t = 0 \tag{2.1.10}$$

Proof. First assume A satisfies equation (2.1.10). Applying the generalized Itô formula to $A \circ X$ we have

$$\begin{aligned} A_t\circ X_t - A_{t'}\circ X_{t'} &= \int_{t'}^t A(X_s, ds) + \int_{t'}^t \nabla A_s\,dX_s + \int_{t'}^t \tfrac{1}{2}\partial^2_{ij}A\,d\left\langle X^{(i)}\,,\,X^{(j)}\right\rangle + \\ &\qquad + \left\langle \int_{t'}^t \partial_i A(X_s, ds)\,,\, X^{(i)}_t - X^{(i)}_{t'} \right\rangle \end{aligned}$$

$$= \int_{t'}^{t} A(X_s, ds) + \int_{t'}^{t} (\nabla A_s) b\, ds + \int_{t'}^{t} (\nabla A_s)\sigma dW_s + \tag{2.1.11}$$
$$+ \int_{t'}^{t} \tfrac{1}{2} a_{ij} \partial_{ij}^2 A_s\, ds + \left\langle \int_{t'}^{t} \partial_i A(X_s, ds)\,,\, X_t^{(i)} - X_{t'}^{(i)} \right\rangle.$$

Differentiating equation (2.1.10) in space, we immediately see that the martingale part of $\partial_i A$ is $-\int (\nabla \partial_i A_t \sigma + \nabla A_t \partial_i \sigma)\, dW$. Since the joint quadratic variation term in (2.1.11) depends only on the martingale part of $\partial_i A$, we can compute it explicitly by

$$\left\langle \int_{t'}^{t} \partial_i A(X_s, ds)\,,\, X^{(i)} \right\rangle = -\int_{t'}^{t} \left(\partial_{ij}^2 A_s \sigma_{jk} \sigma_{ik} + \partial_j A_s (\partial_i \sigma_{jk}) \sigma_{ik} \right)\, ds$$
$$= -\int_{t'}^{t} \left(a_{ij} \partial_{ij}^2 A_s + \partial_j A_s (\partial_i \sigma_{jk}) \sigma_{ik} \right)\, ds. \tag{2.1.12}$$

Thus equation (2.1.11) reduces to

$$A_t \circ X_t - A_{t'} \circ X_{t'} = \int_{t'}^{t} A(X_s, ds) + \int_{t'}^{t} (\nabla A_s) \sigma\, dW + \tag{2.1.13}$$
$$+ \int_{t'}^{t} \left[(\nabla A) b - \tfrac{1}{2} a_{ij} \partial_{ij}^2 A_s - \partial_j A_s (\partial_i \sigma_{jk}) \partial_i \sigma_{ik} \right]\, ds.$$

Now using equation (2.1.10) we have

$$\int_{t'}^{t} A(X_s, ds) = \int_{t'}^{t} \left[-(\nabla A) b + \tfrac{1}{2} a_{ij} \partial_{ij}^2 A_s + \partial_j A_s (\partial_i \sigma_{jk}) \partial_i \sigma_{ik} \right]\, ds + \tag{2.1.14}$$
$$- \int_{t'}^{t} (\nabla A_s) \sigma\, dW_s.$$

Substituting (2.1.14) in equation (2.1.13) we conclude $A_t \circ X_t - A_{t'} \circ X_{t'} = 0$, thus concluding the proof.

Conversely, assume that $A_t \circ X_t$ is constant in time. We again proceed by applying the generalized Itô formula to $A_t \circ X_t$ to obtain (2.1.11). Now we use Lemma 2.1.1 to compute the martingale part of $\partial_i A$, and hence obtain equation (2.1.13). By our assumption $A_t \circ X_t$ is constant in time, hence the left hand side of (2.1.13) must always be 0. Since this holds for arbitrary t', t, and X_t is a homeomorphism, A must satisfy equation (2.1.10). □

Remark. We remark that the $\frac{1}{2} a_{ij} \partial_{ij}^2$ term in (2.1.11) has a positive sign, which is the anti-diffusive sign forward in time. The quadratic variation term in (2.1.11) contains the term $-a_{ij} \partial_{ij}^2$, thus correcting this sign, and giving us a dissipative stochastic PDE for A as we expect.

Corollary 2.1.3. *If A_t is the spatial inverse of X_t, then A_t satisfies equation* (2.1.10)

Proof. Since b and σ are assumed to be spatially C^2, the flow X must also be spatially C^2. Thus A is spatially C^2, and the corollary follows from Proposition (2.1.2). $\square$

Corollary 2.1.4. *Let ϑ be spatially C^2, and differentiable in time, and A_t be the spatial inverse of X_t. Then the process $\theta = \vartheta \circ A$ satisfies the stochastic PDE*

$$d\theta_t + \left[(b \cdot \nabla)\theta_t - \tfrac{1}{2}a_{ij}\partial^2_{ij}\theta_t - \partial_j\theta_t(\partial_i\sigma_{jk})\sigma_{ik}\right] dt + (\nabla\theta_t)\sigma\, dW_t = \partial_t\vartheta\big|_{A_t} dt \tag{2.1.15}$$

Proof. The corollary follows directly from Corollary 2.1.3 and the Itô formula:

$$\begin{aligned}
d\theta_t &= \partial_t\vartheta\big|_{A_t} dt + \nabla\vartheta\big|_{A_t} dA_t + \tfrac{1}{2}\partial^2_{ij}\vartheta\big|_{A_t} d\left\langle A^{(i)}_t\,,\, A^{(j)}_t\right\rangle \\
&= \left[\partial_t\vartheta\big|_{A_t} - \nabla\vartheta\big|_{A_t}(\nabla A_t)b + \tfrac{1}{2}a_{ij}\nabla\vartheta\big|_{A_t}\partial^2_{ij}A_t + \nabla\vartheta\big|_{A_t}\partial_j A_t(\partial_i\sigma_{jk})\sigma_{ik}\right] dt + \\
&\qquad + \tfrac{1}{2}\partial^2_{ij}\vartheta\big|_{A_t}(\partial_k A^i_t)a_{kl}(\partial_l A^j_t)\, dt - \nabla\vartheta\big|_{A_t}(\nabla A_t)\sigma\, dW_t \\
&= \left[\partial_t\vartheta\big|_{A_t} - (b \cdot \nabla)\theta_t + \tfrac{1}{2}a_{ij}\partial^2_{ij}\theta_t - \partial_j\theta_t(\partial_i\sigma_{jk})\sigma_{ik}\right] dt - (\nabla\theta_t)\sigma\, dW_t
\end{aligned}$$ $\square$

Corollary 2.1.4 shows that the SPDE

$$d\theta_t + \left[(b \cdot \nabla)\theta_t - \tfrac{1}{2}a_{ij}\partial^2_{ij}\theta_t - \partial_j\theta_t(\partial_i\sigma_{jk})\sigma_{ik}\right] dt + (\nabla\theta_t)\sigma\, dW_t = f(\theta)\, dt$$

can be solved by considering the random characteristics of the flow (2.1.6), the spatially parametrized ODE (2.1.4), and setting $\theta = \vartheta(A)$. To solve a (deterministic) second order (linear) PDE using these random characteristics, we take the expected value of the above equation and notice that the martingale part has mean 0.

Theorem 2.1.5. *Suppose f is an affine linear function, b' is C^2 in space, C^0 in time and (a_{ij}) is a strictly positive definite symmetric matrix which is C^2 in space and C^0 in time.*

Define σ to be a C^2 in space, C^0 in time matrix such that $\sigma\sigma^* = (a_{ij})$, and define $b_j = b'_j - (\partial_i\sigma_{jk})\sigma_{ik}$. As before let X be the stochastic flow defined by (2.1.6) – (2.1.7), *and ϑ the solution of the spatially parametrized ODE* (2.1.4) – (2.1.5). *Then $\bar\theta = \mathbf{E}\vartheta(X^{-1})$ satisfies the PDE*

$$\partial_t\bar\theta + (b' \cdot \nabla)\bar\theta - \tfrac{1}{2}a_{ij}\partial^2_{ij}\bar\theta = f(\bar\theta)$$

with initial data θ_0.

Proof. Note first that since f is affine linear, $\mathbf{E}f(\theta) = f(\mathbf{E}\theta)$. The proof follows by taking the expected value of the Itô integral of equation (2.1.15). $\square$

Remark. If (a_{ij}) is spatially constant, then $b' = b$, and hence our flow X has the same drift as the characteristics of the first order PDE (2.1.1).

Remark. If $(a_{ij}) \equiv 0$ then the above method of finding a solution reduces to the method of characteristics for the first order PDE (2.1.1).

Example 2.1.6. Suppose the diffusion matrix $\sigma = \sqrt{2\nu}\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. Then the Itô derivative of A is given by

$$dA + (b \cdot \nabla)A\, dt - \nu \triangle A\, dt + \sqrt{2\nu}\nabla A\, dW = 0.$$

Further if ϑ is defined by (2.1.4) – (2.1.5), and $\theta = \vartheta(A)$, then

$$d\theta + (b \cdot \nabla)\theta\, dt - \nu \triangle \theta\, dt + \sqrt{2\nu}\nabla A\, dW = f(\theta)\, dt.$$

2.2 The viscous Burgers equations

As a precursor to the stochastic Lagrangian formulation of the Navier-Stokes equations, we develop here a stochastic Lagrangian representation of the viscous Burgers equations. The absence of the pressure term in the Burgers equations admits an extremely simple Weber formula, avoiding many complications that arise for the Navier-Stokes equations.

We recall the inviscid Burger's equation is

$$\partial_t u + (u \cdot \nabla)\, u = 0$$

with initial data

$$u(x, 0) = u_0(x).$$

The absence of a pressure term causes the velocity to be actively transported by the fluid flow. Thus if X is the fluid flow map, $u \circ X$ is constant in time and hence we recover the velocity u from the instantaneous flow map by

$$u_t = u_0 \circ (X_t^{-1}).$$

As in section 2.1, we use A to denote spatial inverse of the flow map X. In keeping with the convention used by Constantin [10], we use the term *'Weber formula'* to refer to an expression that recovers the velocity field u from the flow map X and initial data u_0.

Explicitly, the system

$$\begin{aligned} \dot{X} &= u \\ A &= X^{-1} \\ u &= u_0(A) \end{aligned}$$

with initial data

$$X_0(a) = a$$

is equivalent to the inviscid Burgers equation before the formation of shocks.

We now obtain a solution to the viscous Burgers equations by using the random characteristics as in section 2.1, and averaging the inviscid Weber formula. We remark that our representation is implicit as it involves computing the drift of the stochastic flow by averaging a functional of the stochastic flow. This is not abnormal as the method of characteristics for the *inviscid* Burgers equation has the same implicit nature.

Theorem 2.2.1. *Let W be a n-dimensional Wiener process, $\alpha \in (0,1)$, $k \geqslant 2$ and $u_0 \in C^{k,\alpha}$. Let the pair u, X be a solution of the stochastic system*

$$dX = u\,dt + \sqrt{2\nu}\,dW \tag{2.2.1}$$

$$A = X^{-1} \tag{2.2.2}$$

$$u = \mathbf{E}[u_0 \circ A] \tag{2.2.3}$$

with initial data

$$X_0(a) = a. \tag{2.2.4}$$

For boundary conditions, we demand that either u and $X - I$ are spatially periodic, or that u and $X - I$ decay at infinity. Then u satisfies the viscous Burgers equation

$$\partial_t u + (u \cdot \nabla)\,u - \nu \triangle u = 0 \tag{2.2.5}$$

with initial data u_0. Here $\mathbf{E}$ denotes the expected value with respect to the Wiener measure and I is the identity function.

Proof. The proof of this theorem follows from Theorem 2.1.5 by setting $b = u$, $\sigma = \sqrt{2\nu}\mathbb{I}$ and $\vartheta = u_0$. $\square$

Remark 2.2.2. The spatial invertibility of X follows from standard theory of stochastic flows [17]. However since the diffusion matrix is spatially constant, we can see this as follows: Taking the (spatial) gradient of equation (2.2.1), we obtain a deterministic ODE for ∇X. A direct computation now shows

$$\det(\nabla X) = \exp\left(\int_0^t \nabla \cdot u_s\big|_{X_s}\,ds\right)$$

and hence X is locally invertible and orientation preserving for all time. Global invertibility now follows since our boundary conditions ensure X_t is properly homotopic to the identity map and hence has degree 1.

Remark 2.2.3. If a solution to the system (2.2.1) – (2.2.4) exists on the time interval $[0, T]$, then our proof will show that u satisfies equation (2.2.5) on this time interval. Though global existence for (2.2.5) is known, the fixed point methods developed in Chapter 3 will only yield a local existence result for (2.2.1) – (2.2.4).

Remark 2.2.4. Conversely, given a (global) solution u of (2.2.5) which is either spatially periodic or decays at infinity, standard theory shows that (2.2.1) has a global solution. Now uniqueness of strong solutions for linear parabolic equations and Corollary (2.1.4) shows that (2.2.3) is satisfied for all time.

Remark 2.2.5. Bounded domains can be treated by considering the backward flow and the stopped process.

2.3 The Weber formula for inviscid fluids.

To use the idea behind Theorem 2.2.1 to obtain a stochastic representation of the incompressible Navier-Stokes equations, we need to find a Weber formula for the inviscid problem: the incompressible Euler equations. This has been developed by Constantin in [10]. The proof of our stochastic formulation of the Navier-Stokes equations does not rely on this theorem, however we still present a proof here as it motivates our formulation (2.4.1) – (2.4.4).

We recall the incompressible Euler equations are

$$\partial_t u + (u \cdot \nabla)\, u + \nabla p = 0 \tag{2.3.1}$$

$$\nabla \cdot u = 0. \tag{2.3.2}$$

Theorem 2.3.1 (Constantin)**.** *Let $k \geqslant 1$ and $u_0 \in C^{k,\alpha}$ be divergence free. Then u satisfies the incompressible Euler equations* (2.3.1) – (2.3.2) *with initial data u_0 if and only if the pair of functions u, X satisfies the system*

$$\dot{X} = u \tag{2.3.3}$$

$$A = X^{-1} \tag{2.3.4}$$

$$u = \mathbf{P}[(\nabla^* A)\,(u_0 \circ A)] \tag{2.3.5}$$

with initial data

$$X(a,0) = a. \tag{2.3.6}$$

Here $\mathbf{P}$ is the Leray-Hodge projection [8, 25] on divergence free vector fields. We impose either periodic boundary conditions and demand that u and $X - I$ are spatially periodic, or demand that u and $X - I$ decay sufficiently rapidly at infinity (I is the identity map).

Proof. Assume first that u satisfies the Euler equations (2.3.1) – (2.3.2). Let $\tilde{p} = p \circ X$, $\tilde{q} = \int_0^t (\frac{1}{2}|\dot{X}|^2 + \tilde{p})$ and $q = \tilde{q} \circ A$. Now

$$\begin{aligned}
&& \ddot{X} &= \dot{u}\big|_X + (u \cdot \nabla)u\big|_X \\
&& &= -\nabla p\big|_X \\
\Longrightarrow && (\nabla^* X)\ddot{X} &= -\nabla \tilde{p} \\
\Longrightarrow && \partial_t \left[(\nabla^* X)\dot{X}\right] &= \tfrac{1}{2}\nabla|\dot{X}|^2 - \nabla\tilde{p} \\
\Longrightarrow && (\nabla^* X_t) u_t \circ X_t - u_0 &= \nabla \int_0^t \left(\tfrac{1}{2}|\dot{X}|^2 - \tilde{p}\right) \\
\Longrightarrow && u_t \circ X_t &= (\nabla^* X_t)^{-1} u_0 + (\nabla^* X_t)^{-1} \nabla \tilde{q}
\end{aligned}$$

$$
\begin{aligned}
&& &= \nabla^* A_t\big|_{X_t} u_0 + \nabla^* A_t\big|_{X_t} \nabla \tilde{q} \\
\Longrightarrow && u_t &= (\nabla^* A_t) u_0 \circ A_t + (\nabla^* A_t) \nabla \tilde{q}\big|_{A_t} \\
\Longrightarrow && u_t &= (\nabla^* A_t) u_0 \circ A_t + \nabla q \\
&& &= \mathbf{P}\left[(\nabla^* A_t) u_0 \circ A_t\right]
\end{aligned}
$$

The other half of the theorem follows from the computations carried out in section 2.5, or from our representation of the Navier-Stokes equations by setting $\nu = 0$. ☐

Remark 2.3.2. In the presence of an external force f, we only need to replace u_0 in equation (2.3.5) with φ defined by

$$\varphi_t = u_0 + \int_0^t (\nabla^* X_s) f(X_s, s)\, ds.$$

Remark 2.3.3. This formulation was extended (deterministically) for viscous fluids in [11]. However the deterministic formulation is significantly more complicated than the stochastic formulation we present here. The Itô correction eliminates the need for the commutator coefficients arising in the diffusive Lagrangian formulation, and this is discussed in Section 2.4.1.

2.4 A stochastic formulation of the Navier-Stokes equations

With the method of random characteristics (Section 2.1) and the inviscid Weber formula (Theorem 2.3.1), the formulation of the incompressible Navier-Stokes equations is immediate: add noise to the characteristics, use the inviscid Weber formula (2.3.5) and average.

Theorem 2.4.1. *Let $\nu > 0$, W be an n-dimensional Wiener process, $k \geqslant 1$ and $u_0 \in C^{k+1,\alpha}$ be a given deterministic divergence free vector field. Let the pair u, X satisfy the stochastic system*

$$dX = u\, dt + \sqrt{2\nu}\, dW \tag{2.4.1}$$

$$A = X^{-1} \tag{2.4.2}$$

$$u = \mathbf{EP}\left[(\nabla^* A)\,(u_0 \circ A)\right] \tag{2.4.3}$$

with initial data

$$X(a, 0) = a. \tag{2.4.4}$$

We impose boundary conditions by requiring u and $X - I$ are either spatially periodic, or decay sufficiently at infinity. Then u satisfies the incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)\, u - \nu \triangle u + \nabla p = 0$$
$$\nabla \cdot u = 0$$

with initial data u_0.

Remark 2.4.2. Remarks 2.2.2, 2.2.3 and 2.2.4 are also applicable here.

Remark 2.4.3. Domains with boundary can be considered using the backward flow and the stopped process. However we are presently unable to provide a self contained local existence proof, or consider the vanishing viscosity limit for domains with boundary. For the whole spaces, we address these questions in Chapter 3.

Remark 2.4.4. In the presence of a deterministic external force f, we only need to replace u_0 in equation (2.4.3) with φ defined by

$$\varphi_t = u_0 + \int_0^t (\nabla^* X) f(X_s, s)\, ds.$$

Clearly, if the forcing is random and independent of the Wiener process W, then our procedure provides a representation of the stochastically forced Navier-Stokes equations. We prove Theorem 2.4.1 in the presence of a deterministic external force at the end of this section.

Remark 2.4.5. The construction above can be modified to provide a stochastic representation of the LANS-alpha (or Camassa-Holm) equations. The inviscid Camassa-Holm [5, 14] equations are

$$\begin{gathered}\partial_t v + (u \cdot \nabla)\, v + (\nabla^* u) v + \nabla p = 0 \\ u = (1 - \alpha^2 \triangle)^{-1} v \\ \nabla \cdot v = 0\end{gathered}$$

Lemma 2.5.3 gives a formula to recover v from the inverse of the flow map. Thus we obtain a stochastic representation of the viscous Camassa-Holm equations by replacing (2.4.3) in (2.4.1) – (2.4.4) with

$$v = \mathbf{EP}\left[(\nabla^* A)\, u_0 \circ A\right] \tag{2.4.5}$$

$$u = (1 - \alpha^2 \triangle)^{-1} v. \tag{2.4.6}$$

The velocity v will now satisfy the viscous equation

$$\partial_t v + (u \cdot \nabla)\, v + (\nabla^* u) v - \nu \triangle v + \nabla p = 0.$$

We draw attention to the fact that the diffusive term is $\nu \triangle v$ and not $\nu \triangle u$. However, we do not derive the relation $u = (1 - \alpha^2 \triangle)^{-1} v$; and any other translation-invariant filter $u = Tv$ would work as well.

Physically, Theorem 2.4.1 rigorously provides us an interpretation of viscous fluids as ideal inviscid fluids plus Brownian motion. We postpone the proof of this theorem to the end of this section, and study a few consequences first.

The first consequence we mention is that we have a self contained proof for the local existence of the stochastic system (2.4.1) – (2.4.4) (for boundary less domains). This is developed in Chapter 3. We remark that our estimates, and existence time are independent of viscosity, and that the theorem and proof also work when the viscosity $\nu = 0$, giving us a local existence theorem about the Euler and Navier-Stokes equations alike. In Chapter 3, we also discuss the rate of convergence of the system (2.4.1) – (2.4.4) to the Euler equations as $\nu \to 0$.

We now turn our attention to more physical consequences. The nature of our formulation causes most identities for the Euler equations (in the Eulerian-Lagrangian form) to be valid in the above stochastic formulation after averaging. We begin by presenting identities for the vorticity.

Proposition 2.4.6. *Let $\omega = \nabla \times u$ be the vorticity, and $\omega_0 = \nabla \times u_0$ be the initial vorticity. Then*

$$\omega = \mathbf{E}\left[((\nabla X)\,\omega_0) \circ A\right]. \tag{2.4.7}$$

If the flow is two dimensional then the above formula reduces to

$$\omega = \mathbf{E}\left[\omega_0 \circ A\right]. \tag{2.4.8}$$

Remark 2.4.7. In the presence of an external force f, we have to replace ω_0 in equations (2.4.7) and (2.4.8) with ϖ defined by

$$\varpi_t = \omega_0 + \int_0^t (\nabla X_s)^{-1} g(X_s, s)\, ds$$

where $g = \nabla \times f$. For two dimensional flows this reduces to

$$\varpi_t = \omega_0 + \int_0^t g(X_s, s)\, ds.$$

We draw attention to the fact that these are exactly the same as the expressions in the inviscid case.

One method of proving Proposition 2.4.6 is to directly differentiate (2.4.3), and use the fact that $\mathbf{P}$ vanishes on gradients. We do not carry out the details here, but instead provide a proof using the Itô formula.

Proof of Proposition 2.4.6. We will show that ω defined by equation (2.4.7) satisfies the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\,\omega - \nu \triangle \omega = (\omega \cdot \nabla) u. \tag{2.4.9}$$

Notice first that since the diffusion matrix is spatially constant, ∇X is differentiable in time. We set $\vartheta = (\nabla X)\omega_0$, $\tilde{\omega} = \vartheta \circ A$ and apply Corollary 2.1.4 (or Example 2.1.6) to obtain

$$\begin{aligned} d\tilde{\omega} + (u\cdot\nabla)\tilde{\omega}\,dt - \nu\triangle\tilde{\omega}\,dt + \sqrt{2\nu}\nabla\tilde{\omega}\,dW &= \nabla\partial_t X\big|_{A_t}\omega_0(A)\,dt \\ &= (\nabla u)(\nabla X)\big|_{A}\omega_0(A)\,dt \\ &= (\nabla u)\tilde{\omega}\,dt \end{aligned}$$

Integrating and taking expected values shows that $\omega = \mathbf{E}\tilde{\omega}$ satisfies the vorticity equation (2.4.9) with initial data ω_0. The proposition follows now follows from the uniqueness of strong solutions. □

Remark 2.4.8. Since u is divergence free, equation (2.4.7) is algebraically equivalent to (2.4.3). Since our proof of Proposition 2.4.6 shows that ω satisfies the vorticity equation (2.4.9), which is algebraically equivalent to the Navier-Stokes equations, Proposition 2.4.6 can be used to provide a quick proof of Theorem 2.4.1. We however choose to provide a proof that directly uses the random characteristics as in Section 2.1.

Remark 2.4.9. The two dimensional vorticity equation does not have the stretching term $(\omega\cdot\nabla)u$, and thus the vorticity is transported by the fluid flow. With our formulation the two dimensional vorticity equation is immediately obtained by observing that the third component of the flow map satisfies $X_3(a,t) = a_3$.

Remark 2.4.10. Although it is evident from equations (2.4.3) and (2.4.7), we explicitly point out that the source of growth in the velocity and vorticity fields arises from the gradient of the noisy flow map X. The Beale-Kato-Majda [1] criterion guarantees if the vorticity ω stays bounded, then no blow up can occur in the Euler equations. In the case of the Navier-Stokes equations, well-known criteria for regularity exist and they can be translated in criteria for the average of the stochastic flow map. As is well known, the vorticity of a two dimensional fluid stays bounded, which immediately follows from equation (2.4.8).

Finally we mention the conservation of circulation. For this we need to consider a stochastic velocity $\tilde{u}$ defined by

$$\tilde{u} = \mathbf{P}\left[(\nabla^* A)\,(u_0\circ A)\right]. \tag{2.4.10}$$

Notice immediately that $u = \mathbf{E}\tilde{u}$ and $\tilde{u}_0 = u_0$. As the Navier-Stokes equations are dissipative, we should not expect circulation to be conserved. However the circulation of the stochastic velocity $\tilde{u}$ is conserved by the stochastic flow.

Proposition 2.4.11. *If* Γ *is a closed curve in space, then*

$$\oint_{X(\Gamma)} \tilde{u}\cdot dr = \oint_{\Gamma} u_0\cdot dr.$$

Proof. By definition of $\mathbf{P}$, there exists a function q so that

$$\tilde{u} = (\nabla^* A)(u_0\circ A) + \nabla q$$

$$\begin{aligned} \implies \qquad \nabla^* X\big|_A \tilde{u} &= u_0 \circ A + \nabla^* X\big|_A \nabla q \\ \implies \quad (\nabla^* X)(\tilde{u} \circ X) &= u_0 + \nabla(q \circ X). \end{aligned}$$

Hence

$$\begin{aligned} \oint_{X(\Gamma)} \tilde{u} \cdot dr &= \int_0^1 (\tilde{u} \circ X \circ \Gamma) \cdot (\nabla X\big|_\Gamma \Gamma')\, dt \\ &= \int_0^1 (\nabla^* X\big|_\Gamma)(\tilde{u} \circ X \circ \Gamma) \cdot \Gamma'\, dt \\ &= \oint_\Gamma (u_0 + \nabla(q \circ X)) \cdot dr = \oint_\Gamma u_0 \cdot dr. \end{aligned}$$

We remark that the above proof is exactly the same as a proof showing circulation is conserved in inviscid flows. ☐

We conclude this section by proving that the stochastic system (2.4.1) – (2.4.4) is indeed a representation of the Navier-Stokes equations.

Proof of Theorem 2.4.1. We provide the proof in the presence of an external force f, as stated in Remark 2.4.4. We begin by remarking that Theorem 3.2.6 guarantees local existence and well posedness of solutions to (2.4.1) – (2.4.4) (for regular initial data). Thus the processes A and X are spatially regular enough to apply the generalized Itô formula [17].

Now let $v = \varphi \circ A$, $w = (\nabla^* A)v$. Notice that Corollary 2.1.4 (or Example 2.1.6) give us the Itô derivatives of A and v respectively. Thus applying the Itô formula, we compute the Itô derivative of w:

$$\begin{aligned} dw_i &= (\partial_i A) \cdot dv + d(\partial_i A) \cdot v + d \left\langle \partial_i A_j \,,\, v_j \right\rangle \\ &= \partial_i A \cdot \left[(-u \cdot \nabla) v + \nu \triangle v + (\nabla^* X)\big|_A f \right] dt - \sqrt{2\nu} \partial_i A \cdot (\nabla v\, dW) + \\ &\quad + v \cdot \left[- \left((\partial_i u) \cdot \nabla \right) A - (u \cdot \nabla) \partial_i A + \nu \triangle \partial_i A \right] dt - \sqrt{2\nu} v \cdot (\nabla \partial_i A\, dW) \\ &\quad + 2\nu \partial^2_{ki} A_j \partial_k v_j\, dt. \end{aligned}$$

Making use of the identities

$$\begin{aligned} (u \cdot \nabla) w_i &= \partial_i A \cdot [(u \cdot \nabla) v] + [(u \cdot \nabla) \partial_i A] \cdot v \\ \triangle w_i &= \partial_i A \cdot \triangle v + \triangle \partial_i A \cdot v + 2 \partial_{ki} A_j \partial_k v_j \\ \partial_i u_k w_k &= v \cdot \left[(\partial_i u \cdot \nabla) A \right] \\ \partial_i A \cdot \left[(\nabla^* X)\big|_A f \right] &= f \end{aligned}$$

and

$$\partial_k w_i = \partial_i A_j \partial_k v_j + v_j \partial_{ki} A_j$$

we conclude

$$dw = [-(u\cdot\nabla)w + \nu\triangle w - (\nabla^* u)w + f]\,dt - \sqrt{2\nu}\nabla w\,dW. \tag{2.4.11}$$

Now from equation (2.4.3) we see

$$\begin{aligned}
u &= \mathbf{E}w + \nabla q\\
\Longrightarrow u - u_0 &= \mathbf{E}\int_0^t [-(u\cdot\nabla)w + \nu\triangle w - (\nabla^* u)w + f] + \nabla q\\
&= \int_0^t [-(u\cdot\nabla)(u-\nabla q) + \nu\triangle(u-\nabla q) - (\nabla^* u)(u-\nabla q) + f] + \nabla q\\
&= \int_0^t [-(u\cdot\nabla)u + \nu\triangle u + f] + \nabla p
\end{aligned}$$

where

$$p = q + \int_0^t \left[(u\cdot\nabla)q - \nu\triangle q - \tfrac{1}{2}|u|^2\right].$$

Differentiating immediately yields the theorem. □

2.4.1 A comparison with the diffusive Lagrangian formulation.

The computations above illustrate the connection between the stochastic Lagrangian formulation (2.4.1) – (2.4.4) presented here, and the deterministic diffusive Lagrangian formulation. We briefly discuss this below. In [11] the Navier-Stokes equations were shown to be equivalent to the system

$$\begin{gathered}
\partial_t\tilde{A} + (u\cdot\nabla)\,\tilde{A} - \nu\triangle\tilde{A} = 0\\
u = \mathbf{P}[(\nabla^*\tilde{A})v]\\
\partial_t\tilde{v}_\beta + (u\cdot\nabla)\,\tilde{v}_\beta - \nu\triangle\tilde{v}_\beta = 2\nu C^i_{j,\beta}\partial_j\tilde{v}_i\\
C^p_{j,i} = (\nabla\tilde{A})^{-1}_{ki}\partial_k\partial_j\tilde{A}_p
\end{gathered}$$

with initial data

$$\begin{gathered}
\tilde{A}(x,0) = 0\\
\tilde{v}(x,0) = u_0(x).
\end{gathered}$$

We see first that $\tilde{A} = \mathbf{E}A$. The commutator coefficients C^α_{ij} in the evolution of $\tilde{v}$ compensate for the first order terms in $\triangle((\nabla^*\tilde{A})\tilde{v})$. In the stochastic formulation these arises naturally from the generalized Itô formula as the joint quadratic variation term $\langle\partial_i A_j, v_j\rangle$. More explicitly, the equations

$$2\nu(\nabla^*\tilde{A})C^j_{k,\cdot}\partial_k\tilde{v}_j = 2\nu\partial^2_{kj}\tilde{A}_i\partial_k\tilde{v}_j$$

$$\text{and} \qquad d\left\langle \partial_i A_j\,,\, v_j \right\rangle = 2\nu \partial^2_{kj} A_i \partial_k v_j\, dt$$

illustrate the connection between the two representations.

2.5 A proof of the stochastic representation of the Navier-Stokes equations using pointwise solutions

In this section we provide an alternate proof that the system (2.4.1) – (2.4.4) is equivalent to the Navier-Stokes equations (Theorem 2.4.1). We do this by constructing pointwise (in the probability space) solutions to the SDE (2.4.1). This idea has been used by LeBris and Lions in [18] using a generalization of the $W^{1,1}$ theory. In our context however, the velocity u is spatially regular enough for us to explicitly construct the pointwise solution without appealing to the generalized $W^{1,1}$ theory.

Definition 2.5.1. Given a (divergence free) velocity u, we define the operator $\mathcal{D}$ by

$$\mathcal{D}v = \partial_t v + (u \cdot \nabla)\, v$$

Lemma 2.5.2. *The commutator* $[\mathcal{D}, \nabla]$ *is given by*

$$[\mathcal{D}, \nabla] f = \mathcal{D}(\nabla f) - \nabla(\mathcal{D}f) = -\nabla^* u \nabla f$$

Proof. By definition,

$$\begin{aligned}
[\mathcal{D}, \nabla] f &= \mathcal{D}(\nabla f) - \nabla(\mathcal{D}f) \\
&= (u \cdot \nabla)\nabla f - \nabla\left[(u \cdot \nabla) f\right] \\
&= (u \cdot \nabla)\nabla f - (u \cdot \nabla)\nabla f - \nabla^* u \nabla f
\end{aligned}$$

□

Lemma 2.5.3. *Given a divergence-free velocity u, let X and A be defined by*

$$\begin{gathered}
\dot{X} = u(X) \\
X(a, 0) = a \\
A = X^{-1}
\end{gathered}$$

We define v by the evolution equation

$$\mathcal{D}v = \Gamma.$$

with initial data v_0. If w is defined by

$$w = \mathbf{P}\left[(\nabla^* A)\ v\right],$$

then the evolution of w is given by the system

$$\partial_t w + (u \cdot \nabla) w + (\nabla^* u) w + \nabla p = (\nabla^* A)\Gamma$$

$$\nabla \cdot w = 0$$
$$w_0 = \mathbf{P} v_0$$

Proof. By definition of the Leray-Hodge projection, there exists a function p such that

$$\begin{aligned} w &= \nabla^* A\, v - \nabla p \\ &= v_i \nabla A_i - \nabla p \\ \implies \qquad \mathcal{D} w &= (\mathcal{D} v_i) \nabla A_i + v_i \mathcal{D} \nabla A_i - \mathcal{D} \nabla p \\ &= \Gamma_i \nabla A_i - v_i (\nabla^* u) \nabla A_i - \nabla \mathcal{D} p + (\nabla^* u) \nabla p \\ &= (\nabla^* A) \Gamma - (\nabla^* u)(v_i \nabla A_i + \nabla p) - \nabla \mathcal{D} p \\ &= (\nabla^* A) \Gamma - (\nabla^* u) w - \nabla \mathcal{D} p. \end{aligned}$$
□

Corollary 2.5.4. *If u, X, A are as above, and we define w by*

$$w = \mathbf{P}\left[(\nabla^* A)\, u_0 \circ A \right]. \tag{2.5.1}$$

Then w evolves according to

$$\mathcal{D} w + (\nabla^* u) w + \nabla p = 0 \tag{2.5.2}$$
$$\nabla \cdot w = 0 \tag{2.5.3}$$
$$w(x, 0) = w_0(x) \tag{2.5.4}$$

Proof. The proof follows from Lemma 2.5.3 by setting $v = u_0 \circ A$ and $\Gamma = 0$. □

Remark 2.5.5. If $w = u$ in equation 2.5.1, then u satisfies the Euler equations. This is the part of Theorem 2.3.1 that was not proved in Section 2.3. This follows immediately because $(\nabla^* u) u = \frac{1}{2} \nabla |u|^2$, and thus the term $(\nabla^* u) w$ in equation 2.5.2 can be combined with the pressure, yielding the Euler equations.

We are now ready to provide an alternate proof of Theorem 2.4.1.

Alternate proof of Theorem 2.4.1. For simplicity and without loss of generality we take $\nu = \frac{1}{2}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $W : [0, \infty) \times \Omega \to \mathbb{R}^3$ a three dimensional Wiener process. Define u^ω and Y^ω by

$$u^\omega(x, t) = u(x + W_t(\omega), t)$$

and

$$\dot{Y}^\omega = u^\omega(Y^\omega)$$
$$Y^\omega(a, 0) = a$$

Although u^ω is not Lipschitz in time, it is certainly uniformly (in time) Lipschitz in space. Thus the regular Picard iteration will produce solutions of this equations. Finally notice that the map X defined by

$$X(a,t,\omega) = Y^\omega(a,t,\omega) + W_t(\omega)$$

solves the SDE (2.4.1).

Let B^ω be the spatial inverse of Y^ω. Notice that

$$\begin{aligned} X(B^\omega(x - W_t, t), t) &= Y^\omega(B^\omega(x - W_t, t), t) + W_t \\ &= x \end{aligned}$$

and hence

$$A = \tau_{W_t} B^\omega \tag{2.5.5}$$

where τ_x is the translation operator defined by

$$\tau_x f(y) = f(y - x) \tag{2.5.6}$$

We define w^ω by

$$w^\omega = \mathbf{P}\left[(\nabla^* B^\omega)\, u_0 \circ B^\omega\right].$$

By Lemma 2.5.3, the function w^ω evolves according to

$$\partial_t w^\omega + (u^\omega \cdot \nabla) w^\omega + (\nabla^* u^\omega) w^\omega + \nabla q^\omega = 0 \tag{2.5.7}$$

$$\nabla \cdot w^\omega = 0 \tag{2.5.8}$$

$$w^\omega(x, 0) = u_0(x). \tag{2.5.9}$$

Now using equation (2.4.3) we have

$$\begin{aligned} u &= \mathbf{EP}\left[(\nabla^* A)\, u_0 \circ A\right] \\ &= \mathbf{EP}\left[(\nabla^* \tau_{W_t} B^\omega)\, u_0 \circ \tau_{W_t} B^\omega\right] \\ &= \mathbf{EP}\left[\tau_{W_t}\left((\nabla^* B^\omega)\, u_0 \circ B^\omega\right)\right] \\ &= \mathbf{E}\tau_{W_t}\mathbf{P}\left[(\nabla^* B^\omega)\, u_0 \circ B^\omega\right] \\ &= \mathbf{E}\tau_{W_t} w^\omega \end{aligned} \tag{2.5.10}$$

Our assumption $u_0 \in C^{k+1,\alpha}$ along with Theorem 3.2.6 guarantee that w^ω is spatially regular enough to apply the generalized Itô formula [17] to $w^\omega(x - W_t, t)$, and we have

$$\begin{aligned} w^\omega(x - W_t, t) - u_0(x) = &\int_0^t w^\omega(x - W_s, ds) - \int_0^t \nabla w^\omega\big|_{x - W_s, s} dW_s + \\ &+ \tfrac{1}{2}\int_0^t \triangle w^\omega\big|_{x - W_s, s} ds + \left\langle \int_0^t \partial_j w^\omega(x - W_s, ds)\,,\, x_j - W_t^j \right\rangle. \end{aligned}$$

Notice that the process w^ω is C^1 in time (since the time derivative is given by equation (2.5.7)), and hence bounded variation. Thus the joint quadratic variation term vanishes. Taking expected values and using (2.5.10) we conclude

$$u(x,t) - u_0(x) = \mathbf{E}\int_0^t w^\omega(x - W_s, ds) + \tfrac{1}{2}\int_0^t \triangle u(x,s)\, ds \tag{2.5.11}$$

Using equation (2.5.7) and the definition of the Itô integral we have

$$\begin{aligned}
\mathbf{E}\int_0^t w^\omega(x - W_s, ds) &= \mathbf{E}\int_0^t \partial_t w^\omega\big|_{x-W_s,s} ds \\
&= -\mathbf{E}\int_0^t \left[(u^\omega \cdot \nabla) w^\omega + (\nabla^* u^\omega) w^\omega + \nabla q^\omega\right]_{x-W_s,s} ds \\
&= -\mathbf{E}\int_0^t \Big[(u(x,s)\cdot\nabla)\, w^\omega\big|_{x-W_s,s} + (\nabla^* u(x,s))\, w^\omega\big|_{x-W_s,s} \\
&\qquad + \nabla q^\omega\big|_{x-W_s,s}\Big]\, ds \\
&= -\int_0^t \Big[(u(x,s)\cdot\nabla)\, u\big|_{x,s} + (\nabla^* u(x,t))\, u\big|_{x,s} + \\
&\qquad + \nabla \mathbf{E} q^\omega(x - W_s, s)\Big]\, ds \\
&= -\int_0^t \Big[(u(x,s)\cdot\nabla)\, u\big|_{x,s} + \nabla q'\big|_{x,s}\Big]\, ds \qquad (2.5.12)
\end{aligned}$$

where q' is defined by

$$q' = \tfrac{1}{2}\nabla|u|^2 + \mathbf{E}\tau_{W_t} q^\omega$$

Using equations (2.5.12) in (2.5.11) (along with the observation that the joint quadratic variation term is 0) we obtain

$$u(x,t) - u_0(x) = -\int_0^t \left[(u\cdot\nabla)u + \tfrac{1}{2}\triangle u + \nabla q\right]_{x,s} ds$$

Equations (2.5.8) and (2.5.10) show that u is divergence free, concluding the proof. □

2.6 Stochastic representations of the reaction diffusion, and semi-linear transport equations

In this section we produce a stochastic representation of second order semi-linear transport equations, and study specifically the reaction diffusion equations. Our representation again involves an implicit fixed point similar to that used for our representation of the Navier-Stokes equations.

In the absence of the diffusive second order term, the method of characteristics provides a solution to the first order PDE (2.1.1) by boosting particle trajectories with the solution of a non-linear (spatially parametrized) ODE. For the second order dissipative PDE, we obtain the non-linearity by boosting the random characteristics of the stochastic flow X by a function that involves an average of a functional of X. We begin by extending Theorem 2.1.5 to semi-linear transport equations.

Theorem 2.6.1. *Suppose f is a C^2 function, b' is C^2 in space, C^0 in time and (a_{ij}) is a strictly positive definite symmetric matrix which is C^2 in space and C^0 in time.*

Define σ to be a C^2 in space, C^0 in time matrix such that $\sigma\sigma^* = (a_{ij})$, and define $b_j = b'_j - (\partial_i \sigma_{jk})\sigma_{ik}$. If the pair of processes X, θ satisfy the system of equations

$$dX = b\,dt + \sigma\,dW \tag{2.6.1}$$

$$\theta_t = \vartheta_t(A_t) \tag{2.6.2}$$

with initial data $X_0(a) = a$, where A is the spatial inverse of X and ϑ is defined by

$$\vartheta_t = \theta_0 + \int_0^t f\left(\bar{\theta}_s \circ X_s\right)\,ds.$$

where $\bar{\theta}$ denotes the expected value of θ, then $\bar{\theta}$ satisfies the PDE

$$\partial_t \bar{\theta} + (b' \cdot \nabla)\bar{\theta} - \tfrac{1}{2} a_{ij} \partial^2_{ij} \bar{\theta} = f(\bar{\theta})$$

with initial data θ_0.

Proof. From Corollary 2.1.4 we know

$$d\theta_t + \left[(b' \cdot \nabla)\theta_t - \tfrac{1}{2} a_{ij} \partial^2_{ij} \theta_t\right]\,dt + (\nabla\theta_t)\sigma\,dW_t = f(\bar{\theta}_t)\,dt$$

and the theorem follows by taking the expected value of the Itô integral. □

Remark 2.6.2. If $\sigma \equiv 0$, then ϑ would be a solution of the ODE equation (2.1.4), as in the method of characteristics. When $\sigma \neq 0$, we obtained the non-linear term $f(\bar{\theta})$ by boosting the inverse flow A by the function ϑ along the random particle trajectories. The computation of ϑ however is implicit, as it involves $\bar{\theta}$, unlike the case when $\sigma \equiv 0$.

In order to understand the physical significance of the dependence of ϑ on $\bar{\theta}$, we consider the reaction diffusion equations as an example. Consider the system of equations

$$\partial_t \bar{N} - \nu_1 \triangle \bar{N} = -\bar{N}\bar{\theta} \tag{2.6.3}$$

$$\partial_t \bar{\theta} - \nu_2 \triangle \bar{\theta} = \bar{N}\bar{\theta}, \tag{2.6.4}$$

with specified initial data N_0, θ_0. Here $\bar{N}$ represents the concentration (amount) of fuel, and θ the temperature.

When the diffusion rates ν_1 and ν_2 are equal, and if we assume $\theta_0 + N_0 \equiv 1$ adding (2.6.3) and (2.6.4) immediately gives $\bar{\theta} + \bar{N} \equiv 1$, and hence (2.6.3) – (2.6.4) reduce to the Kolmogorov equation

$$\partial_t \bar{\theta} - \nu \triangle \bar{\theta} = \bar{\theta}(1 - \bar{\theta}).$$

For our purposes however it is more illustrative to think of the Kolmogorov equation as the coupled system (2.6.3) – (2.6.4).

We obtain a stochastic representation as follows: Let X, Y be the stochastic flows defined by

$$X_t(a) = a + \sqrt{2\nu_1} W_t \tag{2.6.5}$$

$$Y_t(a) = a + \sqrt{2\nu_2} W_t \tag{2.6.6}$$

where W is a Wiener process. Let $A = X^{-1}$, $B = Y^{-1}$, and define

$$N_t = \left[\exp\left(- \int_0^t \bar{\theta}_s \circ X_s \right) N_0 \right] \circ A_t \tag{2.6.7}$$

$$\theta_t = \left[\exp\left(\int_0^t \bar{N}_s \circ Y_s \right) \theta_0 \right] \circ B_t. \tag{2.6.8}$$

Here $\bar{\theta}$ and $\bar{N}$ denote the expected values of N and θ respectively. Then by Corollary 2.1.4, we have

$$\begin{aligned} dN - \nu_1 \triangle N \, dt + \sqrt{2\nu_1} \nabla N \, dW &= -\bar{\theta} N \, dt \\ d\theta - \nu_2 \triangle \theta \, dt + \sqrt{2\nu_2} \nabla \theta \, dW &= \theta \bar{N} \, dt. \end{aligned}$$

Integrating and taking expected values, we immediately see that $\bar{N}$ and $\bar{\theta}$ satisfy equations (2.6.3) – (2.6.4) with initial data N_0 and θ_0 respectively.

Notice again that we obtain θ by boosting trajectories of the stochastic flow Y by a functional of the *expected value* of the concentration N. This is physically what we expect, as the likely hood of the reaction to occur at a given time should depend on the average concentration, and not the concentration along the individual Wiener paths.

We explore this concept more rigorously in Section 3.3, for the Navier-Stokes equations. We consider a system where the stochastic flow is driven by a velocity which depends on a functional of the stochastic flow map, and not it's average. Though this system is a super-linear approximation of the Navier-Stokes equations, we show that this system has characteristics that are more similar to the Euler equations, and not the Navier-Stokes equations as we might be inclined to believe.

CHAPTER 3
LOCAL EXISTENCE, AND A NON-AVERAGED MODEL OF THE NAVIER-STOKES EQUATIONS

In this chapter we provide a self contained proof of local existence and well posedness for the Navier-Stokes equations in boundaryless domains using the stochastic formulation (2.4.1) – (2.4.4). The estimates developed allow us to quickly compute the rate of convergence to the Euler equations as $\nu \to 0$.

Finally in Section 3.3 we consider the system of equations where the velocity u is given by the inviscid Weber formula, *without* averaging the noise. Our first instinct would be to believe that the average of this system behaves like a perturbation of the Navier-Stokes equations. We will show however that this system, while a good approximation of the Navier-Stokes equations, behaves more like a perturbation of the Euler equations.

3.1 The Weber operator and bounds.

In this section we define and obtain estimates for the Weber operator which will be central to all subsequent results. We begin by establishing the notational convention we use throughout this chapter. We let $\mathcal{I}$ denote the cube $[0, L]^3$ with side of length L. We define the (scale invariant) Hölder norms and semi-norms on $\mathcal{I}$ by

$$
\begin{aligned}
|u|_\alpha &= \sup_{x,y\in\mathcal{I}} L^\alpha \frac{|u(x)-u(y)|}{|x-y|^\alpha}\\
\|u\|_{C^k} &= \sum_{|m|\leqslant k} L^{|m|} \sup_{\mathcal{I}} |D^m u|\\
\|u\|_{k,\alpha} &= \|u\|_{C^k} + \sum_{|m|=k} L^k \, |D^m u|_\alpha
\end{aligned}
$$

where D^m denotes the derivative with respect to the multi index m. We let C^k denote the space of all k-times continuously differentiable spatially periodic functions on $\mathcal{I}$, and $C^{k,\alpha}$ denote the space of all spatially periodic $k+\alpha$ Hölder continuous functions. The spaces C^k and $C^{k,\alpha}$ are endowed with the norms $\|\cdot\|_{C^k}$ and $\|\cdot\|_{k,\alpha}$ respectively.

We use I to denote the identity function on $\mathbb{R}^3$ or $\mathcal{I}$ (depending on the context), and use $\mathbb{I}$ to denote the identity matrix.

Definition 3.1.1. We define the Weber operator $\mathbf{W} : C^{k,\alpha} \times C^{k+1,\alpha} \to C^{k,\alpha}$ by

$$\mathbf{W}(v, \ell) = \mathbf{P}\left[(\mathbb{I} + \nabla^* \ell)\, v\right]$$

where $\mathbf{P}$ is the Leray-Hodge projection [8] onto divergence free vector fields.

Remark 3.1.2. The range of $\mathbf{W}$ is $C^{k,\alpha}$ because multiplication by a $C^{k,\alpha}$ function is bounded on $C^{k,\alpha}$, and $\mathbf{P}$ is a classical Calderon-Zygmund singular integral operator [25] which is bounded on Hölder spaces.

Remark. In the whole space, or with periodic boundary conditions, the Leray-Hodge projection commutes with derivatives. This is not true for arbitrary domains [9].

Formally it seems that $\mathbf{W}(v,\ell)$ should have one less derivative than ℓ. However we prove below that $\mathbf{W}(v,\ell)$ has as many derivatives as ℓ. The reason being, when we differentiate $\mathbf{W}(v,\ell)$, we can use 'integration by parts' to express the right hand side only in terms of first order derivatives.

Lemma 3.1.3 (Integration by parts)**.** *If $u, v \in C^{1,\alpha}$ then*

$$\mathbf{P}\left[(\nabla^* u)\, v\right] = -\mathbf{P}\left[(\nabla^* v)\, u\right]$$

Proof. This follows immediately from the identity

$$(\nabla^* u)\, v + (\nabla^* v)\, u = \nabla(u \cdot v)$$

and the fact that $\mathbf{P}$ vanishes on gradients. ☐

Corollary 3.1.4. *If $k \geqslant 1$ and $v, \ell \in C^{k,\alpha}$ then $\mathbf{W}(v,\ell) \in C^{k,\alpha}$ and*

$$\|\mathbf{W}(v,\ell)\|_{k,\alpha} \leqslant c\left(1 + \|\nabla \ell\|_{k-1,\alpha}\right)\|v\|_{k,\alpha}.$$

Proof. Notice first that $\mathbf{W}(v,\ell) \in C^{k-1,\alpha}$ by Remark 3.1.2. Now

$$\begin{aligned}\partial_i \mathbf{W}(v,\ell) &= \mathbf{P}\left[\partial_i v + (\nabla^* \partial_i \ell) v + \nabla^* \ell\, \partial_i v\right] \\ &= \mathbf{P}\left[\partial_i v - \nabla^* v\, \partial_i \ell + \nabla^* \ell\, \partial_i v\right].\end{aligned}$$

Now the right hand side has only first order derivatives of ℓ and v, hence $\nabla \mathbf{W}(v,\ell) \in C^{k-1,\alpha}$ and the proposition follows. ☐

Proposition 3.1.5. *If $k \geqslant 1$ and $\ell_1, \ell_2 \in C^{k,\alpha}$ and $v_1, v_2 \in C^{k,\alpha}$, are such that*

$$\|\nabla \ell_i\|_{k-1,\alpha} \leqslant d$$

for $i = 1, 2$, then there exists $c = c(k, d, \alpha)$ such that

$$\|\mathbf{W}(v_1,\ell_1) - \mathbf{W}(v_2,\ell_2)\|_{k,\alpha} \leqslant c\left(\|v_2\|_{k,\alpha}\|\nabla \ell_1 - \nabla \ell_2\|_{k-1,\alpha} + \|v_1 - v_2\|_{k,\alpha}\right).$$

Proof. We deduce this proposition from Corollary 3.1.4 as follows:

$$\begin{aligned}
&\mathbf{W}(v_1,\ell_1)-\mathbf{W}(v_2,\ell_2) = \mathbf{P}\left[(\mathbb{I}+\nabla^*\ell_1)v_1-(\mathbb{I}+\nabla^*\ell_2)v_2\right]\\
&\qquad = \mathbf{P}\left[(\mathbb{I}+\nabla^*\ell_1)(v_1-v_2)+\nabla^*(\ell_1-\ell_2)v_2\right]\\
\implies\quad &\|\mathbf{W}(v_1,\ell_1)-\mathbf{W}(v_2,\ell_2)\|_{k,\alpha} \leqslant c\big([1+\|\nabla\ell\|_{k-1,\alpha}]\,\|v_1-v_2\|_{k,\alpha}+\\
&\qquad +\|\nabla\ell_1-\nabla\ell_2\|_{k-1,\alpha}\,\|v_2\|_{k,\alpha}\big) \qquad \square
\end{aligned}$$

3.2 Local existence for the stochastic formulation.

In this section we prove local in time $C^{k,\alpha}$ existence for the stochastic formulation of the Navier-Stokes equations. We conclude this section by using the stochastic formulation (2.4.1) – (2.4.4) to study the convergence of the Navier-Stokes equations (in boundaryless domains) as $\nu \to 0$. We begin with a few preliminary results.

Lemma 3.2.1. *If $k \geqslant 1$, then there exists a constant $c = c(k,\alpha)$ such that*

$$\|f\circ g\|_{k,\alpha} \leqslant c\,\|f\|_{k,\alpha}\left(1+\|\nabla g\|_{k-1,\alpha}\right)^{k+\alpha}$$

$$\|f\circ g_1 - f\circ g_2\|_{k,\alpha} \leqslant c\,\|\nabla f\|_{k,\alpha}\left(1+\|\nabla g_1\|_{k-1,\alpha}+\|\nabla g_2\|_{k-1,\alpha}\right)^{k+1}\cdot \\ \cdot\,\|g_1-g_2\|_{k,\alpha}$$

and

$$\|f_1\circ g_1 - f_2\circ g_2\|_{k,\alpha} \leqslant c\left(1+\|\nabla g_1\|_{k-1,\alpha}+\|\nabla g_2\|_{k-1,\alpha}\right)^{k+1}\cdot \\ \cdot\left[\|f_1-f_2\|_{k,\alpha}+\min\left\{\|\nabla f_1\|_{k,\alpha},\|\nabla f_2\|_{k,\alpha}\right\}\|g_1-g_2\|_{k,\alpha}\right].$$

The proof of Lemma 3.2.1 is elementary and not presented here. We subsequently use the above lemma repeatedly without reference or proof.

Lemma 3.2.2. *Let X be a Banach algebra. If $x \in X$ is such that $\|x\| \leqslant \rho < 1$ then $1+x$ is invertible and $\|(1+x)^{-1}\| \leqslant \frac{1}{1-\rho}$. Further if in addition $\|y\| \leqslant \rho$ then*

$$\left\|(1+x)^{-1}-(1+y)^{-1}\right\| \leqslant \frac{1}{(1-\rho)^2}\,\|x-y\|$$

Proof. The first part of the Lemma follows immediately from the identity $(1+x)^{-1} = \sum(-x)^n$. The second part follows from the first part and the identity

$$(1+x)^{-1}-(1+y)^{-1} = (1+x)^{-1}(y-x)(1+y)^{-1}. \qquad \square$$

We generally use Lemma 3.2.2 when X is the space of $C^{k,\alpha}$ periodic matrices.

Lemma 3.2.3. *Let $X_1, X_2 \in C^{k+1,\alpha}$ be such that*

$$\|\nabla X_1 - \mathbb{I}\|_{k,\alpha} \leqslant d < 1 \quad \textit{and} \quad \|\nabla X_2 - \mathbb{I}\|_{k,\alpha} \leqslant d < 1.$$

Let A_1 and A_2 be the inverse of X_1 and X_2 respectively. Then there exists a constant $c = c(k, \alpha, d)$ such that

$$\|A_1 - A_2\|_{k,\alpha} \leqslant c \, \|X_1 - X_2\|_{k,\alpha}$$

Proof. Let $c = c(k, \alpha, d)$ be a constant that changes from line to line (we use this convention implicitly throughout this paper). Note first $\nabla A = (\nabla X)^{-1} \circ A$, and hence by Lemma 3.2.2

$$\|\nabla A\|_{C^0} \leqslant \left\|(\nabla X)^{-1}\right\|_{C^0} \leqslant c.$$

Now using Lemma 3.2.2 to bound $\|(\nabla X)^{-1}\|_\alpha$ we have

$$\|\nabla A\|_\alpha = \left\|(\nabla X)^{-1} \circ A\right\|_\alpha \leqslant \left\|(\nabla X)^{-1}\right\|_\alpha (1 + \|\nabla A\|_{C^0}) \leqslant c$$

When $k \geqslant 1$, we again bound $\|(\nabla X)^{-1}\|_{k,\alpha}$ by Lemma 3.2.2. Taking the $C^{k,\alpha}$ norm of $(\nabla X)^{-1} \circ A$ we have

$$\|\nabla A\|_{k,\alpha} \leqslant \left\|(\nabla X)^{-1}\right\|_{k,\alpha} \left(1 + \|\nabla A\|_{k-1,\alpha}\right)^k.$$

So by induction we can bound $\|\nabla A\|_{k,\alpha}$ by a constant $c = c(k, \alpha, d)$. The Lemma now follows immediately from the identity

$$\begin{aligned} A_1 - A_2 &= (A_1 \circ X_2 - I) \circ A_2 \\ &= (A_1 \circ X_2 - A_1 \circ X_1) \circ A_2 \end{aligned}$$

and Lemma 3.2.1. $\square$

Lemma 3.2.4. *Let $u \in C([0, T], C^{k+1,\alpha})$ and X satisfy the SDE* (2.4.1) *with initial data* (2.4.4)*. Let $\lambda = X - I$ and $U = \sup_t \|u(t)\|_{k+1,\alpha}$. Then there exists $c = c(k, \alpha, \|u\|_{k+1,\alpha})$ such that for short time*

$$\|\nabla\lambda(t)\|_{k,\alpha} \leqslant \frac{cUt}{L} e^{cUt/L} \qquad \textit{and} \qquad \|\nabla\ell(t)\|_{k,\alpha} \leqslant \frac{cUt}{L} e^{cUt/L}.$$

Proof. From equation (2.4.1) we have

$$X(x, t) = x + \int_0^t u(X(x, s), s) \, ds + \sqrt{2\nu} B_t$$

$$\Longrightarrow \quad \nabla X(t) = I + \int_0^t (\nabla u) \circ X \cdot \nabla X. \tag{3.2.1}$$

Taking the C^0 norm of equation (3.2.1) and using Gronwall's Lemma we have

$$\|\nabla\lambda(t)\|_{C^0} = \|\nabla X(t) - I\|_{C^0} \leqslant e^{Ut/L} - 1.$$

Now taking the $C^{k,\alpha}$ norm in equation (3.2.1) we have

$$\|\nabla\lambda(t)\|_{k,\alpha} \leqslant c\int_0^t \|\nabla u\|_{k,\alpha}\left(1 + \|\nabla\lambda\|_{k-1,\alpha}\right)^k\left(1 + \|\nabla\lambda\|_{k,\alpha}\right).$$

The bound for $\|\nabla\lambda\|_{k,\alpha}$ now follows from the previous two inequalities, induction and Gronwall's Lemma. The bound for $\|\nabla\ell\|_{k,\alpha}$ then follows from Lemma 3.2.3.

We draw attention to the fact that the above argument can only bound $\nabla\lambda$, and not λ. Fortunately, our results only rely on a bound of $\nabla\lambda$. □

Lemma 3.2.5. *Let $u, \tilde{u} \in C([0,T], C^{k+1,\alpha})$ be such that*

$$\sup_{0\leqslant t\leqslant T}\|u(t)\|_{k+1,\alpha} \leqslant U \quad \textit{and} \quad \sup_{0\leqslant t\leqslant T}\|\tilde{u}(t)\|_{k+1,\alpha} \leqslant U.$$

Let $X, \tilde{X}$ be solutions of the SDE (2.4.1)–(2.4.4) with drift u and $\tilde{u}$ respectively, and let A and $\tilde{A}$ be the spatial inverse of X and $\tilde{X}$ respectively. Then there exists $c = c(k,\alpha,U)$ and a time $T = T(k,\alpha,U)$ such that

$$\left\|X(t) - \tilde{X}(t)\right\|_{k,\alpha} \leqslant ce^{cUt/L}\int_0^t \|u - \tilde{u}\|_{k,\alpha} \tag{3.2.2}$$

$$\left\|A(t) - \tilde{A}(t)\right\|_{k,\alpha} \leqslant ce^{cUt/L}\int_0^t \|u - \tilde{u}\|_{k,\alpha} \tag{3.2.3}$$

for all $0 \leqslant t \leqslant T'$.

Proof. We first use Lemma 3.2.4 to bound $\|\nabla X - \mathbb{I}\|_{k,\alpha}$ and $\|\nabla\tilde{X} - \mathbb{I}\|_{k,\alpha}$ for short time T'. Now

$$\begin{aligned} X(t) - \tilde{X}(t) &= \int_0^t u\circ X - \tilde{u}\circ\tilde{X} \\ \implies \qquad \left\|X(t) - \tilde{X}(t)\right\|_{k,\alpha} &\leqslant \int_0^t \left\|u\circ X - \tilde{u}\circ\tilde{X}\right\|_{k,\alpha} \\ &\leqslant c\int_0^t \left(\|u - \tilde{u}\|_{k,\alpha} + \frac{U}{L}\left\|X - \tilde{X}\right\|_{k,\alpha}\right) \end{aligned}$$

and inequality (3.2.2) follows by applying Gronwall's Lemma. Inequality (3.2.3) follows immediately from (3.2.2) and Lemma 3.2.3. □

We are now ready to prove local existence for the Navier-Stokes equations using the stochastic formulation (2.4.1) – (2.4.4).

Theorem 3.2.6. *Let $k \geqslant 1$ and $u_0 \in C^{k+1,\alpha}$ be divergence free. There exists a time $T = T(k, \alpha, L, \|u_0\|_{k+1,\alpha})$, but independent of viscosity, and a pair of functions $\lambda, u \in C([0,T], C^{k+1,\alpha})$ such that u and $X = I + \lambda$ satisfy the system (2.4.1) – (2.4.4). Further $\exists U = U(k, \alpha, L, \|u_0\|_{k+1,\alpha})$ such that $t \in [0,T] \implies \|u(t)\|_{k+1,\alpha} \leqslant U$.*

Proof. Let U be a large constant, and T a small time, both of which will be specified later. Define as before $\mathcal{U}$ and $\mathcal{L}$ by

$$\mathcal{U} = \left\{ u \in C([0,T], C^{k+1,\alpha}) \;\middle|\; \|u(t)\|_{k+1,\alpha} \leqslant U,\ \nabla \cdot u = 0 \text{ and } u(0) = u_0 \right\}$$

$$\text{and} \quad \mathcal{L} = \left\{ \ell \in C([0,T], C^{k+1,\alpha}) \;\middle|\; \|\nabla \ell(t)\|_{k,\alpha} \leqslant \tfrac{1}{2}\ \forall t \in [0,T] \text{ and } \ell(\cdot, 0) = 0 \right\}.$$

We clarify that the functions u and ℓ are required to be spatially $C^{k+1,\alpha}$, and need only be continuous in time.

Now given $u \in \mathcal{U}$ we define X_u to be the solution of equation (2.4.1) with initial data (2.4.4) and $\lambda_u = X_u - I$ be the Eulerian displacement. We define A_u by equation (2.4.2) and let $\ell_u = A_u - I$ be the Lagrangian displacement. Finally we define $W : \mathcal{U} \to \mathcal{U}$ by

$$W(u) = \mathbf{EW}(u_0 \circ A_u, \ell_u).$$

We aim to show that $W : \mathcal{U} \to \mathcal{U}$ is Lipschitz in the weaker norm

$$\|u\|_{\mathcal{U}} = \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{k,\alpha}$$

and when T is small enough, we will show that W is a contraction mapping.

Let c be a constant that changes from line to line. By Corollary 3.1.4 we have

$$\begin{aligned} \|W(u)\|_{k+1,\alpha} &\leqslant c\mathbf{E}\left[\left(1 + \|\nabla \ell_u\|_{k,\alpha}\right) \|u_0 \circ A_u\|_{k+1,\alpha} \right] \\ &\leqslant c \|u_0\|_{k+1,\alpha} \sup_{\Omega} \left(1 + \|\nabla \ell_u\|_{k,\alpha}\right)^{k+2}. \end{aligned} \tag{3.2.4}$$

Here Ω is the probability space on which our processes are defined. We remark that Lemma 3.2.4 gives us a bound on $\|\nabla \ell_u\|_{k,\alpha}$. A bound on $\mathbf{E}\|\nabla \ell_u\|_{k,\alpha}$ instead would not have been enough.

Now we choose $U = c(\frac{3}{2})^{k+2}\|u_0\|_{k+1,\alpha}$, and then apply Lemma 3.2.4 to choose T small enough to ensure $\ell_u, \lambda_u \in \mathcal{L}$. Now inequality 3.2.4 ensures that $W(u) \in \mathcal{U}$. Now if $u, \tilde{u} \in \mathcal{U}$, Lemma 3.2.5 guarantees

$$\|\ell_u(t) - \ell_{\tilde{u}}(t)\|_{k,\alpha} \leqslant c e^{cUt/L} \int_0^t \|u - \tilde{u}\|_{k,\alpha}.$$

Thus applying Proposition 3.1.5 we have

$$\begin{aligned}
\|W(u)(t) - W(\tilde{u})(t)\|_{k,\alpha} &\leqslant c\Big(\tfrac{U}{L}\,\|\ell_u(t) - \ell_{\tilde{u}}(t)\|_{k,\alpha} + \\
&\qquad + \|u_0 \circ A_u(t) - u_0 \circ A_{\tilde{u}}(t)\|_{k,\alpha}\Big) \\
&\leqslant \frac{cU}{L}\,\|\ell_u(t) - \ell_{\tilde{u}}(t)\|_{k,\alpha} \\
&\leqslant \frac{cU}{L} e^{cUt/L} \int_0^t \|u - \tilde{u}\|_{k,\alpha}\,.
\end{aligned}$$

So choosing $T = T(k, \alpha, L, U)$ small enough we can ensure W is a contraction.

The existence of a fixed point of W now follows by successive iteration. We define $u_{n+1} = W(u_n)$. The sequence (u_n) converges strongly with respect to the $C^{k,\alpha}$ norm. Since $\mathcal{U}$ is closed and convex, and the sequence (u_n) is uniformly bounded in the $C^{k+1,\alpha}$ norm, it must have a weak limit $u \in \mathcal{U}$. Finally since W is continuous with respect to the weaker $C^{k,\alpha}$ norm, the limit must be a fixed point of W, and hence a solution to the system (2.4.1) – (2.4.4). $\square$

We conclude this section by studying the rate of convergence of (2.4.1) – (2.4.4) to the Euler equations (2.3.1) – (2.3.2) in the limit $\nu \to 0$. Our method will only work for boundaryless domains, and we remark again that the vanishing viscosity limit for domains with boundary is not well understood.

Proposition 3.2.7. *Let $k \geqslant 1$ and $u_0 \in C^{k+1,\alpha}$ be divergence free, and U, T be as in Theorem 3.2.6. For each $\nu > 0$ we let u_ν be the solution of the system* (2.4.1) – (2.4.4) *on the time interval* $[0, T]$. *Making T smaller if necessary, let u be the solution to the Euler equations* (2.3.1) – (2.3.2) *with initial data u_0 defined on the time interval* $[0, T]$. *Then there exists a constant $c = c(k, \alpha, U, L)$ such that for all $t \in [0, T]$ we have*

$$\|u(t) - u_\nu(t)\|_{k,\alpha} \leqslant \tfrac{cU}{L}\sqrt{\nu t}$$

Proof. We use a subscript of ν to denote quantities associated to the solution of viscous problem (2.4.1) – (2.4.4), and unsubscripted letters to denote the corresponding quantities associated to the solution of the Eulerian-Lagrangian formulation of the Euler equations (2.3.3) – (2.3.6). We use the same notation as in the proof of Theorem 3.2.6.

Now from the proof of Theorem 3.2.6 we know that for short time $\ell_\nu, \ell \in \mathcal{L}$. Using Lemma 3.2.3 and making T smaller if necessary, we can ensure $\lambda_\nu, \lambda \in \mathcal{L}$. We begin by estimating $\mathbf{E}\|\lambda_\nu - \lambda\|_{k,\alpha}$:

$$\lambda_\nu(t) - \lambda(t) = \int_0^t [u_\nu \circ X_\nu - u \circ X] + \sqrt{2\nu} B_t$$

$$\implies \quad \|\lambda_\nu(t) - \lambda(t)\|_{k,\alpha} \leqslant c\left(\int_0^t \left[\|u_\nu - u\|_{k,\alpha} + \tfrac{U}{L}\,\|\lambda_\nu - \lambda\|_{k,\alpha}\right] + \sqrt{\nu}|B_t|\right)$$

and so by Gronwall's lemma

$$\|\lambda_\nu(t) - \lambda(t)\|_{k,\alpha} \leqslant c\left(\sqrt{\nu}|B_t| + \int_0^t \|u_\nu - u\|_{k,\alpha}\right) e^{cUt/L}.$$

Using Lemma 3.2.3 and taking expected values gives

$$\mathbf{E}\,\|\ell_\nu(t) - \ell(t)\|_{k,\alpha} \leqslant c\left(\sqrt{\nu t} + \int_0^t \|u_\nu - u\|_{k,\alpha}\right) e^{cUt/L}. \tag{3.2.5}$$

To estimate the difference $u_\nu - u$, we use (2.4.3), and (2.3.5) to obtain

$$\begin{aligned}
& & u_\nu - u &= \mathbf{E}\mathbf{W}(u_0 \circ A_\nu, \ell_\nu) - \mathbf{W}(u_0 \circ A, \ell) \\
\Longrightarrow & & \|u_\nu - u\|_{k,\alpha} &\leqslant c\mathbf{E}\left(\tfrac{U}{L}\,\|\ell_\nu - \ell\|_{k,\alpha} + \|u_0 \circ A_\nu - u_0 \circ A\|_{k,\alpha}\right) \\
& & &\leqslant \tfrac{cU}{L}\mathbf{E}\,\|\ell_\nu - \ell\|_{k,\alpha} \\
\Longrightarrow & & \|u_\nu(t) - u(t)\|_{k,\alpha} &\leqslant \tfrac{cU}{L} e^{cUt/L}\left(\sqrt{\nu t} + \int_0^t \|u_\nu - u\|_{k,\alpha}\right)
\end{aligned}$$

and the theorem follows from Gronwall's lemma. □

3.3 A non-averaged model and super-linear approximation

In this section we consider the system of equations

$$dX' = u'\,dt + \sqrt{2\nu}\,dW \tag{3.3.1}$$
$$A' = (X')^{-1} \tag{3.3.2}$$
$$u' = \mathbf{P}\left[(\nabla^* A')\,(u_0' \circ A')\right] \tag{3.3.3}$$

with initial data

$$X'(a, 0) = a. \tag{3.3.4}$$

The system (3.3.1) – (3.3.4) differs from the stochastic Lagrangian formulation of the Navier-Stokes equations (2.4.1) – (2.4.4) in the Weber formula: the procedure for recovering the velocity u from the flow map X involves only knowledge of one realization of the Wiener process, and not an average with respect to the Wiener measure, as is the case with the system (2.4.1) – (2.4.4).

The implicit dependence of X on the average in the system (2.4.1) – (2.4.4) and in Theorem 2.6.1 poses numerous analytical difficulties: For instance it is meaningless to talk about solutions of the system (2.4.1) – (2.4.4) which are only defined with probability $\alpha \in (0, 1)$. Further when considering the system (2.4.1) – (2.4.4) in a domain with boundary, it is again meaningless to restrict our attention to only those Wiener paths that avoid the boundary. Such questions are of course meaningful and interesting for the system (3.3.1) – (3.3.4). We this as motivation we study the system (3.3.1) – (3.3.4) in this section.

We will show that for short time, the system (3.3.1) – (3.3.4) is a super-linear approximation of the Navier-Stokes equations, however this system behaves more like a perturbation of the Euler equations as opposed to a perturbation of the Navier-Stokes equations. This will be made precise later in the section.

We begin by remarking that all the bounds and the local existence result proved in Section 3.2 for the system (2.4.1) – (2.4.4) apply verbatim to the system (3.3.1) – (3.3.4). Since the proofs are almost identical, we do not reproduce them here, and throughout this section we repeatedly use analogues of the results from Section 3.2.

The first result we prove shows that stopping and resetting the system (3.3.1) – (3.3.4) at time s, will not give rise to a different velocity u' after time s.

Proposition 3.3.1 (Semigroup property). *Let X', u' satisfy the system* (3.3.1) – (3.3.4) *with initial data u'_0. If $s \leqslant t$, we define $X'_{s,t}$ to be the solution of the SDE* (3.3.1) *at time t, with initial data $X_{s,s}(a) = a$. Let $A'_{s,t} = (X'_{s,t})^{-1}$, then for all s, t we have*

$$u'_t = \mathbf{P}\left[\left(\nabla^* A'_{s,t}\right)\,\left(u'_s \circ A'_{s,t}\right)\right]$$

Proof. If $r < s < t$, we know $X'_{s,t} \circ X'_{r,s} = X'_{r,t}$ and hence $A'_{r,s} \circ A'_{s,t} = A'_{r,t}$. Now by definition of the $\mathbf{P}$, there exists a function q such that

$$u'_s = \left(\nabla^* A'_{0,s}\right) u'_0 \circ A'_{0,s} + \nabla q$$

hence

$$\begin{aligned}\left(\nabla^* A'_{s,t}\right) u'_s \circ A'_{s,t} &= \left(\nabla^* A'_{s,t}\right)\left(\nabla^* A'_{0,s}\big|_{A'_{s,t}}\right)\left(u_0 \circ A'_{0,s} \circ A'_{s,t}\right) + \\ &\qquad + \left(\nabla^* A'_{s,t}\right)\nabla q\big|_{A'_{s,t}} \\ &= \left(\nabla^*\left[A'_{0,s} \circ A'_{s,t}\right]\right)\left(u_0 \circ A'_{0,t}\right) + \nabla\left[q \circ A'_{s,t}\right].\end{aligned}$$

Taking the Leray-Hodge projection of both sides finishes the proof. □

We now show that a solution to (3.3.1) – (3.3.4) is a super-linear approximation to the Navier-Stokes equations.

Theorem 3.3.2. *Let u, X satisfy the system* (2.4.1) – (2.4.4)*, and u', X' satisfy the system* (3.3.1) – (3.3.4) *with initial data $u_0 \in C^{k+1,\alpha}$. Then there exists a constant c and a time $T > 0$, depending only on $k, \alpha, L, \|u_0\|_{k+1,\alpha}$ such that whenever $s, t < T$ we have*

$$\|u_t - \mathbf{E}u'_t\|_{k,\alpha} \leqslant c\left(\frac{\|u_0\|^2_{k+1,\alpha}\, t}{L}\right)^{3/2} \tag{3.3.5}$$

Remark. The quantities $\|u_0\|$ and L appear on the right hand side only for dimensional correctness. Since c is allowed to depend on both L and $\|u_0\|$, they can be inserted and removed as we please, and the main content of (3.3.5) is that $\|u'_t - \mathbf{E}u'_t\| \leqslant O(t^{3/2})$.

Proof. Let $\lambda' = X' - I'$, and $\ell' = A' - I$ be the Lagrangian displacements associated to the system (3.3.1) – (3.3.4). We remark that the estimates proved in section 3.2, in particular Lemma 3.2.4 are true (with identical proofs) for the system (3.3.1) – (3.3.4). Further since u' depends on X' only through it's spatial gradient, the bounds on u' guaranteed by Theorem 3.2.6 will be almost sure bounds.

We first show that $\mathbf{E}\|u'_t - \mathbf{E}u'_t\|_{k,\alpha} \leqslant O(\sqrt{t})$. Note that by Theorem 3.2.6, we know that there exists T, U such that

$$\sup_{0\leqslant t\leqslant T} \|u_t\|_{k+1,\alpha} \leqslant U \qquad \text{and} \qquad \sup_{0\leqslant t\leqslant T} \|u'_t\|_{k+1,\alpha} \leqslant U.$$

From equation (3.3.3) we have

$$u'_t - \mathbf{E}u'_t = \mathbf{P}\left[u_0(A'_t) - \mathbf{E}u_0(A'_t)\right] + \mathbf{P}\left[(\nabla^* \ell'_t)\,(u_0 \circ A'_t) - \mathbf{E}(\nabla^* \ell'_t)\,(u_0 \circ A'_t)\right].$$

We now take the expected value of the $C^{k,\alpha}$ norm of the right. By Lemma 3.2.4, the second term is $O(t)$, and by standard diffusion theory [16] (or by repeating the proof of Lemma 3.2.4), the first term is $O(\sqrt{t})$. Thus for short time,

$$\mathbf{E}\left\|u'_t - \mathbf{E}u'_t\right\|_{k,\alpha} \leqslant cU\sqrt{\tfrac{Ut}{L}} \tag{3.3.6}$$

For the remainder of this proof we use $\bar{u}'$ or $\mathbf{E}u'$ interchangeably, to denote the expected value of u'. Now we estimate $\mathbf{E}\|X - X'\|_{k,\alpha}$:

$$\begin{aligned}
X_t - X'_t &= \int_0^t \left[u_s(X_s) - u'_s(X'_s)\right]\, ds \\
\implies \qquad \mathbf{E}\left\|X_t - X'_t\right\|_{k,\alpha} &\leqslant \mathbf{E}\int_0^t \left\|u_s(X_s) - u'_s(X'_s)\right\|_{k,\alpha}\, ds \\
&\leqslant c\mathbf{E}\int_0^t \Big(\left\|u_s(X_s) - \bar{u}'_s(X'_s)\right\|_{k,\alpha} + \\
&\qquad\qquad + \left\|\bar{u}'_s(X'_s) - u'_s(X'_s)\right\|_{k,\alpha} \Big)\, ds \\
&\leqslant c\int_0^t \Bigg(\left\|u_s - \bar{u}'_s\right\|_{k,\alpha} + \tfrac{U}{L}\mathbf{E}\left\|X_s - X'_s\right\|_{k,\alpha} + \\
&\qquad\qquad\qquad + U\sqrt{\tfrac{Us}{L}}\Bigg)\, ds.
\end{aligned}$$

We used (3.3.6) to obtain the last inequality. Applying Gronwall's Lemma we obtain

$$\mathbf{E}\left\|X_t - X'_t\right\|_{k,\alpha} \leqslant c\left[L\left(\tfrac{Ut}{L}\right)^{3/2} + \int_0^t \left\|u_s - \bar{u}'_s\right\|_{k,\alpha}\right] e^{Ut/L}$$

and by Lemma 3.2.3 this gives

$$\mathbf{E}\,\|A_t - A'_t\|_{k,\alpha} \leqslant c\left[L\left(\tfrac{Ut}{L}\right)^{3/2} + \int_0^t \|u_s - \bar{u}'_s\|_{k,\alpha}\right] e^{Ut/L}. \tag{3.3.7}$$

We are now ready to estimate $\|u - \bar{u}'\|$. From equations (2.4.3) and (3.3.3) we have

$$\|u_t - \bar{u}'_t\|_{k,\alpha} = \|\mathbf{EP}\left[(\nabla^* A_t)(u_0 \circ A_t) - (\nabla^* A'_t)(u_0 \circ A'_t)\right]\|_{k,\alpha}$$

and applying Proposition 3.1.5 and Lemma 3.2.1 we obtain

$$\|u_t - \bar{u}'_t\|_{k,\alpha} \leqslant c\tfrac{U}{L}\,\|A_t - A'_t\|_{k,\alpha}$$

and using (3.3.7) and Gronwall's Lemma, we are done. □

We now find a SPDE governing the evolution of u'.

Proposition 3.3.3. *If u' is a solution of* (3.3.1) – (3.3.4), *then u' satisfies the SPDE*

$$du' + (u' \cdot \nabla)u'\,dt - \nu\triangle u'\,dt + d(\nabla p) + \sqrt{2\nu}\nabla u'\,dW = 0 \tag{3.3.8}$$

$$\nabla \cdot u' = 0 \tag{3.3.9}$$

Proof. We proceed as in the proof of Theorem 2.4.1. We set $w' = (\nabla^* A)u_0 \circ A$, and a computation similar to the one leading up to equation (2.4.11) yields

$$dw' = \left[-(u' \cdot \nabla)w' + \nu\triangle w' - (\nabla^* u')w'\right]dt - \sqrt{2\nu}\nabla w'\,dW \tag{3.3.10}$$

From equation (3.3.3) we know that there exists q' such that $u' = w' + \nabla q'$. Hence

$$\begin{aligned}
du' &= dw' + d(\nabla q') \\
du' &= \left[-(u' \cdot \nabla)w' + \nu\triangle w' - (\nabla^* u')w'\right]dt - \sqrt{2\nu}\nabla w'\,dW + d(\nabla q') \\
&= \left[-(u' \cdot \nabla)(u' - \nabla q') + \nu\triangle(u' - \nabla q') - (\nabla^* u')(u' - \nabla q')\right]dt + \\
&\qquad - \sqrt{2\nu}\nabla(u' - \nabla q')\,dW + d(\nabla q) \\
&= \left[-(u' \cdot \nabla)u' + \nu\triangle u'\right]dt + d(\nabla p')
\end{aligned}$$

where

$$p' = q' + \int_0^t \left[(u' \cdot \nabla)q' - \nu\triangle q' - \tfrac{1}{2}|u'|^2\right]ds + \sqrt{2\nu}\int_0^t \partial_j q'\,dW_s^{(j)}.$$ □

Though evident, we draw attention to the striking similarity between the system 3.3.8 – 3.3.9 and the Navier-Stokes equations. We will now show that despite this similarity, the $\mathbf{E}u'$ does not behave like a solution of the Navier-Stokes equations. The covariance between u' and $\nabla u'$ in the term $\mathbf{E}(u' \cdot \nabla)u'$ will alter the behaviour of the solution and as mentioned earlier, the system (3.3.1) – (3.3.4) behaves more like the Euler equations. We now show that the system (3.3.1) – (3.3.4) can be thought of as a random translate of the solution of the Euler equations.

Theorem 3.3.4. *Let v be a solution of the Euler equations* (2.3.1) – (2.3.2) *with initial data u_0, and u' a solution to the stochastic system* (3.3.1) – (3.3.4)*. Then*

$$u'(x,t) = v(x - \sqrt{2\nu} W_t, t) \tag{3.3.11}$$

One method of proof would be to apply the Itô formula to $v(x - \sqrt{2\nu} W_t, t)$, and see that u' satisfies the SPDE (3.3.8), and use uniqueness. We provide a different proof here, which does not rely on the uniqueness for the SPDE (3.3.8).

Proof. We use the computations involved in the alternate proof of Theorem 2.4.1 as given in Section 2.5. We define u^ω, w^ω as we did in the proof of Theorem 2.4.1 in Section 2.5. Following the computations leading up to (2.5.10), we obtain

$$u' = \tau_{\sqrt{2\nu} W_t} w^\omega$$

and hence

$$w^\omega(x,t) = u'(x + W_t, t) = u^\omega(x,t).$$

Thus equation (2.5.7) gives us

$$\partial_t u^\omega + (u^\omega \cdot \nabla)\, u^\omega + \nabla p^\omega = 0$$

where $p^\omega = q^\omega + \frac{1}{2}|u^\omega|^2$. Thus u^ω satisfies the Euler equations with initial data u_0, and by uniqueness of strong solutions we have

$$v(x,t) = u^\omega(x,t) = u(x + \sqrt{2\nu} W_t, t)$$

concluding the proof. $\square$

We conclude by studying the vanishing viscosity limit of (3.3.1) – (3.3.4).

Proposition 3.3.5. *Let $u_0 \in C^{2,\alpha}$, and v be a solution of the Euler equations with initial data u_0. Suppose $T > 0$ is such that for each $\nu > 0$, the system* (3.3.1) – (3.3.4) *has a solution u'_ν defined on the interval* $[0, T]$*. Then*

$$\lim_{\nu \to 0} u'_\nu(x,t) = v(x,t)$$

almost surely.

Proof. The proof is immediate from the identity (3.3.11). $\square$

Remark 3.3.6. Our assumption $u_0 \in C^{2,\alpha}$ is necessary to ensure that we have a local existence theorem for (3.3.1) – (3.3.4) uniformly in ν, and that the identity (3.3.11) holds.

We believe that result and proof will still be true for domains with boundary. The intuitive justification for this is because for a given realization of the Wiener process W, we can always choose ν small enough so that the path of the diffusion X' will not exit the domain in time T. In this case identity (3.3.11) is still valid, and thus our proof of Proposition 3.3.5 should go through.

This will mean that the 'boundary layer' of the system (3.3.1) – (3.3.4) will not affect the interior flow as $\nu \to 0$. This is of course an open question for the Navier-Stokes equations.

REFERENCES

[1] J. T. Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for the* 3-*D Euler equations*. Comm. Math. Phys. **94** (1984), 61–66.

[2] R. N. Bhattacharya, L. Chen, S. Dobson, R. B. Guenther, C. Orum, M. Ossiander, E. Thomann, E. C. Waymire, *Majorizing kernels and stochastic cascades with applications to incompressible Navier-Stokes equations.* Trans. Amer. Math. Soc. **355** (2003), no. 12, 5003–5040 (electronic).

[3] B. Busnello, F. Flandoli, M. Romito *A probabilistic representation for the vorticity of a three-dimensional viscous fluid and for general systems of parabolic equations.* Proc. Edinb. Math. Soc. (2) **48** (2005), no. 2, 295–336.

[4] B. Busnello, *A probabilistic approach to the two-dimensional Navier-Stokes equations.* Ann. Probab. **27** (1999), no. 4, 1750–1780.

[5] S. Chen, C. Foias, D. Holm, E. Olson, E. Titi, S. Wynne, *Camassa-Holm equations as a closure model for turbulent channel and pipe flow.* Phys. Rev. Lett. **81** (1998), no. 24, 5338–5341.

[6] A. Chorin, *Numerical study of slightly viscous flow.* J. Fluid Mech. **57** (1973), no. 4, 785–796.

[7] A. Chorin, *Vortex models and boundary layer instability.* SIAM J. Sci. Statist. Comput. **1** (1980), no. 1, 1–21.

[8] A. Chorin, J. Marsden, *A Mathematical Introduction to Fluid Mechanics.* Springer (2000).

[9] P. Constantin, C. Foias *Navier-Stokes Equations.* University of Chicago Press (1988).

[10] P. Constantin, *An Eulerian-Lagrangian approach for incompressible fluids: local theory.* J. Amer. Math. Soc. **14** (2001) no. 2, 263–278 (electronic).

[11] P. Constantin, *An Eulerian-Lagrangian Approach to the Navier-Stokes equations.* Comm. Math. Phys. **216** (2001) no. 3, 663–686.

[12] A. Friedman, *Stochastic Differential Equations and Applications, Volume 1.* Academic Press (1975).

[13] J. Goodman, *Convergence of the random vortex method.* Comm. Pure Appl. Math. **40** (1987), no. 2, 189–220.

[14] D. Holm, J. Marsden, T. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories.* Adv. Math. **137** (1998), no. 1, 1–81.

[15] G. Iyer, *A stochastic perturbation of inviscid flows.* Eprint: math.AP/0505066

[16] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus.* Graduate Texts in Mathematics **113** (1991).

[17] H. Kunita, *Stochastic flows and stochastic differential equations.* Cambridge Studies in Advanced Mathematics, **24** (1997).

[18] C. LeBris, P. L. Lions, *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications.* Ann. Mat. Pura Appl. (4) **183** (2004), no. 1, 97–130.

[19] J. LeGall, *Spatial Branching Processes, Random Snakes and Partial Differential Equations.* Lectures in Mathematics, Birkhäuser (1999).

[20] Y. LeJan, A. S. Sznitman, *Stochastic cascades and 3-dimensional Navier-Stokes equations.* Probab. Theory Related Fields **109** (1997), no. 3, 343–366.

[21] D. Long, *Convergence of the random vortex method in two dimensions.* J. Amer. Math. Soc. **1** (1988), no. 4, 779–804.

[22] A. Majda, A. Bertozzi, *Vorticity and Incompressible Flow.* Cambridge University Press (2002).

[23] C. Peskin, *A random-walk interpretation of the incompressible Navier-Stokes equations.* Comm. Pure Appl. Math. **38** (1985), no. 6, 845–852

[24] S. Pope, *Stochastic Lagrangian models of velocity in homogeneous turbulent shear flow.* Physics of fluids **14** (2002), no. 5, 1696–1702.

[25] E. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press (1970).